Global phase diagram in the quantum Hall effect

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We report recent progress in determining the global behavior of the two-dimensional electron gas in a high magnetic field. Specifically, we have: (i) derived a law of corresponding states which allows us to construct a global phase diagram and calculate many interrelations between transport coefficients; (ii) derived a "selection rule" governing the allowed continuous transitions between pairs of quantum Hall liquid states; and (iii) identified the "insulating state," which we have named the *Hall insulator,* as a state in which, as the temperature $T\rightarrow 0$, $\rho_{xx}\rightarrow\infty$, σ_{xx} and $\sigma_{xy}\rightarrow 0$, but ρ_{xy} tends to a constant value, roughly B/nec. Each of these results has many testable experimental consequences.

I. INTRODUCTION

In this paper, we report on our progress in determining the global behavior of the two-dimensional electron gas in a high magnetic field. Our results are summarized in the schematic phase diagrams in Figs. 1(a)–1(c). The logic that leads to these figures is fairly simple, although the specific calculations become somewhat complex. To begin with, in this introduction, we will describe the logic and motivate our conclusions physically, but will not provide detailed derivations. In the subsequent two sections, we will discuss the derivation of the various concrete results we have obtained which support our picture. These sections are fairly technical, and can be skipped by the reader who is interested in results but is uninterested in their derivation. In Sec. IV and the Appendix we discuss the issue of the universality of the conductivity at the critical point between quantum Hall liquids. Finally, in Sec. V, we discuss specific experimentally testable consequences of our global picture.

A. Thermodynamic considerations

In the presence of disorder, as the temperature T tends to zero, there are two known *stable* phases of the two-dimensional electron gas in a transverse magnetic field, B: (1) the insulating state, in which $\sigma_{xx}\rightarrow 0$ and $\rho_{xx}\rightarrow\infty$ as $T\rightarrow 0$ and (2) the quantum Hall liquid, in which $\sigma_{xx}\rightarrow 0$, $\rho_{xx}\rightarrow 0$, and σ_{xy} takes a quantized value $\sigma_{xy}=(e^2/h)s_{xy}$, where s_{xy} is a particular rational fraction and e^2/h is the quantum of conductance. From a quantum-thermodynamic point of view, the particular rational value of s_{xy} characterizes the particular quantum Hall liquid state.[1] We shall argue in Sec. III that the insulating state is *not unique* in the presence of a magnetic field, but is rather characterized by its Hall resistance as well, so that as $T\rightarrow 0$, $\sigma_{xx}\rightarrow 0$ and $\sigma_{xy}\rightarrow 0$, in such a way that $\sigma_{xy}\propto(\sigma_{xx})^2$, so that $\rho_{xy}\rightarrow\rho_{xy}(0)$, a constant, finite value which we expect to be roughly its classical value, $\rho_{xy}(0)\approx B/nec$, where n is the area density of electrons and c is the speed of light. This state therefore differs fundamentally from a band insulator, or a Mott insulator, in which the dc $\rho_{xy}\rightarrow\infty$. We have named this state the "Hall insulator." In the phase diagrams in Fig. 1, the quantum Hall liquid and the Hall insulator are the only thermodynamic phases exhibited.

There appears experimentally[2] to be a third possible stable phase, for which, to date, no satisfactory theoretical understanding exists. This is sometimes known as "the $\nu=\frac{1}{2}$ anomaly." This state is observed in very high mobility heterojunctions for magnetic fields such that ν, the electron density per magnetic flux quantum, is $\nu\approx\frac{1}{2}$. It appears to have a finite (metallic) value of σ_{xx} in the $T\rightarrow 0$ limit and an unquantized value of σ_{xy}. Typically, $\sigma_{xx}(0)$ is considerably smaller than e^2/h and has a rather sharp minimum as a function of magnetic field at precisely $\nu=\frac{1}{2}$, while $\sigma_{xy}\approx nec/B$. [The residual conductance $\sigma_{xx}(0)$ decreases as the sample mobility increases.] It is not clear whether the $\nu=\frac{1}{2}$ anomaly corresponds to a true stable phase of matter in the thermodynamic sense. Rather, it could be associated with a crossover behavior governed by the properties of an unstable metallic fixed point at $\nu=\frac{1}{2}$. Metallic behavior is certainly observed in the absence of magnetic fields in high mobility heterojunctions[3] down to the lowest experimentally accessible temperatures, although there is every reason to believe that at asymptotically low temperatures they would be insulating.[4] This is due to the slow (logarithmic) flow of the conductivity under the action of the renormalization group in the so-called weak localization regime, where the flow occurs in the neighborhood of the unstable (metallic) fixed point. Similar crossover behavior could be re-

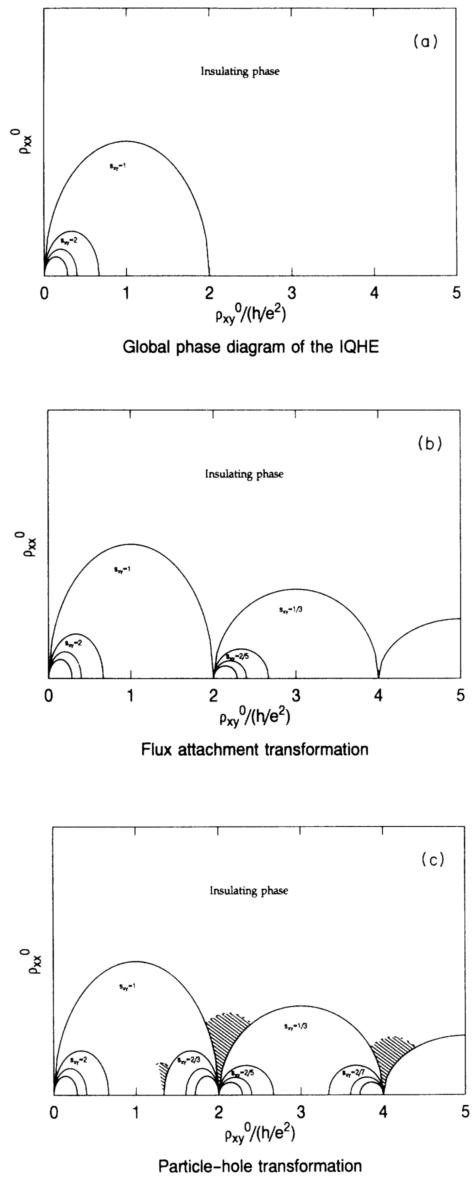

FIG. 1. (a) Phase diagram for the integer quantum Hall effect. $\rho_{xx}^{(0)}$ and $\rho_{xy}^{(0)}$ are, respectively, microscopic measures of the strength of the disorder and the magnetic field. We have drawn the phase boundaries according to the naive scaling theory of the integer quantum Hall effect described in the text:

$$\rho_{xy}^{(0)}/[(\rho_{xy}^{(0)})^2+(\rho_{xx}^{(0)})^2]=\sigma_{xy}^{(0)}=(e^2/h)(n+\tfrac{1}{2}) \ .$$

This expression assumes the accuracy of a two-parameter scaling theory, and so ignores the possible effects of other irrelevant operators. However, for the purposes of the present study, it is only the topology of the phase diagram which matters. (b) Phase diagram for the integer and fractional quantum Hall effect obtained by applying the flux attachment transformation to the phase diagram in (a). The relative heights of the $\nu=1$ and $\frac{1}{3}$ lobes are arbitrary, as discussed in the text. (c) Phase diagram obtained by applying the particle-hole transformation to the phase diagram in (c). Also shown as shaded regions are the region where Hall metal (the "$\nu=\frac{1}{2}$ anomaly") behavior is observed experimentally, and regions related to it by the law of corresponding states.

sponsible for the observed properties of the $\nu=\frac{1}{2}$ anomaly. The $\nu=\frac{1}{2}$ anomaly appears in the phase diagram in Fig. 1(c) as a shaded area, indicating a region in which the system exhibits metallic behavior down to very low temperatures. Note that similar regions are indicated in the vicinity of $\nu=\frac{1}{4}$ and $\frac{3}{4}$. The existence of these and an infinite hierarchy of related "Hall metal" regions is a consequence of the law of corresponding states, which we will discuss in Sec. I B.

