9.13. **Displacement-current flux**

The electric field between the plates equals $\sigma/\epsilon_0$, so the displacement current is $J_\text{d} = \epsilon_0(\partial E/\partial t) = d\sigma/dt$. The flux through $S'$ is therefore $\Phi = J_\text{d}A = (d\sigma/dt)A$, where $A$ is the area of each plate. Hence,

$$\Phi = \frac{d\sigma}{dt}A = \frac{d(\sigma A)}{dt} = \frac{dQ}{dt} = I, \tag{593}$$

as desired. We haven't paid attention to signs, but if the right plate in Fig. 9.4 is positive, and if the capacitor is discharging, then the displacement current points to the right. (The $\mathbf{E}$ field between the plates points to the left, but it is decreasing, so $\partial\mathbf{E}/\partial t$ points to the right.) The displacement-current flux therefore passes from left to right through $S'$, just as the real-current flux passes from left to right through $S$. The total flux through the closed volume bounded by $S$ and $S'$ is zero, as it should be, because a closed surface has no boundary, so the line integral of $\mathbf{B}$ around this (non-existent) boundary is zero.

9.14. **Sphere with a hole**

Very close to the wire, the magnetic field is $B = \mu_0 I/2\pi r$. Therefore $\int_C \mathbf{B}\cdot d\mathbf{s} = (\mu_0 I/2\pi r)(2\pi r) = \mu_0 I$. On the right-hand side of Maxwell's equation, the term involving $\mathbf{J}$ is zero because no current pierces the surface $S$ (the sphere-minus-hole). To calculate the term involving $\partial\mathbf{E}/\partial t$, we know that the electric field at points on the surface $S$ is $E = Q/4\pi\epsilon_0 R^2$, where $Q$ is the point charge and $R$ is the radius of the sphere. Hence $dE/dt = (dQ/dt)/4\pi\epsilon_0 R^2 = I/4\pi\epsilon_0 R^2$. Integrating this over the surface of the sphere brings in a factor of $4\pi R^2$. Remembering the factor of $\mu_0\epsilon_0$ out front, the right-hand side of Maxwell's equation equals $\mu_0\epsilon_0(I/4\pi\epsilon_0 R^2)(4\pi R^2) + 0 = \mu_0 I$, in agreement with the left-hand side.

9.15. **Field inside a discharging capacitor**

Written in terms of the displacement current, the integral law reads

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_S (\mathbf{J}_\mathrm{d} + \mathbf{J}) \cdot d\mathbf{a}. \tag{594}$$

Since $s \ll b$ we can neglect the edge fields, in which case the displacement current $\mathbf{J}_\mathrm{d}$ is uniformly distributed in the gap. The integral of $\mathbf{J}_\mathrm{d}$ over the area of the plates equals the conduction current $I$ in the wire (see Exercise 9.13). The fraction of $\int \mathbf{J}_\mathrm{d} \cdot d\mathbf{a} = I$ that is enclosed in a circle through $P$, centered on the axis, is $\pi r^2/\pi b^2$. The integral law applied to this circle therefore gives (with the conduction current $\mathbf{J} = 0$ inside the capacitor)

$$2\pi r B = \mu_0 \left( I \frac{r^2}{b^2} \right) + 0 \quad \Longrightarrow \quad B = \frac{\mu_0 I r}{2\pi b^2}, \tag{595}$$

as desired. The similarity of this calculation to the calculation of the $\mathbf{E}$ field in Fig. 7.16 is the following. If we solve the problem straight from Maxwell's equation, without invoking the definition of the displacement current, we can write (with $\mathbf{J} = 0$ inside the capacitor)

$$\int_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} \quad \Longrightarrow \quad 2\pi r B = \mu_0 \epsilon_0 \frac{d\Phi_\mathrm{E}}{dt}, \tag{596}$$

where $\Phi_\mathrm{E}$ is the flux of the electric field through the given surface. This equation is exactly analogous to Faraday's law of induction, which we used in the example of Fig. 7.16 (among many other places),

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \quad \Longrightarrow \quad 2\pi r E = -\frac{d\Phi_\mathrm{B}}{dt}. \tag{597}$$

The similarity arises because of the symmetry of the two "curl" Maxwell's equations; and also because there is no current $\mathbf{J}$ of real electric charges inside the capacitor in the present problem, and likewise there is no current of real magnetic charges in Fig. 7.16 (or anywhere else) because magnetic monopoles don't exist (as far as we know).

