PHYSICS 4C | FINAL EXAM | FALL QUARTER 2019
PROF. HIRSCH | | DECEMBER 11, 2019

Formulas:

$F = k\frac{q_1 q_2}{r^2}$ Coulomb's law ; $k = \frac{1}{4\pi\varepsilon_0} = 8.99\times 10^9 \mathrm{N\cdot m^2/C^2}$; $\varepsilon_0 = 8.85\times 10^{-12} C^2/Nm^2$

Electric field: $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{kq}{r^2}\hat{r}$; $\vec{E}(\vec{r}) = k\int \rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}d^3r'$; $\vec{F} = q_0\vec{E}$

Linear, surface, volume charge density: $dq = \lambda\, d\ell$, $dq = \sigma\, da$, $dq = \rho\, dv$

Electric field of infinite: line of charge: $E = \frac{\lambda}{2\pi\varepsilon_0 r}$; sheet of charge: $E = \frac{\sigma}{2\varepsilon_0}$

$\vec{\nabla}\times\vec{E} = 0$ (electrostatics) Divergence theorem: $\int d^3r\, \vec{\nabla}\cdot\vec{F} = \oint \vec{F}\cdot d\vec{a}$

Gauss law: $\Phi = \oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int d^3r\, \rho(\vec{r})$; Φ=electric flux

Energy: $U = k\frac{q_1 q_2}{r_{12}}$ $U = \frac{\varepsilon_0}{2}\int E(\vec{r})^2 d^3r$ Work: $W = \int \vec{F}\cdot d\vec{s}$

Electric potential: $\phi(P_2) - \phi(P_1) = -\int_{P_1}^{P_2}\vec{E}\cdot d\vec{s}$ $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

Point charge: $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ Dipole: $\phi(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

$U = \frac{1}{2}\int \rho\,\phi\, d^3r$ $\vec{E} = -\vec{\nabla}\phi$; $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$; $\vec{\nabla}\times\vec{E} = 0$

Capacitors: $Q = C\phi$, $U = \frac{Q^2}{2C}$; Planar: $C = \frac{\varepsilon_0 A}{s}$ electric field: $E = \sigma/\varepsilon_0$

$I = \int \vec{J}\cdot d\vec{a}$, $I = \frac{dQ}{dt}$, $\vec{J} = nq\vec{u}$; $div\vec{J} = -\frac{\partial\rho}{\partial t}$; Power: $P = I^2R$; $P = \varepsilon I$

V=IR; $\vec{J} = \sigma\vec{E}$; $\vec{E} = \rho\vec{J}$; $R = \rho\frac{L}{A}$; $\sigma = \frac{ne^2\tau}{m_e}$; $Q(t) = C\varepsilon(1-e^{-t/RC})$; $Q(t) = Q_0 e^{-t/RC}$

Lorentz force: $\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$; force on wire: $d\vec{F} = Id\vec{\ell}\times\vec{B}$; solenoid: $B = \mu_0 nI$

Stokes' theorem: $\oint_C \vec{F}\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$; vector potential: $\vec{\nabla}\times\vec{A} = \vec{B}$

Ampere's law: $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I_{enc} = \mu_0\int_S \vec{J}\cdot d\vec{a}$; $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$; $\vec{\nabla}\cdot\vec{B} = 0$

Field of a wire: $B = \frac{\mu_0 I}{2\pi r}\hat{\theta}$; Force between wires: $F = \frac{\mu_0 I_1 I_2 \ell}{2\pi r}$; cyclotron: $\omega = \frac{qB}{m}$

loop (axis): $B = \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}}$; Biot-Savart: $d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{\ell}\times\hat{r}}{r^2}$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$

Faraday's law $\varepsilon = \frac{1}{q}\int \vec{f}\cdot d\vec{s} = -\frac{d\Phi}{dt} = \oint \vec{E}\cdot d\vec{l}$; $\Phi = \int \vec{B}\cdot d\vec{a}$; $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Mutual inductance: $M_{21} = \frac{N_2\Phi_{21}}{I_1}$; $\varepsilon_2 = -M_{21}\frac{dI_1}{dt}$; $M_{21} = M_{12} = M$