There are several *unstable* behaviors of the two-dimensional electron gas that are important to mention: (1) For zero disorder, it is easy to see that as $T\to 0$, $\sigma_{xx}\to 0$, and

$$\sigma_{xy}\to nec/B=(e^2/h)\nu \ ,$$

where $\nu=nhc/eB$ is the density of electrons per unit magnetic flux quantum, or "the filling fraction." However, it is well known that in two dimensions, in the absence of a magnetic field, any amount of disorder localizes all states and so the metallic state is necessarily unstable under the introduction of disorder. Similarly, in the presence of a magnetic field, it is believed that (except for the set of measure zero of densities $\nu=s_{xy}$) the zero-disorder state is unstable with respect to the introduction of a small amount of disorder. As far as the phase diagrams in Fig. 1 are concerned, these unstable phases lie on the horizontal axis, where the limit of zero disorder occurs. Near this axis, the phase diagram becomes infinitely complicated, reflecting the infinite number but zero density of stable quantum Hall liquid phases in the zero-disorder limit. (2) Again, for zero disorder and low enough electron density, we expect a Wigner crystal phase, with spontaneously broken translational symmetry. Moreover, unlike all the other phases mentioned to this point, one would expect there to be a finite temperature transition (presumably of the Kosterlitz-Thouless-Halperin-Nelson-Young[5] variety) into this phase. This phase would also be insulating. However, once again, any finite amount of disorder in two dimensions is expected to eliminate this phase.[6] Thus, while in the presence of disorder it is possible to have short-range Wigner crystal order, at long distances and low temperatures the only known insulating phase is the Hall insulating phase. For the remainder of this paper we will completely ignore the Wigner crystal phase. (3) For zero disorder, there is also the possibility of a more exotic phase, the Hall crystal phase,[7] which has Wigner crystalline long-range order but, nonetheless, finite Hall conductance. Again, in the presence of any disorder, the Wigner crystalline long-range order is destroyed, so this phase is unstable to the introduction of any disorder.

B. Law of corresponding states

The fundamental building block on which we have based our theory is a law of corresponding states. This law is a set of approximate relations between the properties of a given system at one value of the magnetic field and the properties of the same system at another value of the magnetic field. Here, we describe the qualitative physics underlying the law of corresponding states and give heuristic arguments that support them. Below, we will derive them, in a well-defined approximate sense, from the Chern-Simons formulation of the problem and will obtain specific relations [see Eq. (3)] between the transport coefficients in the different states related by the correspondence. (The formal statement of the law of corresponding states, discussed in Sec. II, is that there is an exact equivalence between a given system at one value of the filling factor ν and a *similar* system at a corresponding value of ν'. The similar systems may have slightly different values of the disorder potential, be at slightly different temperatures, or have slightly different interactions so that an appropriate measure of the microscopic physics, $\sigma^{(b)}$, also defined below, is held fixed in going from ν to ν'.)

(i) To start with, let us consider the simplest of these relations, *the Landau-level addition transformation:*

$$\nu \leftrightarrow \nu+1 \ , \tag{1a}$$

where we have used the symbol $\leftrightarrow$ to imply that there exists a correspondence between the properties of the system at the two different values of ν. This piece of the law of corresponding states has a simple interpretation; it derives from the observation that a filled Landau level is an inert background which does not much affect the physics in higher-lying, partially filled Landau levels. Suppose we know the ground-state wave function for a particular system at one value of ν, including the effects of some specified disorder potential and some appropriate electron-electron interactions. We can construct an approximate ground state for the system at $\nu+1$, by applying the Landau-level raising operator to the ground-state wave function for density ν and then multiplying the resulting wave function by the creation operator for a filled lowest Landau level. Among other things, this transformation relates the different integer quantum Hall liquid states.

(ii) The second relation is the *particle-hole transformation* for $\nu<1$:

$$\nu \leftrightarrow 1-\nu \ . \tag{1b}$$

This piece of the law also has a simple interpretation. If we happen to know the ground state (or some other state) of the system at a given density ν, we can construct a good approximate eigenstate of the system at density $1-\nu$, by applying to a new vacuum state, corresponding to a filled Landau level, a function of electron annihilation operators which create the state of density ν for the quasiholes. While in the thermodynamic limit this is an exact symmetry[8] for certain model Hamiltonians restricted to the lowest Landau level, we expect it to be a good approximate symmetry more generally. As an example, this transformation relates the $\nu=\frac{1}{3}$ and $\frac{2}{3}$ states.

(iii) The third, and most interesting, relation is the *flux attachment transformation:*

$$\nu^{-1} \leftrightarrow \nu^{-1}+2 \ . \tag{1c}$$

Among other things, this piece of the law relates integer and fractional quantum Hall liquid states, such as $\nu=1$ to $\nu=\frac{1}{3}$ and $\nu=2$ to $\nu=\frac{2}{5}$. This law was first proposed as a way to understand the hierarchy of ideal incompressible fractional quantum Hall liquid states by Jain[9] (see also Greiter and Wilczek[10(a)]) and from a different perspective by MacDonald, Aers, and Daharma-Wardana[10(b)] from an analysis of the particle-hole transformation in the lowest Landau level. It was argued by Jain, Kivelson, and Trevedi[11(a)] (JKT) that this law is more generally applicable to compressible, as well as incompressible, states. Indeed, JKT showed that explicit wave functions can be constructed corresponding to this transformation which have good variational energies. They also showed that, for certain model systems in the presence of disorder, the transformation is exact, even in a regime where the system is compressible, so long as the system is deep in the quantum Hall liquid phase. More recently, Dev and Jain[11(b)] have constructed trial ground states for systems with small numbers of particles ($N \leq 8$) at magnetic fields such that the filling factor ν lies between $\nu=\frac{1}{3}$ and $\frac{2}{5}$ by applying a flux attachment transformation to the corresponding states with ν between $\nu=1$ and 2; they found better than 98% overlap between these trial states and the exact ground state obtained by numerical diagonalization. The flux attachment transformation can be understood intuitively on the basis of the mean-field arguments applied by Laughlin[12] in the context of anyon superconductivity and by two of us[19] in the context of the quantum Hall effect: Consider the problem of electrons in a given magnetic field B. Imagine attaching some flux to each particle, so in addition to the physical interactions, they also interact via an Aharonov-Bohm phase. If the amount of flux ϕ is equal to two flux quanta, $\phi=2\phi_0=2hc/e$, the Aharonov-Bohm interaction can be gauged away, so that the flux has exactly no effect on the physics; the problems with and without the added flux are identical. Now imagine treating the added flux in a mean-field approximation, where it is represented by a uniform magnetic field of magnitude $2n\phi_0$, where n is the mean areal electron density. Then at mean-field level, there is an equivalence between the problems of the electrons in a magnetic field of magnitude B and electrons in a field of magnitude $B+2n\phi_0$. In terms of the filling factor, the correspondence is between the two filling factors related in Eq. 1(c).

By applying these laws of corresponding states multiple times, all the spin-polarized quantum Hall liquid states that have been observed experimentally to date can be related to the fundamental $s_{xy}=1$ quantum Hall liquid state. Thus, a consequence of the law of corresponding states is that if we understand the long-wavelength electromagnetic properties of the $s_{xy}=1$ quantum Hall liquid state, we understand all spin-polarized quantum Hall liquid states and if we understand the transition between the $s_{xy}=1$ state and other states, we understand all transitions between any quantum Hall liquid state and another state. Similarly, the existence of the $\nu=\frac{1}{2}$ anomaly implies the existence of similar states at $\nu=\frac{1}{4},\frac{3}{4}$, etc., with analogous properties which can be computed in terms of the properties of the $\nu=\frac{1}{2}$ anomaly. The law of corresponding states has been discussed previously in terms of wave functions by Jain[9] and JKT; its explicit implementation as a relation between transport coefficients in Eq. (3) below is our principal result. Most of our findings follow directly.