9.18. **Associated B field**

The wave is traveling in the $-\hat{\mathbf{z}}$ direction, as shown by the sign in $(z+ct)$; if $t$ increases, then $z$ must decrease to keep the same value of $(z+ct)$. $\mathbf{B}$ is perpendicular to both this direction and to $\mathbf{E}$. So $\mathbf{B}$ must point in the $\pm(\hat{\mathbf{x}}-\hat{\mathbf{y}})$ direction. But since we know that $\mathbf{E}\times\mathbf{B}$ points in the direction of the wave's velocity, which is $-\hat{\mathbf{z}}$, we must pick the "+" sign, as you can quickly verify with the right-hand rule. The magnitude of $\mathbf{B}$ is $1/c$ times the magnitude of $\mathbf{E}$, so the desired $\mathbf{B}$ field is

$$\mathbf{B} = (E_0/c)(\hat{\mathbf{x}}-\hat{\mathbf{y}})\sin[(2\pi/\lambda)(z+ct)]. \tag{602}$$

With $E_0 = 20\,\text{V/m}$, we have $B_0 = E_0/c = (20\,\text{V/m})(3\cdot 10^8\,\text{m/s}) = 6.67\cdot 10^{-8}$ T. The amplitudes of the $\mathbf{E}$ and $\mathbf{B}$ waves are actually $\sqrt{2}$ times $E_0$ and $B_0/c$, respectively, because the magnitude of the $(\hat{\mathbf{x}}\pm\hat{\mathbf{y}})$ vectors is $\sqrt{2}$.

9.19. **Find the wave**

It is given that $\mathbf{E}\perp\hat{\mathbf{z}}$. And we know that $\mathbf{E}\perp\mathbf{v}$, where $\mathbf{v}\propto -\hat{\mathbf{x}}$ here. So $\mathbf{E}$ must point in the $\pm\hat{\mathbf{y}}$ direction. Let's pick $+\hat{\mathbf{y}}$. The other direction would simply change the sign of $E_0$; the sign is arbitrary, since the trig function switches signs anyway. So we have (a sine would work just as well)

$$\mathbf{E} = \hat{\mathbf{y}}E_0\cos(kx+\omega t), \tag{603}$$

where $\omega = 2\pi\nu = 6.28\cdot 10^8\,\text{s}^{-1}$ and $k=\omega/c = 2.09\,\text{m}^{-1}$. The sign inside the cosine is a "+" because the wave is traveling in the negative $x$ direction. Since $\mathbf{E}\times\mathbf{B}$ points in the direction of $\mathbf{v}$, which is $-\hat{\mathbf{x}}$, and since $B_0 = E_0/c$, the $\mathbf{B}$ field must take the form,

$$\mathbf{B} = -\hat{\mathbf{z}}(E_0/c)\cos(kx+\omega t). \tag{604}$$

9.23. **Field in a box**

We immediately see that $\nabla \cdot \mathbf{E} = 0$, because $E_z$ has no $z$ dependence. And also $\nabla \cdot \mathbf{B} = 0$, because the $\partial B_x/\partial x$ and $\partial B_y/\partial y$ terms cancel. So two of Maxwell's equations are satisfied. For the other two, we can calculate the curls via the usual determinant method,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ E_x & E_y & E_z \end{vmatrix}. \tag{610}$$