Self - inductance: $L = \frac{N\Phi_B}{I}$; $\varepsilon_L = -L\frac{dI}{dt}$; $\frac{L}{l} = \mu_0 n^2 A$ for solenoid ; energy $U = \frac{1}{2}LI^2$

energy density: magnetic $u = \frac{1}{2}\frac{B^2}{\mu_0}$; electric $u = \frac{1}{2}\varepsilon_0 E^2$, $\mu_0 = 4\pi\times 10^{-7}\, kg\, m/C^2$

LR circuit: $I = \frac{V_0}{R}(1-e^{-t/\tau_L})$ (rise) ; $I = I_0 e^{-t/\tau_L}$ (decay) ; $\tau_L = L/R$

LRC circuit: $Q(t) = Q_0 e^{-\frac{R}{2L}t}\cos(\omega t+\phi)$; $\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$; $\omega_0 = 1/\sqrt{LC}$

AC circuit, LRC: $\varepsilon = \varepsilon_0 e^{i\omega t}$; $I = I_0 e^{i(\omega t+\phi)}$; $I = \varepsilon / Z$; $\tan\phi = -\frac{Z_L + Z_C}{iR}$

$Z_L = i\omega L$; $Z_C = 1/i\omega C$; $Z = R + Z_L + Z_C$; $I_0 = \varepsilon_0/|Z|$; $\bar{P} = I_{rms}V_{rms}\cos\phi$

Ampere- Maxwell law: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} + \mu_0\vec{J}_d$

Electromagnetic wave in vacuum: $c = 1/\sqrt{\mu_0\varepsilon_0} = \omega/k$

$\vec{E} = \hat{z}E_0\sin(ky-\omega t)$; $\vec{B} = \hat{x}B_0\sin(ky-\omega t)$; propagates in direction $\vec{E}\times\vec{B}$

Poynting vector: $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$; $\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$; $u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$

Electric dipole moment: $\vec{p} = \int d^3r\rho(\vec{r})\vec{r}$ field: $E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}$; $E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$

$\vec{N} = \vec{p}\times\vec{E}$; $U = -\vec{p}\cdot\vec{E}$; $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$; $P = Np = dipole\ moment\ /\ volume$

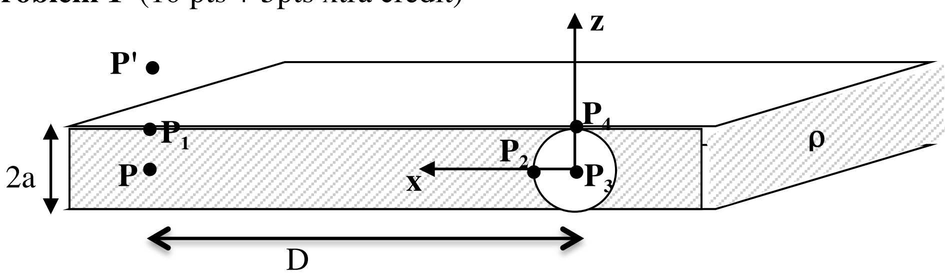
Magnetic dipole moment: $\vec{m} = I\,\vec{a}$ field $B_r = \frac{\mu_0 m\cos\theta}{2\pi r^3}$; $B_\theta = \frac{\mu_0 m\sin\theta}{4\pi r^3}$

$\vec{N} = \vec{m}\times\vec{B}$; $U = -\vec{m}\cdot\vec{B}$; $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$ orbital motion: $\vec{m} = (q/2M)\vec{L}$

$M = Nm$ magnetization = magnetic moment/volume (q=charge, M=mass)

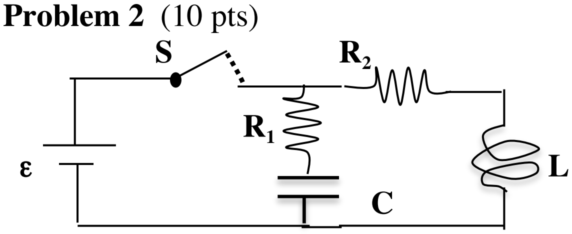
8 problems. Please write clearly and explain your reasoning.