C. Selection rules and the global phase diagram

In Fig. 1 we construct the schematic $T=0$ phase diagram in the disorder–magnetic-field plane. For simplicity, we have drawn the phase diagram for spinless electrons, so there are no differences between the even and odd integer quantum Hall liquid states, as there would be if we included spin. We label the disorder axis ρ_{xx}^0, indicating some microscopic measure of the longitudinal resistance, and the magnetic-field axis with the dimensionless measure of the magnetic-field strength, $e^2\rho_{xy}^0/h$.

In Fig. 1(a), we represent the phase diagram for the integer quantum Hall effect. We know that in the zero magnetic-field limit for any value of the disorder, or for any value of the magnetic field with strong enough disorder, or for strong enough magnetic field for a given value of the disorder, the system should be insulating. Thus, the entire "outer reaches" of the phase diagram consist of an insulating state in which $\rho_{xx}\to\infty$ as $T\to 0$. We know that for vanishing disorder, the boundary between the insulating state and the $s_{xy}=1$ state occurs at $\nu=\frac{1}{2}$, where the Fermi energy passes below the delocalized states at the middle of the Landau band, and that similarly the boundary between the $s_{xy}=1$ and 2 states occurs at $\nu=\frac{3}{2}$. Indeed, we know that the value of s_{xy} simply counts[13,14] the number of delocalized levels below the Fermi energy. Thus, unless two delocalized levels merge (which we believe cannot happen, generically), an important selection rule can be immediately deduced concerning possible continuous phase transitions in the integer quantum Hall effect:

across each phase boundary, s_{xy} *must change by* ± 1 . (2a)

Among other things, this implies that there can only be a direct transition between the insulating state and the $s_{xy}=1$ liquid; all other integer quantum Hall liquid states must lie in a region of the phase diagram completely surrounded by the $s_{xy}=1$ phase. It is well known that at fixed magnetic field, as a function of increasing disorder, there is little effect of the disorder on the location of the delocalized states until the broadening of the Landau bands becomes comparable to $\hbar\omega_c$. For still larger disorder, the delocalized states begin "floating up" in energy,[14] until eventually all of the delocalized states pass through the Fermi level. The unique result of these considerations is a phase diagram with the general structure of the one shown in Fig. 1(a).

We do not know precisely how the phase boundaries coalesce near the origin of the diagram where $B\to 0$ and $\rho_{xx}^0\to 0$. However, a simple expression for the shape of the phase boundary can be made using a simple renormalization-group argument similar to those used by Khemelinskii and Laughlin:[14] According to the two-parameter scaling theory of the integer quantum Hall effect, the critical value of σ_{xy} at the transition between the $s_{xy}=n$ and $n-1$ plateaus is $(e^2/h)(n-\frac{1}{2})$. Moreover, at the critical point, σ_{xy} is constant under renormalization, which means that

$$\sigma_{xy}^{(0)}=\rho_{xy}^{(0)}/[(\rho_{xy}^{(0)})^2+(\rho_{xx}^{(0)})^2]=(e^2/h)(n+\tfrac{1}{2}) .$$

This argument is inescapable in a two-parameter scaling theory, but in the strong disorder limit, where all sorts of irrelevant variables can affect the scaling at intermediate length scales, the location (but not the topology) of the phase boundaries may be determined by more complicated microscopic considerations.

In Fig. 1(b) we apply the flux attachment transformation to the phase diagram in Fig. 1(a). As you can see, $\nu=\frac{1}{3}$ is to $\nu=\frac{2}{5}$ as $\nu=1$ to $\nu=2$. Transitions, indeed reentrant transitions, between the insulating state and the $\nu=\frac{1}{3}$ state or the $\nu=\frac{1}{5}$ state are permitted, but all other quantum Hall liquid states are separated from the insulating state by a regime of at least one other quantum Hall liquid phase. Figure 1 illustrates the topology of the phase diagram; the detailed shape of the phase boundary can change as a result of electron-electron interactions and other microscopic details. In particular, the relative heights of the $s_{xy}=1$ and $\frac{1}{3}$ lobes are of no particular significance.

In Fig. 1(c) we apply the particle hole transformation to the phase diagram in Fig. 1(b). We stop here because the phase diagram is already so complicated. The reader can easily verify that this series of transformations can be iterated to produce the entire phase diagram with all the appropriate quantum Hall states. (Of course, for physically relevant interactions, eventually iteration of the law of corresponding states will lead us to construct regions of the phase diagram corresponding to Hall liquid phases that are never stable, so we need not iterate forever. See the Comment below.) On this phase diagram, we have also included a shaded region indicating the regime of the $\nu=\frac{1}{2}$ anomaly. As stated previously, this shaded region does not necessarily indicate the location of a true $T=0$ phase in the thermodynamic sense, but rather may be a regime in which metallic behavior is observed down to a very low crossover temperature. As is also shown in the figure, if this shaded region exists near $\nu=\frac{1}{2}$, there must be another such region, related by the flux attachment transformation, near $\nu=\frac{1}{4}$ and $\frac{3}{4}$, etc.

Finally, we note that the important selection rule concerning possible continuous transitions that we derived from the integer quantum Hall effect has a natural generalization for the fractional quantum Hall effect:

any continuous transition between two states must be related by the law of corresponding states to a transition in which s_{xy} *changes by* ± 1. (2b)

Thus, for example, a continuous transition between the $s_{xy}=\frac{1}{3}$ and the $s_{xy}=\frac{2}{5}$ states is allowed, but not between $s_{xy}=\frac{1}{3}$ and $s_{xy}=\frac{3}{7}$.

Comment. It is important to stress that the law of corresponding states relates the properties of the two-dimensional electron gas at two different values of ν only if some appropriate *long-wavelength* measure of the disorder and the electron-electron interactions is held fixed. (We will return to discuss this point more explicitly in Sec. II D below.) It is easy to think of examples where, for fixed *microscopic* interactions and disorder, the properties of the system at the two different values of ν related by the law of corresponding states will not correspond. For example, if at $\nu=1$ a given system exhibits quantum Hall liquid behavior, this does not guarantee that, at all ν related by multiple iterations of the law of corresponding states, the system will exhibit quantum Hall liquid behavior. Multiple applications of the integer addition transformation will eventually lead to a regime where either the inelastic scattering rate or the effective Rydberg is large[14(b)] compared to the spacing between Landau levels, leading to a breakdown of the quantum Hall effect. Repeated applications of the flux attachment transformation will eventually lead to such low values of ν that the quantum Hall liquid behavior will be terminated by either the effect of the disorder of the appearance of a Wigner crystal. The phase diagram in Fig. 1(c) already partially reflects the diminished stability of higher-order quantum Hall liquids. However, in reality, both the effective interactions and disorder can have nontrivial magnetic-field dependence, so on the phase diagrams in Fig. 1, a given physical system (with fixed microscopic disorder and electron-electron interactions) could follow a complicated trajectory, as a function of increasing B, with a tendency to rise to higher values of the effective disorder in the large magnetic-field regime at the right-hand side of the phase diagram. In order to estimate the stability of the various quantum Hall liquid phases, as opposed to the topology of the global phase diagram and the nature of the phase transitions, it is necessary to carry out detailed microscopic calculations for physical interactions, as in the calculations of MacDonald, Aers, and Daharma-Wardana and co-workers and Jain[10(b)] and co-workers.[9,11(b)]

D. The role of the short-distance cutoff

It is important to note that in this discussion we have implicitly assumed that $l(B)$, the magnetic length, is the shortest length scale in the problem, since $1/l(B)$ is the ultraviolet cutoff for the Chern-Simons field theory. For weak localization theory it is always assumed that the elastic mean free path l_{el} is the shortest length in the problem and that $l(B)$ is large compared to l_{el}. It is not clear that the physics in these two extremely different limits is simply connected. It is possible that a phase transition can occur as a function of $l_{el}/l(B)$ from a Hall insulator to an Anderson insulator. The nature of this transition would be beyond the scope of the present theory. It is with this piece of physics in mind that we have refrained from trying to understand the $\nu=\frac{1}{2}$ anomaly as being related by the law of corresponding states to the zero-field, weak localization state at $\nu=0$.