You can verify that the various derivatives are

$$\begin{aligned} \nabla \times \mathbf{E} &= kE_0\big(-\hat{\mathbf{x}}\cos kx \sin ky + \hat{\mathbf{y}} \sin kx \cos ky\big)\cos\omega t, \\ \frac{\partial \mathbf{E}}{\partial t} &= -\omega\hat{\mathbf{z}}E_0 \cos kx \cos ky \sin\omega t, \\ \nabla \times \mathbf{B} &= -2k\hat{\mathbf{z}}B_0 \cos kx \cos ky \sin\omega t, \\ \frac{\partial \mathbf{B}}{\partial t} &= \omega B_0\big(\hat{\mathbf{x}}\cos kx \sin ky - \hat{\mathbf{y}} \sin kx \cos ky\big)\cos\omega t. \end{aligned} \tag{611}$$

Therefore, $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ gives $kE_0 = \omega B_0$. And $\nabla \times \mathbf{B} = (1/c^2)\partial\mathbf{E}/\partial t$ gives $2kB_0 = \omega E_0/c^2$. These two requirements quickly yield $\omega = \sqrt{2}ck$ and $E_0 = \sqrt{2}cB_0$, as desired. (Technically, $\omega = -\sqrt{2}ck$ and $E_0 = -\sqrt{2}cB_0$ also work, but these relations yield the same wave, as you can verify.)

The fields don't depend on $z$, so to determine what they look like, let's consider the square cross section of the box in the $xy$ plane. At all times, $\mathbf{E}$ is zero on the boundary of the box where $(x, y) = (\pm\pi/2k, \pm\pi/2k)$. At a given instant in time, $\cos\omega t$ takes on a specific value, so $\mathbf{E}$ is proportional to $\hat{\mathbf{z}}\cos kx \cos ky$. This function is maximum at the origin. The plot of $E_z \propto \cos kx \cos ky$ is basically a bump above the $xy$ plane (or a valley below the $xy$ plane at times when $\cos\omega t$ is negative). The bump oscillates up and down according to $\cos\omega t$. The level curves of constant $E_z$ are given by $\cos kx \cos ky = C$. You can show with a Taylor series that these level curves are circles near the origin. So the curves start off as circles and end up as squares. They are shown roughly in Fig. 149. Since $\mathbf{E}$ has only a $z$ component, it points perpendicular to the page.

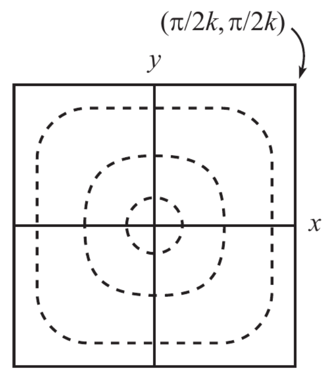

**Figure 149**

$\mathbf{B}$ isn't quite as clean, but it's easy to get a handle on its values along the $x$ and $y$ axes, and along the 45° lines, and also along the boundary of the box. Some sample vectors at times when $\sin\omega t = 1$ are shown in Fig. 150. All the vectors oscillate in phase according to $\sin\omega t$.

REMARK: That's all that was required, but we can say a little more about the fields. For small $x$ and $y$, we can use $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ to obtain $\mathbf{B} \approx kB_0(\hat{\mathbf{x}}y - \hat{\mathbf{y}}x)\sin\omega t$. The field lines associated with this $\mathbf{B}$ field are circles, because the vector $\mathbf{B} \propto (y, -x)$ is always perpendicular to the radial vector $(x, y)$. Alternatively, since the tangent to the field line is in the direction of $\mathbf{B}$, we can separate variables and integrate $dy/dx = B_y/B_x = -x/y$ to obtain $x^2 + y^2 = C$, where $C$ is a constant. The $\mathbf{B}$ field goes to zero at the origin.