Problem 1 (10 pts + 5pts xtra credit)



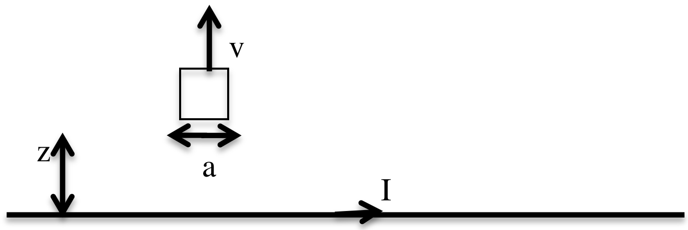
Consider an infinite slab of thickness 2a with uniform volume charge density ρ>0. A spherical hole of radius a is carved out as shown in the figure. Use a coordinate system so that the z axis is perpendicular to the slab, and the origin is at the center of the spherical hole. Assume the electric potential at point P=(D, 0, 0) is 0, where D>>a, and the electric potential at point P'=(D, 0, 2a) is -3V. Find the electric potential (magnitude and sign) at:
(a) P_1=(D,0,a)
(b) P_2=(a,0,0)
(c) P_3=(0,0,0)
(d) P_4=(0,0,a)
(e) If you put a point negative charge very near the center of the hole that can move inside the hole, will it end up at P_2, P_3 or P_4?
(f) Same as (e) for a point positive charge.
Hint: first write expressions for the electric field of the slab alone, then of the sphere alone with appropriate charge density, then use superposition.

Problem 2 (10 pts)



In the circuit shown in the figure, R_1=R_2=R, L is the self-inductance of the inductor and C the capacitance of the capacitor. Initially C is uncharged. At time t=0 the switch S is closed.
(a) Immediately after S is closed, what is the current through R_1 and what is the current throught R_2?
(b) A long time after S is closed, what is the current through R_1 and what is the current throught R_2?
(c) A long time after S is closed, what is the charge on the capacitor?
(d) A long time after S is closed, it is opened again. What is the current through R_1 right after S is opened again?
(e) It is found that a while after S is opened again, the charge on the capacitor plates changes sign. Give a necessary condition on the parameters for this to happen for the circuit under consideration.

Problem 3 (10 pts)



A straight long wire carries constant current I. A square wire loop of side length a is at distance z(t) from the wire, and is moving away from the straight wire at constant speed v as shown in the figure. The loop has resistance R. Note that two sides of the loop are parallel to the long wire, two are perpendicular. Note also that a is not necessarily small compared to z. Ignore the self-inductance of the loop.

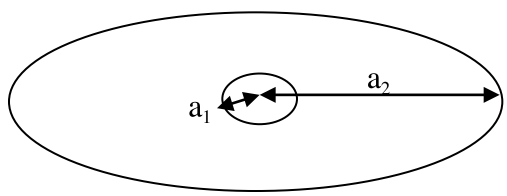
(a) Find an expression for the current i induced in the loop. Is it clockwise or counterclockwise? Explain your reasoning.

(b) Find an expression for the energy per unit time (power) P dissipated in the loop.

(c) Using the formula for the magnetic force on a current carrying wire, find an expression for the net force F that has to be applied in order to pull the loop away from the long wire at constant speed v.

(d) Compare the expressions you found for P and for F times v and explain why they are the same or different.