E. Relation to other global theories

To illustrate the severity of the constraints placed on any global theory of the quantum Hall effect by the law of corresponding states, it is useful to ask whether other theories of the quantum Hall effect are consistent with it. As discussed originally by JKT, the hierarchical theories of the incompressible quantum Hall liquids (whether constructed using the notion of a condensation of quasiparticles, as was done by Halperin[15] and Haldane,[16] or by constructing explicit wave functions based on the integer quantum quantum Hall states, as was done by Jain[9]) are fully consistent with the law of corresponding states. By contrast, the conjectured global renormalization-group flow diagram constructed by Laughlin *et al.*[17] based on an analogy with the scaling theory of the integer quantum Hall effect is inconsistent with the law of corresponding states. For instance, in the theory of Laughlin *et al.*,[17] the possibility exists of a continuous transition between the $s_{xy}=\frac{1}{3}$ and $\frac{2}{3}$ states, which is incompatible with the selection rules discussed above and, indeed, has never been observed experimentally. Thus, from the topology of the implied phase diagram alone, it is possible to conclude that the theory of Laughlin *et al.*[17] is inconsistent with the law of corresponding states.

Strong indirect evidence for the selection rules discussed above can be obtained by studying the edge-state structure of various Hall liquid states. For instance, studies by MacDonald and Johnson[18] of the interface between $\nu=\frac{2}{3}$ and 0 states show that there always exist two edges, one between $\nu=\frac{2}{3}$ and 1 and a second between $\nu=1$ and 0. Indeed, using the law of corresponding states and our knowledge of the edge-state structure in the integer quantum Hall effect, we can easily predict the edge-state structure of arbitrary quantum Hall liquids.

II. THE LAW OF CORRESPONDING STATES

We have discussed the qualitative contents and consequences of the law of corresponding states in Sec. I. In this section we will be more quantitative. In particular, we will write down the implications of the three principal correspondences on the low-energy and long-wavelength electromagnetic properties of the system and derive expressions which allow the conductivity tensor of a two-dimensional electron gas to be parametrized by those of a related Bose system. Finally, we will show that these expressions embody the law of corresponding states.

A. The three principal correspondences

The quantitative statement corresponding to the law of corresponding states is summarized in the three relations:

$$\sigma_{xx}(\nu)\leftrightarrow\sigma_{xx}(\nu+1), \quad \sigma_{xy}(\nu)\leftrightarrow\sigma_{xy}(\nu+1)-e^2/h \ , \tag{3a}$$

$$\sigma_{xx}(\nu)\leftrightarrow\sigma_{xx}(1-\nu), \quad \sigma_{xy}(\nu)\leftrightarrow e^2/h-\sigma_{xy}(1-\nu) \ , \tag{3b}$$

and

$$\begin{aligned} &\rho_{xx}\left[\frac{\nu}{1+2\nu}\right]\leftrightarrow\rho_{xx}(\nu) \ , \\ &\rho_{xy}\left[\frac{\nu}{1+2\nu}\right]\leftrightarrow\rho_{xy}(\nu)+2h/e^2 \ . \end{aligned} \tag{3c}$$

We have used the symbol $\leftrightarrow$ instead of an equal sign to signify "correspondence" rather than equality between the transport coefficients at different filling fractions. The related transport coefficients are, in fact, equal to each other only if a specific long-wavelength measure of the electron-electron interactions and the strength of the disorder (defined below) is held fixed as the filling factor is changed. In realistic experimentally relevant circumstances, as the filling factor is varied, the effective strength of the disorder potential changes as well, so that we do not expect strict equality between transport coefficients in this relation. Of course, where the behavior is independent of microscopic details, the relations in Eq. (3) must hold as strict equalities. For example, at temperature $T=0$ in a quantum Hall liquid state, the relations in Eq. (3) become strict equalities, so if at some density ν near 2, the system exhibits $s_{xy}=2$ quantum Hall liquid behavior, $\sigma_{xx}=0$, and $\sigma_{xy}=2(e^2/h)$, then according to Eq. (3c) at the corresponding density ν' near $\frac{2}{5}$, the system should have $\rho_{xx}=0$ and $\rho_{xy}=\frac{5}{2}(h/e^2)$, i.e., $s_{xy}=\frac{2}{5}$ quantum Hall liquid behavior. Note that at $T=0$, remarkably, the relations in Eq. (3) can also relate the conductivity of a quantum Hall liquid to that of an insulator. For instance, if, for ν near 1, a system exhibits $s_{xy}=1$ quantum Hall liquid behavior, then according to Eq. (3a), for the corresponding density ν' near 0, insulating behavior with $\sigma_{xx}=0$ and $\sigma_{xy}=0$ should be observed. Finally, assuming that the critical conductivity is universal, then Eqs. (3) should be read as equalities when they relate the critical densities at which a continuous transition between two states occurs. The full nature of the correspondence in Eqs. (3) will be clarified in the derivation below.

B. Chern-Simmons bosons

To derive the law of corresponding states, we exploit the *exact* mapping between the two-dimensional electron gas and a bosonic field coupled to a fluctuating "statistical" gauge field with a Chern-Simons action. We refer to the bosons as Chern-Simons bosons to remind ourselves that, while they are certainly bosons, the physical particles they represent actually have other statistics (in this case, Fermi statistics). We will also use the term Chern-Simons bosons to designate a similar bosonic representation of the quasiparticles, which we exploit in Sec. II C below. We use the same name, although in a somewhat different context, since it is fundamental to our approach that we treat the two problems on an equal footing.

The Hamiltonian of the two-dimensional (spinless) electron gas in a transverse magnetic field is

$$H=\frac{1}{2m^*}\sum_j\left[-i\hbar\nabla_j-\frac{e}{\mathbf{c}}\mathbf{A}(\mathbf{r}_j)\right]^2+\frac{1}{2}\sum_{i\neq j}e^2V(\mathbf{r}_i-\mathbf{r}_j)+\sum_j[eU(\mathbf{r}_j)-eA_0(\mathbf{r}_j)]\ , \tag{4}$$

where the sum over j runs over all particles, V is the electron-electron repulsion, and U is a disorder potential. Counter terms involving the interaction between the electrons and the external electromagnetic vector potential A_μ with a uniform positively charged background are left implicit. Up to this point, we have retained all factors of e, h, and c. From this point forward in the derivation, for notational clarity, we will adopt units so that $\hbar=e/c=1$.