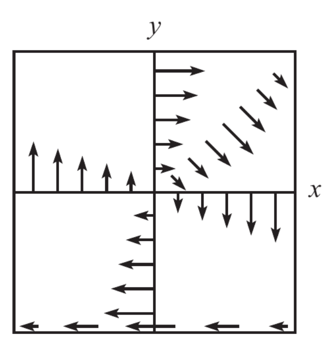

**Figure 150**

What do the **B** field lines look like for general $x$ and $y$ values? Again, since the tangent to the field line is in the direction of **B**, we have the general relation,

$$\frac{dy}{dx} = \frac{B_y}{B_x} = -\frac{\sin kx \cos ky}{\cos kx \sin ky}. \tag{612}$$

Separating variables and integrating gives $\ln(\cos ky) = -\ln(\cos kx) + D$, where $D$ is a constant. Exponentiating gives $\cos kx \cos ky = C$, where $C = e^D$ is another constant. Small values of $C$ yield near-squares close to the boundary of the box, and values close to 1 yield the small near-circles close to the origin we found above. Note that the $\cos kx \cos ky = C$ curves of the **B** field lines are also the curves of constant $E_z$, which we found above and plotted in Fig. 149. This can be traced to the fact that if **E** has only a $z$ component, then $\nabla \times \mathbf{E}$ is perpendicular to $\nabla E_z$, as you can verify.

9.25. **Microwave background radiation**

As shown in Section 9.6, the average energy density $\mathcal{U}$ of a sinusoidal electromagnetic wave is $\mathcal{U} = \epsilon_0 E_0^2/2 = \epsilon_0 E_{\text{rms}}^2$. So we have

$$E_{\text{rms}}^2 = \frac{\mathcal{U}}{\epsilon_0} = \frac{4 \cdot 10^{-14}\,\text{J/m}^3}{8.85 \cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg m}^3}} = 4.5 \cdot 10^{-3}\,\text{V}^2/\text{m}^2 \quad \Longrightarrow \quad E_{\text{rms}} = 0.067 \text{ V/m}. \tag{614}$$

If the 1 kilowatt radiated by the transmitter is spread out over a sphere of radius $R$, then the power density at radius $R$ equals $S = (10^3\,\text{W})/4\pi R^2$. The energy density is then $\mathcal{U} = S/c$. We therefore want

$$\frac{1}{c} \cdot \frac{10^3\,\text{W}}{4\pi R^2} = 4 \cdot 10^{-14}\,\text{J/m}^3 \quad \Longrightarrow \quad R = 2600 \text{ m}, \tag{615}$$

or 2.6 km. However, the power is undoubtedly emitted in at least a somewhat directed manner, so the distance from an actual radio transmitter would be larger than this.

9.26. **An electromagnetic wave**

(a) The fields are

$$\mathbf{E} = \hat{\mathbf{y}} E_0 \sin(kx + \omega t), \qquad \text{and} \qquad \mathbf{B} = -\hat{\mathbf{z}}(E_0/c)\sin(kx + \omega t). \tag{616}$$

We immediately see that $\nabla \cdot \mathbf{E} = 0$ (because the lone $y$ component of **E** has no $y$ dependence) and $\nabla \cdot \mathbf{B} = 0$ (because the lone $z$ component of **B** has no $z$ dependence). So two of Maxwell's equations are satisfied. For the other two, you can verify that

$$\begin{aligned} \nabla \times \mathbf{E} &= \hat{\mathbf{z}} k E_0 \cos(kx + \omega t), & \frac{\partial \mathbf{E}}{\partial t} &= \hat{\mathbf{y}} \omega E_0 \cos(kx + \omega t), \\ \nabla \times \mathbf{B} &= \hat{\mathbf{y}} k (E_0/c) \cos(kx + \omega t), & \frac{\partial \mathbf{B}}{\partial t} &= -\hat{\mathbf{z}} \omega (E_0/c) \cos(kx + \omega t). \end{aligned} \tag{617}$$

Therefore, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ requires $k = \omega/c$. And (using $\mu_0 \epsilon_0 = 1/c^2$) $\nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t$ requires $k/c = (1/c^2)\omega$, which again says that $k = \omega/c$.