Problem 4 (10 pts)



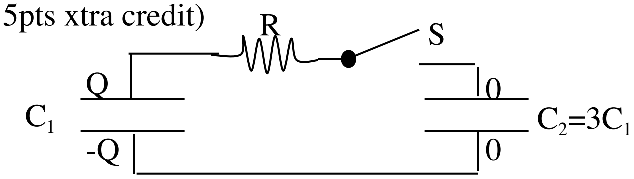
Consider two concentric rings of metallic wire of radii a_1 and a_2, with $a_2=10a_1$. Their resistivities are the same, ρ, and the cross-sectional area of the metallic wire is A, the same for both rings.

(a) Find their mutual inductance M.

(b) When a time-dependent current $I(t)=I_0\, t^2/\tau^2$ circulates through the inner ring, the induced current at the outer ring at time t=1s is 3mA. If instead that current I(t) circulates through the outer ring, what would be the current induced in the inner ring at time t=1s?

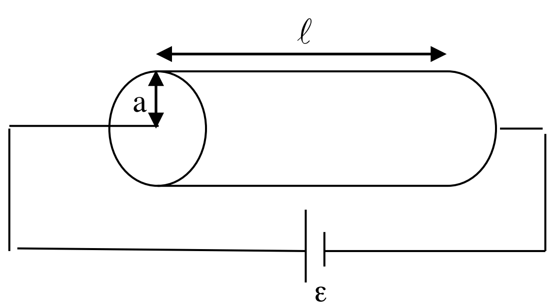
(c) In the latter case, what would be the current induced in the inner ring at time t=2s?

Problem 5 (10 pts + 5pts xtra credit)



Consider the circuit shown in the figure. Initially the left capacitor C_1 has charge Q, the right capacitor $C_2=3C_1$ is uncharged. At time t=0 the switch S is closed.
(a) Find the charge in each capacitor a long time after the switch is closed.
(b) Write a differential equation in terms of $Q_1(t)$, $Q_2(t)$ and I(t), that describes the time evolution of the charge in the capacitors and the current I through the resistor.
(c) Rewrite the differential equation so that it only involves $Q_2(t)$ and its time derivative(s), and does not involve neither $Q_1(t)$ nor I(t) explicitly.
(d) Solve the differential equation, e.g. by solving the homogeneous equation and finding a particular solution. Find the function $Q_2(t)$ describing the charge on the capacitor C_2 as function of time. Your result should give $Q_2(t=0)=0$ and for $Q_2(t=oo)$ the same result that you found in (a).
(e) Find an expression for the total energy dissipated in the resistor in this process in terms of Q (the initial charge) and C_1 only.
(f) Repeat (e) using a different method and verify that energy is conserved.

Problem 6 (10 pts)



A cylinder of radius a, length ℓ and resistivity ρ is connected to a battery with emf ε so that a current flows parallel to the cylinder axis.
(a) What is the current density J in the cylinder?
(b) How much power (resistance heating) is dissipated in the region $r<r_0$, where r is measured from the cylinder axis, and $r_0<a$?
(c) What is the magnitude of the Poyinting vector at $r=r_0$?
(d) Explain how (b) can be calculated using the Poynting vector by a detailed calculation.

Problem 7 (10 pts)

The electric field of an electromagnetic wave is given by

$$\vec{E} = E_0 \hat{x}\cos(kz + \omega t) + E_0 \hat{y}\sin(kz + \omega t)$$

(a) Find the magnetic field $\vec{B}$.

(b) Show that $\vec{E}$ and $\vec{B}$ in this wave satisfy Faraday's law.

(c) Find the Poynting vector, magnitude and direction.

Problem 8 (10 pts)

A cylinder of radius R, height h, uniform charge density ρ and uniform mass density ρ_m is rotating around its axis with angular velocity ω.

(a) Find the magnetic moment of the cylinder, $\vec{m}$.

(b) Find the angular momentum of the cylinder, $\vec{L}$. Hint: moment of inertia of homogeneous cylinder around its axis is $MR^2/2$, with M the mass.

(c) Write the magnetic moment found in (a) in terms of the angular momentum found in (b) instead of in terms of ω.