It has been shown[19] previously that this problem is exactly equivalent to a problem of charged bosons interacting with both the electromagnetic vector potential and with a "statistical" gauge field a_μ with a Chern-Simons action, in which the coupling $k=2m+1$ is an odd integer. Thus, the Chern-Simons term serves to attach k quanta of statistical flux to each boson. It is important to note that m is an arbitrary integer. This is the essential observation responsible for the third transformation of the law of corresponding states; it reflects the fact that we can attach two quanta of statistical flux to a particle without any observable consequences. The properties of the Chern-Simons bosons is described by the Euclidean Lagrangian

$$S=\sum_j\int dt[\tfrac{1}{2}m^*(\partial_0\mathbf{r}_j)^2]+\int d^2r\,dt\left[\frac{i}{2\theta}\varepsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda-i(a_\mu-\eta A_\mu)J_\mu+\tfrac{1}{2}\eta^2J_0VJ_0+\eta UJ_0+i\eta\mathbf{J}\cdot\mathbf{A}_{\text{ext}}\right]\ , \tag{5a}$$

where $\theta=k$ and $\eta=1$, and $J_\mu=(J_0,\mathbf{J})$, and where

$$J_0(\mathbf{r})=\sum_j q_j\delta(\mathbf{r}-\mathbf{r}_j)$$
and
$$\mathbf{J}(\mathbf{r})=\sum_j q_j(d\mathbf{r}_j/dt)\delta(\mathbf{r}-\mathbf{r}_j)\ , \tag{5b}$$

with $q_j=1$, are, respectively, the particle and current densities. (We include θ, η, and q_j in these equations for later reference.) The coordinates r_j are now interpreted as the positions of point bosons, and appropriate bosonic boundary conditions must be applied to the path integral. Note that the coefficient of the Chern-Simons term θ in the Lagrangian contains the information concerning the statistics of the physical particles. In the present case, θ is an odd integer, so Eq. (5) is a theory of fermions; if θ were an even integer, then Eq. (5) is a theory of bosons; for θ noninteger, Eq. (5) is a theory of anyons. Similarly, η specifies the charge of the physical particles (in units of e), which is, of course, $\eta=1$ for electrons, as stated. Since this Lagrangian is expressed entirely in terms of bosonic particles, it is straightforward to reexpress it in terms of a coherent-state path integral as a field theory by introducing a bosonic field φ such that $|\varphi|^2$ is the density of particles and the Lagrangian density

$$\mathcal{L}=\frac{i}{2\theta}\varepsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma+\varphi^*(-i\partial_0-a_0-\eta A_0)\varphi+\frac{1}{2m^*}|(-i\nabla-\eta\mathbf{A}-\mathbf{a})\varphi|^2+\frac{\eta^2}{2}|\varphi|^2V|\varphi|^2+\eta U|\varphi|^2-A_0\bar{n}\ , \tag{6}$$

where

$$|\varphi|^2V|\varphi|^2 \equiv \int dr'|\varphi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\varphi(\mathbf{r}')|^2$$

and $\bar{n}$ is the charge density of the uniform background. So far, all our manipulations have been formally exact. The properties of the principal quantum Hall liquids, with $s_{xy}=1/k$, can be simply understood by studying the mean-field solutions of this Lagrangian; in particular, it is found that the quantum Hall liquid state corresponds to a superfluid state of the Chern-Simons bosons in which φ develops quasi-long-range order. However, to obtain a richer understanding of the two-dimensional electron gas and, in particular, to derive the law of corresponding states, it is necessary to treat fluctuations in a serious fashion. This is what we do next.

C. Derivation of the law of corresponding states

The external gauge field can always be written as $A_\mu = A_\mu^{\text{ext}} + \delta A_\mu$, where A_μ^{ext} is due to the external magnetic field $\varepsilon^{\mu\nu\sigma}\partial_\nu A_\sigma^{\text{ext}} = \delta^{\mu 0}B$ and δA_μ is a small external perturbation (e.g., an applied voltage) used to probe the system. Similarly, the statistical gauge field can be written as the sum of an average piece and a fluctuating piece, $a_\mu = \bar{a}_\mu + \delta a_\mu$ where

$$\varepsilon^{\mu\nu\sigma}\partial_\nu \bar{a}_\sigma \equiv \delta^{\mu 0} k\phi_0 n(\mathbf{r}) , \tag{7}$$

and $n(\mathbf{r})$ is the self-consistent ground-state density. The matter field φ is a complex field characterized by a magnitude $|\varphi|$ and a phase α, $\varphi = |\varphi| e^{i\alpha}$. There are three types of fluctuations of φ which must be treated separately: (1) small-amplitude fluctuations of $|\varphi|$, (2) "spin-wave" fluctuations corresponding to smooth deformations of the field α, and (3) vortex excitations, that is to say, point defects where $|\varphi|$ vanishes and the curl of α is nonzero. As discussed in Ref. 20, by introducing suitable Hubbard-Stratonivich fields, one can integrate out the spin-wave fluctuations exactly. The result is an effective action describing a gas of quasiparticles (the dressed vortices and antivortices described above) which interact with each other, with the externally applied magnetic field, and with a fluctuating statistical gauge field. In this effective action, far-separated quasiparticles are characterized by their creation energies Δ_+ and Δ_-, their charges $e_+^* = \eta e$ and $e_-^* = -\eta e$, and their statistical angle θ, where $2\pi\theta$ is the change in the Berry's phase when one quasiparticle is adiabatically transported around another. Δ, η, and θ are all properties of the parent states, and, in particular, in a primary quantum Hall liquid, $\eta=\theta=1/k$. Other than this, the effective Coulomb repulsion between two quasiparticles at short distances and, more importantly, the effective disorder potential seen by the quasiparticles are somewhat renormalized by the effect of short-wavelength fluctuations of the modulous $|\varphi|$. Fluctuations of the quasiparticle current dominate the temperature dependence of the quantum Hall liquid. If we represent the quasiparticles as Chern-Simons bosons, then when the Chern-Simons bosons are frozen in an insulating state, the physical particles exhibit the daughter quantum Hall liquid behavior, with $\sigma_{xy}=(e^2/h)s_{xy}$, while if the Chern-Simons bosons condense into a superconducting state, the physical particles exhibit the parent state behavior

$$\sigma_{xy}=(e^2/h)(s_{xy}-\eta^2/\theta) .$$

As discussed in Ref. 21, the same transformations can be carried out at higher levels of the hierarchy, so long as the values of η and θ are suitably computed. Thus, the condensation of quasiparticles provides an appropriate context to understand any continuous transition involving this system.

The result of all these formal manipulations is that an effective action which captures the low-energy, long-wavelength action describing the electromagnetic response of a two-dimensional electron gas at filling factor ν is given by

$$S = \int d^2r\, dt \left[-\frac{i}{2} s_{xy} \varepsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{i}{2\theta}\varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - i(a_\mu - \eta A_\mu) J_\mu \right] + S_{\text{matter}} , \tag{8a}$$

where

$$S_{\text{matter}} = -\mu\eta(N_+ - N_-) + N_+\Delta_+ + N_-\Delta_- + \int d^2r\, dt \left(\tfrac{1}{2}\eta^2 J_0 V J_0 - \eta J_0 U + I\eta \mathbf{J}\cdot\overline{\mathbf{A}} + \cdots\right) , \tag{8b}$$

where the ellipsis represents other (less important) interactions between vortices. The parameter s_{xy} in Eq. (8a) deserves particular attention. Imagine that we fix the magnetic field (and hence ν) and increase the strength of the disorder. The two-dimensional electron gas will land in one of the two states: (i) the quantum Hall liquid at low disorder and (ii) the insulator or another quantum Hall liquid at higher disorder. s_{xy} is defined such that the Hall conductance of the low-disorder ("daughter") quantum Hall liquid is given by $\sigma_{xy}=(e^2/h)s_{xy}$, and θ and η are the statistics and charge parameters of quasiholes in this quantum Hall liquid. Other quantities in Eq. (8) that need explanation are J_μ, the particle three-current of the Chern-Simons bosons associated with the quasiparticles, which has the same definition as in Eq. (5b), but now r_j are the positions of the vortices and $q_j=\pm 1$ for quasiholes (vortices) and quasielectrons (antivortices), respectively; μ is the chemical potential, and Δ_+ and Δ_- are the creation energies of the quasiholes (+) and quasielectrons (−), respectively. The first term in Eq. 8(a) is the electromagnetic gauge action of the reference Hall liquid, and the rest describes the motion of quasiparticles and the coupling between the quasiparticles and the electromagnetic gauge field. Note that the quasiparticles have no effective mass and obey guiding center dynamics. Moreover, θ and η take on different values reflecting the fractional charge and statistics of the quasiparticles, and both V and U are now renormalized interactions in which the effect of short-wavelength fluctuations have already been taken into account. $\overline{\mathbf{A}}(\mathbf{r}) = \mathbf{A}^{\text{ext}}(\mathbf{r}) + \bar{\mathbf{a}}(\mathbf{r})$ is the static mean-field "seen" by the quasiparticles. The partition function can be computed from S by performing the path integral

$$Z=\sum_{N_+,N_-}\left[\frac{1}{N_+!N_-!}\right]\times\sum_P\int_{r_j(\beta)=r_{P(j)}(0)}D[\mathbf{r}_j]\int D[a_\mu]e^{-S}\ . \tag{9}$$

Here, the sum is over all permutations P of the quasiparticles, and we note the Bose boundary conditions for the integral. Given Eqs. (8) and (9), we could integrate out a_μ and J_μ to obtain an effective action in terms of A_μ alone. This effective action summarizes all electromagnetic properties of the two-dimensional electron gas.