(b) The wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{2\pi(3 \cdot 10^8\,\text{m/s})}{10^{10}\,\text{s}^{-1}} = 0.19\,\text{m}. \tag{618}$$

As shown in Section 9.6, the average energy density of a sinusoidal electromagnetic wave is $\epsilon_0 E_0^2/2$, which equals

$$\frac{1}{2}\epsilon_0 E_0^2 = \frac{1}{2}\Big(8.85 \cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\Big)(10^3\,\text{V/m})^2 = 4.4 \cdot 10^{-6}\,\text{J/m}^3. \tag{619}$$

The power density equals the energy density times the speed, so

$$S = \frac{1}{2}\epsilon_0 E_0^2 c = (4.4 \cdot 10^{-6}\,\text{J/m}^3)(3 \cdot 10^8\,\text{m/s}) = 1300\ \text{J/(m}^2\text{s)}. \tag{620}$$

9.28. **Poynting vector and resistance heating**

The electric field inside the wire is given by $E = J/\sigma$. Since the curl of $\mathbf{E}$ is zero, we can draw a thin rectangular loop along the surface to show that the electric field right outside the wire is also $E = J/\sigma$ (and it points in the direction of the current, of course). The magnetic field right outside the wire points tangentially with the usual magnitude of $B = \mu_0 I/2\pi R$, where $R$ is the radius of the wire. $\mathbf{E}$ and $\mathbf{B}$ are perpendicular, and you can show with the right-hand rule that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ points radially into the wire. So the direction is correct; the energy in the wire increases, consistent with the fact that it heats up. The magnitude of $\mathbf{S}$ equals

$$S = \frac{1}{\mu_0}EB = \frac{1}{\mu_0}\frac{J}{\sigma}\frac{\mu_0 I}{2\pi R} = \frac{JI}{2\pi R\sigma}\,. \tag{623}$$

To obtain the power flux into the wire through the surface, we must multiply by $2\pi R\ell$, where $\ell$ is the length of a given section of the wire. So the total energy flow per time into a length $\ell$ of the wire is

$$P_\ell = S \cdot 2\pi R\ell = \frac{JI}{2\pi R\sigma}2\pi R\ell = \frac{JI}{\sigma}\ell = \frac{(I/A)I}{\sigma}\ell = I^2\frac{\ell}{\sigma A} = I^2\frac{\rho\ell}{A} = I^2 R, \tag{624}$$

where $R$ is the resistance of the length $\ell$ of the wire. We have used the fact that the resistivity $\rho$ is given by $\rho = 1/\sigma$. As desired, $P_\ell$ equals the rate of resistance heating in the length $\ell$ of the wire. $P_\ell$ can also be written as $I(IR) = IV$, of course, where $V$ is the voltage drop along the length $\ell$ of the wire.

Alternatively, we never actually had to use the $J/\sigma$ form of $E$. A quicker method is:

$$P_\ell = S \cdot 2\pi R\ell = \frac{1}{\mu_0}E\frac{\mu_0 I}{2\pi R} \cdot 2\pi R\ell = IE\ell = IV, \tag{625}$$

9.30. **Comparing the energy densities**

If $E(t) = E_0 \cos\omega t$, then $\partial E/\partial t = -\omega E_0 \sin\omega t$, so the amplitude of the $B$ field given in Eq. (9.46) is $B_0 = (\epsilon_0\mu_0 r/2)(\omega E_0)$. The ratio of the magnetic energy density to the electric energy density is therefore

$$\frac{\dfrac{B_0^2}{2\mu_0}}{\dfrac{\epsilon_0 E_0^2}{2}} = \frac{\dfrac{1}{2\mu_0}\left(\dfrac{\epsilon_0\mu_0 r}{2}\,\omega E_0\right)^2}{\dfrac{\epsilon_0}{2}E_0^2} = \frac{\mu_0\epsilon_0 r^2\omega^2}{4} = \left(\frac{\pi r}{cT}\right)^2, \qquad (631)$$

where we have used $\omega = 2\pi/T$ and $1/\mu_0\epsilon_0 = c^2$. As desired, this result is small if the period $T$ much larger than $r/c$, which is (half) the time it takes light to travel across the capacitor disks. As in Problem 9.6, we have ignored the high-order feedback effects between $E$ and $B$. These effects are negligible if the current doesn't change too quickly.