We proceed to integrate out degrees of freedom in two stages. First, we integrate out the matter fields J_μ and the short-wavelength high-energy pieces of the statistical gauge field to obtain an effective action for the remaining fields. To quadratic order in the fluctuating gauge fields, this action is of the form

$$S'_{\text{eff}}=\int d^2r\,dt\left[-\frac{i}{2}s_{xy}\varepsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\frac{i}{2\theta}\varepsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda+\tfrac{1}{2}f_{0i}\pi_1 f_{0i}+\tfrac{1}{2}f_{12}\pi_2 f_{12}-i\pi_3\varepsilon_{\mu\nu\lambda}(\eta A_\mu-a_\mu)f_{\mu\lambda}\right]\ . \tag{10}$$

Here

$$f_{\mu\nu}=\partial_\mu(\eta A_\nu-a_\nu)-\partial_\nu(\eta A_\mu-a_\mu)\ ,$$

and π_1, π_2, and π_3 are space-time functions summarizing the linear-response properties of the Chern-Simons bosons. At this point we have made only one approximation, which is to ignore higher-order terms in powers of $f_{\mu\nu}$. We will consider the validity of this approximation in Sec. II D below. Of course, even with this approximation, we have only formally carried out the integration; we have hidden all our ignorance in the unknown response functions π_j.

Now we can integrate out a_μ to obtain the final effective action in terms of A alone:

$$S''_{\text{eff}}=\int d^2r\,dt\left[-i\left[s_{xy}+\Pi_3-\frac{\eta^2}{\theta}\right]\varepsilon_{\mu\nu\lambda}A_\mu F_{\nu\lambda}+\tfrac{1}{2}F_{0j}\Pi_1F_{0j}+\tfrac{1}{2}F_{12}\Pi_2F_{12}\right]\ , \tag{11}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$,

$$\Pi_1=\left[\frac{\eta}{\theta}\right]^2\frac{\pi_1}{D},\qquad \Pi_2=\left[\frac{\eta}{\theta}\right]^2\frac{\pi_2}{D},\qquad \Pi_3=\left[\frac{\eta}{\theta}\right]^2\frac{(\theta^{-1}-\pi_3)}{D}\ , \tag{12}$$

and

$$D\equiv\omega^2\pi_1^2+q^2\pi_1\pi_2+(\theta^{-1}-\pi_3)^2\ .$$

The conductivity tensor (with factors of e and h restored) is given by

$$\sigma_{xx}=\frac{e^2}{h}\lim_{\omega\to0}[\omega\Pi_1(\mathbf{q}=\mathbf{0},\omega)],\qquad \sigma_{xy}=\frac{e^2}{h}\lim_{\omega\to0}\left[s_{xy}+\Pi_3(\mathbf{q}=\mathbf{0},\omega)-\frac{\eta^2}{\theta}\right]\ , \tag{13}$$

which reduces to

$$\sigma_{xx}=\frac{(\eta e)^2}{h}\frac{\sigma_{xx}^{(b)}}{(1-\theta\sigma_{xy}^{(b)})^2+(\theta\sigma_{xx}^{(b)})^2}\ , \tag{14a}$$

$$\sigma_{xy}=\frac{(e)^2}{h}\left[s_{xy}-\eta^2\frac{\theta[(\sigma_{xx}^{(b)})^2+(\sigma_{xy}^{(b)})^2]-\sigma_{xy}^{(b)}}{(1-\theta\sigma_{xy}^{(b)})^2+(\theta\sigma_{xx}^{(b)})^2}\right]\ , \tag{14b}$$

where the dimensionless conductivity tensor of the Chern-Simons bosons is given by

$$\sigma_{xx}^{(b)}=\lim_{\omega\to0}[\omega\pi_1(\mathbf{q}=\mathbf{0},\omega)]\ ,\qquad \sigma_{xy}^{(b)}=\lim_{\omega\to0}[\pi_3(\mathbf{q}=\mathbf{0},\omega)]\ . \tag{15}$$

Now we will show that Eq. (14) embodies the three principles of the law of corresponding states. To show that the first two laws are satisfied by Eq. (14), we note that

$$s_{xy}(\nu+1)=1+s_{xy}(\nu)\ ,\qquad \theta(\nu+1)=\theta(\nu)\ ,\qquad \eta(\nu+1)=\eta(\nu) \tag{16a}$$

and

$$s_{xy}(1-\nu)=1-s_{xy}(\nu)\ ,\qquad \theta(1-\nu)=-\theta(\nu)\ ,\qquad \eta(1-\nu)=-\eta(\nu)\ . \tag{16b}$$

Therefore, Eq. (14) implies

$$\sigma_{xx}(\nu+1)=\sigma_{xx}(\nu)\ \text{ and }\ \sigma_{xy}(\nu+1)=\frac{e^2}{h}+\sigma_{xy}(\nu)\ , \tag{17a}$$

provided that

$$\sigma_{xx}^{(b)}(\nu+1)=\sigma_{xx}^{(b)}\ \text{ and }\ \sigma_{xy}^{(b)}(\nu+1)=\sigma_{xy}^{(b)}(\nu)\ . \tag{17b}$$

Similarly,

$$\sigma_{xx}(1-\nu)=\sigma_{xx}(\nu)\ \text{ and }\ \sigma_{xy}(1-\nu)=\frac{e^2}{h}-\sigma_{xx}(\nu)\ , \tag{18a}$$

provided that

$$\sigma_{xx}^{(b)}(1-\nu)=\sigma_{xx}^{(b)}(\nu),\quad \sigma_{xy}^{(b)}(1-\nu)=-\sigma_{xy}^{(b)}(\nu)\ . \tag{18b}$$

The third law of corresponding states is somewhat more subtle; here we will only illustrate it using a simple example. Consider the case $\nu'^{-1}=\nu^{-1}+2$, in a range of ν such that $s_{xy}(\nu)=1$ and $s_{xy}(\nu')=\frac{1}{3}$. By evaluating Eq. (14)

with the charge and statistics parameters $\theta(\nu)=\eta(\nu)=1$ and $\theta(\nu')=\eta(\nu')=\frac{1}{3}$, we find

$$\rho_{xx}(\nu)=\frac{h}{e^2}\sigma_{xx}^{(b)}(\nu), \quad \rho_{xx}(\nu')=\frac{h}{e^2}\sigma_{xx}^{(b)}(\nu') , \tag{19a}$$

$$\rho_{xy}(\nu)=\frac{h}{e^2}[1-\sigma_{xy}^{(b)}(\nu)],$$
$$\rho_{xy}(\nu')=\frac{h}{e^2}[3-\sigma_{xy}^{(b)}(\nu')] \tag{19b}$$

and therefore that

$$\rho_{xx}(\nu')=\rho_{xx}(\nu) \quad \text{and} \quad \rho_{xy}(\nu')=\rho_{xy}(\nu)+2h/e^2 , \tag{20a}$$

provided that

$$\sigma_{xx}^{(b)}(\nu')=\sigma_{xx}^{(b)}(\nu) \quad \text{and} \quad \sigma_{xy}^{(b)}(\nu')=\sigma_{xy}^{(b)}(\nu) . \tag{20b}$$

Equations (17b), (18b), and (20b) constitute the conditions under which the relations in Eq. (3) are equalities.

D. Regime of validity and further implications

The major approximation which went into deriving the above relations is the neglect in the effective action [Eq. (10)] of the higher-order terms in powers of the fluctuating gauge fields. We consider its validity separately near a critical point and far from a critical point: (1) At the critical point, we can determine the relevance or irrelevance of any term in the effective action by computing its renormalization-group dimension. Since gauge fields cannot acquire an anomalous dimension under renormalization (see the Appendix) and since charge conservation implies that all density fluctuations vanish in the $q\rightarrow 0$ limit, we believe that the terms of greater than second order in the gauge fields do not contribute relevant corrections to the conductivity. (2) Far from a critical point, i.e., deep in one of the phases, there is no reason to expect large-amplitude fluctuations to play an important role so long as the phase is stable to fluctuations. Thus, self-consistently, whenever the system is well described by either the superconducting or insulating phase of the appropriate Chern-Simons bosons, the neglect of large-amplitude density and current fluctuations (i.e., the large-amplitude fluctuations of the statistical gauge fields) is well justified for computing the long-wavelength, low-energy response functions of the system. The second assumption we made is that all the complications of the microscopic physics, which we explicitly ignore when we restrict ourselves to considering the low-energy, long-wavelength properties of the system, can be expressed in terms of the conductivity tensor of the Chern-Simons bosons. In effect, this assumption is related to the fact that all the complicated microscopic processes which lead to a magnetic-field-dependent renormalization of the effective strength of the disorder and the electron-electron interactions occur at short distances, of order the magnetic length, so that it can serve as input to the Chern-Simons Landau-Ginzburg theory which treats all the effects at longer length scales. To be precise, the law of corresponding states relates the properties of the two-dimensional electron gas at different values of the magnetic field, interactions, and disorder under conditions of fixed $|\sigma_{ij}^{(b)}|$.

One further implication follows at once from this, which was noted previously by JKT. If the transition between quantum Hall liquids is continuous, so that there is a localization length $\xi_0(\nu)$ which diverges as $\nu\rightarrow\nu_c$, then there should be a correspondence between $\xi_0(\nu)$ for ν approaching ν_c and $\xi_0(\nu)$ as ν approaches ν_c',

$$\xi(\nu)\sim\xi(\nu') , \tag{21}$$

where ν and ν' are any two densities related by the law of corresponding states and ν_c and ν_c' are the corresponding critical densities. Equation (21) implies that any two transitions related by the law of corresponding states must be in the same universality class.

III. THE EXISTENCE OF THE HALL INSULATOR

To lowest order in the dimensionless conductivity $g=\sigma_{xx}h/e^2$, the scaling theory of weak localization in two dimensions has the remarkable feature[22] that the Hall coefficient $R_H=\sigma_{xy}/\sigma_{xx}^2$ does not renormalize (to logarithmic accuracy). This analysis is valid so long as $\xi_0 \gg l_{\rm in}(T) \gg l_{\rm el}$ and $l(B) \gg l_{\rm in}$, where ξ_0 is the localization length, $l_{\rm in}$ is the inelastic scattering length (which diverges as $T\rightarrow 0$), $l_{\rm el}$ is the elastic mean free path, and $l(B)$ is the magnetic length. Were R_H unrenormalized to all orders in g and for all values of the magnetic field, it would imply that in the insulating state $\rho_{xy}\rightarrow B/nec$, i.e., Anderson insulators would be Hall insulators.

Entirely different physics can occur in the strong-field limit, where $l_{\rm in} \gg l(B)$, as is surely the case at all relevant magnetic fields for quantum Hall devices. It is in this limit that we argue that the existence of the Hall insulator follows from Eq. (14). Before we start, we note that the Hall insulator can only be reached by a continuous transition from the $s_{xy}=1/(2p+1)$ quantum Hall liquid. Therefore, if a two-dimensional electron gas at filling factor ν can be driven directly to a Hall insulator phase by increasing the disorder, it must have

$$s_{xy}(\nu)=\theta(\nu)=\eta(\nu)=1/(2p+1) . \tag{22}$$

Let us first look at the case where $\nu=1/(2p+1)$, i.e., the "center" of the plateau. In that case, there is no net magnetic field acting on the Chern-Simons bosons, and so $\sigma_{xy}^{(b)}=0$ and Eq. (14) reduces to

$$\sigma_{xx}=\frac{(\eta e)^2}{h}\frac{\sigma_{xx}^{(b)}}{1+(\theta\sigma_{xx}^{(b)})^2} ,$$
$$\sigma_{xy}=\frac{e^2}{h}\left[s_{xy}-\eta^2\theta\frac{(\sigma_{xx}^{(b)})^2}{1+(\theta\sigma_{xx}^{(b)})^2}\right] . \tag{23}$$

The transition between the $s_{xy}=1/(2p+1)$ quantum Hall liquid and the Hall insulator is triggered by the condensation of the Chern-Simons bosons. Specifically, in the quantum Hall liquid phase, $\sigma_{xx}^{(b)}=0$. Therefore, $\sigma_{xx}=0$ and $\sigma_{xy}=e^2/h(2p+1)$. However, in the Hall insulator phase, $\sigma_{xx}^{(b)} \to \infty$ and

$$\sigma_{xx} \approx \frac{(\eta e/\theta)^2}{h}(\sigma_{xx}^{(b)})^{-1} \quad \text{and} \quad \sigma_{xy} \approx \frac{(\eta e)^2}{\theta^3 h}(\sigma_{xx}^{(b)})^{-2} . \tag{24}$$

Therefore, $\rho_{xx} \to \infty$ and

$$\rho_{xy} \to (h\theta)/(\eta e)^2 = (h/e^2)(s_{xy})^{-1} = (2p+1)(h/e^2) .$$

Put another way, at the commensurate values of the magnetic field such that $\nu = 1/(2p+1)$ and values of the disorder such that $\rho_{xx} \to \infty$ as $T \to 0$, we have shown that $\rho_{xy} = B/nec$. This establishes the existence of the Hall insulator. For other values of the magnetic field, we have no such clean derivation of the transport properties of the insulating state. However, we see no reason to expect quantization of ρ_{xy} in the insulating state, and therefore we expect the Hall resistance will interpolate smoothly between its values at commensurate values of the magnetic field. Therefore, the Hall insulator is characterized by

$$\rho_{xx} \to \infty \quad \text{and} \quad \rho_{xy} \sim B/nec . \tag{25}$$

One caveat is in order at this point. Equation (25) follows directly from the quadratic form of the effective action in Eq. (10), so the only corrections to it come from higher-order terms in the gauge field. While these are certainly small, we have not yet *proven* that away from the critical point they make no contribution to the low-energy response of the system. Since in the insulating phase, σ_{xx} and σ_{xy} are vanishing rapidly as the temperature tends to zero, even a small contribution, if it vanishes less rapidly as $T \to 0$, can affect this result. Thus, while it seems to us unlikely, we have not ruled out the possibility that at asymptotically low temperatures, there is a crossover to different behavior, e.g., a weakly diverging ρ_{xy}.

IV. THE ISSUE OF THE UNIVERSALITY OF THE CONDUCTIVITY AT THE CRITICAL POINT

It has been noted in many contexts[23,24] that, since the conductivity in two dimensions is a dimensionless number times e^2/h, it is possible that it takes a universal value at a zero-temperature critical point. (This interesting proposal is akin to the original proposal of Mott[25] that in a two-dimensional electron gas, the conductivity takes a universal value at the metal-insulator transition.) For noninteracting electrons Abrahams *et al.*[4] have shown that the metallic phase is unstable in two (or fewer) dimensions; upon renormalization, logarithmic corrections always drive the system insulating. As a result, there can be no metal-insulator transition and hence no quantum critical point. The situation is different for the disordered, interacting boson problem, where there is no doubt that there exist at least two distinct phases: an insulating phase and a superfluid phase. On the basis of scaling arguments, Fisher and co-workers[24] have argued that there exists a zero-temperature critical point between these two phases and that there the conductivity at the critical point should be universal. While the explicit calculations of Fisher and co-workers have been carried out with this particular example in mind, the basic argument that the conductivity should be universal at the transition is more general, and hence should be applicable to the critical point governing the transitions between quantum Hall liquids and between a quantum Hall liquid and an insulator as well. (See the Appendix for a review of the arguments for a universal critical conductance, and some cautionary remarks concerning them.) In the rest of this section, we assume the applicability of these general scaling results, and explore their consequences.

In the previous sections we showed that the transitions in the quantum Hall systems can be understood in terms of the superfluid-to-insulator transitions of the quasiparticle Chern-Simons bosons. Moreover, we presented scaling arguments that the relation between the conductivities of the Bose and Fermi systems is exact at the critical point. Thus, at the critical point, the conductivity tensor of the original electrons is exactly given in terms of the conductivity tensor of the Chern-Simons bosons by Eq. (14). Any universal feature of the Bose conductivities translates into a corresponding universal feature of the actual electronic conductivities.

We first consider the case in which the Chern-Simons bosons are symmetric under charge conjugation, and hence $\sigma_{xy}^{(b)} = 0$. (On a rough, intuitive level, this corresponds to looking at the transition as a function of disorder at a fixed magic value of $\nu = s_{xy}$. For noninteracting electrons in the integer quantum Hall effect, this corresponds to the case in which the disorder potential is particle-hole symmetric.) In this case, the critical values of σ_{xy} and σ_{xx} (σ_{xyc} and σ_{xxc}) are uniquely determined by the critical value of the Bose conductivity, $\sigma_{xxc}^{(b)}$. As shown in Ref. 26, if we also assume that the result of the scaling theory of the integer quantum Hall effect[13,14(a)] that $\sigma_{xyc} = (e^2/h)(n - \frac{1}{2})$ survives in the presence of interactions, we can invert Eq. (14b) with $\theta = \eta = 1$ and $s_{xy} = n$, to find that $\sigma_{xxc}^{(b)} = 1$. It now immediately follows from Eq. (14) that the critical value of the conductivity tensor at an integer or fractional plateau is given by the expression[26]

$$\sigma_{xxc} = \frac{(\eta e)^2}{h} \frac{1}{1+\theta^2}, \quad \sigma_{xyc} = \frac{e^2}{h}\left[s_{xy} - \eta^2\theta \frac{1}{1+\theta^2} \right] . \tag{26}$$

When the Chern-Simons bosons are not charge-conjugation symmetric, we no longer know *a priori* what the value of $\sigma_{xyc}^{(b)}$ should be. We envisage two possibilities: (1) The system always scales to a self-dual fixed point at which $\sigma_{xy}^{(b)} = 0$. Since in the scaling theory of the integer quantum Hall effect, the transition between plateaus is governed by a single fixed point, independent of whether the transition is approached as a function of disorder at fixed magic ν, or as a function of ν (i.e., B) at fixed disorder, this is the scenario suggested by that theory. If this is the case, Eq. (26) should be universally applicable. (2) The transition away from the self-dual point is different from that at the critical point in the same way that the superconductor-to-insulator transition is thought to be different in the presence and absence of a magnetic field.[27] In this case, it is possible that $\sigma_{xyc}^{(b)}$

represents the eigenvalue of a marginal operator, and hence that σ_{xxc} and σ_{xyc} vary continuously along the phase boundary separating two quantum Hall liquid phases or a Hall liquid and an insulating phase. This provides a good example of how the existence of a marginal operator could spoil the universality of the critical conductance. We are currently trying to determine which of these possibilities applies.

V. EXPERIMENTAL IMPLICATIONS

There are a number of important, and unambiguous, experimental implications of the theory described above, even though it is incomplete and there are many experimentally relevant questions which we cannot, at this stage, answer with any degree of certainty. The clearest predictions derive from the topology of the phase diagram in Fig. 1 and from the fact that all the transitions are continuous and in the same universality class.

Transitions between quantum Hall liquids. It has been argued previously by several authors[13,14,28] that in the absence of electron-electron interactions, the transition between integer quantum Hall liquid states should be continuous and in the same universality class regardless of the value of the integer. The Landau-level addition transformation of the law of corresponding states implies that the same universality of the results applies in the presence of electron-electron interactions. However, whereas the one-electron theory is related to a two-dimensional statistical-mechanics problem (a nonlinear σ model), in the presence of interactions, it is clear that the correct theory must be $2+1=3$ dimensional. As a result, in the presence of interactions, we expect the conductivity at the transition to be universal (as discussed above) and the transition to be characterized not only by a diverging correlation length but by a diverging correlation time as well. It has been argued by JKT that the same is true of the transitions between fractional Hall liquid states, i.e., that transitions between fractional quantum Hall liquid states are in the same universality class as transitions between integer states. Their argument is essentially the same as the one outlined here, based on the flux attachment transformation of the law of corresponding states.

Experiments have already been reported by Tsui and co-workers demonstrating that the transitions between plateaus are characterized by a single critical exponent, regardless of whether the transition is between integer plateaus[29] or fractional plateaus.[30] Let us briefly recall the analysis that underlies this result.[29] Consider the zero-temperature conductivity tensor of a system near the transition between two plateaus or between a given plateau and an insulating state. We will perform a finite-size scaling analysis. Since we are studying a quantum critical phenomenon, it is important to consider a system which has finite extent L in space and L_t in (imaginary) time. Finite-size scaling implies that σ_{ab} should be a function of L/ξ and $\omega_0 L_t$, where ξ is the diverging correlation length and ω_0^{-1} is the "correlation time:"

$$\sigma_{ab}(L,L_t,B-B_c)=\sigma_{ab}(L/\xi,\omega_0 L_t) ,$$

where B_c is the value of the magnetic field at which the (zero-temperature) transition occurs. For values of B near to B_c, $\xi\sim|B-B_c|^{-x}$. (The traditional symbol for this exponent is ν, but here ν is the filling factor. The best estimate[31] of x is $x\approx\frac{7}{3}$.) $\omega_0\sim\xi^{-z}$, where z is the dynamic exponent (at present unknown). When L/ξ and $\omega_0 L_t \ll 1$, the system behaves as if it were at its critical point, so $\sigma_{ab}\approx\sigma_{ab}^{(c)}$, the value of σ_{ab} at the critical point (which is likely to be universal, as discussed in Sec. IV above). For L/ξ and $\omega_0 L_t \gg 1$, the system behaves as if it is far from the critical point, so σ_{xy} is quantized and σ_{xx} is exponentially small.

One can, in principle, explore the transition between short- and long-distance behavior in many ways. The two extreme approaches are either (1) hold $\omega_0 L_t \gg 1$ and consider the behavior as a function of L/ξ or (2) hold $L/\xi \gg 1$ and consider the behavior as a function of $\omega_0 L_t$. In either case, the transition occurs when the scaled system size is of order 1.

The relevance of this to experiments at finite temperature is that temperature introduces an effective finite size for the system. Specifically, we can think of the system as having a spatial extent corresponding to the inelastic scattering length $L\approx l_{\rm in}(T)\sim T^{p/2}$ [in the low-field limit (Ref. 32), $p=1$] and an extent in imaginary time $L_t=\hbar/k_B T$. Thus, for a given value of $B-B_c$, a crossover from low-temperature (quantum Hall) to high-temperature (critical) behavior will occur at a crossover temperature $T^*(B-B_c)$ which is equal to the smaller of the temperatures $T_{\rm space}$ and $T_{\rm time}$ defined implicitly by the relations

$$l_{\rm in}(T_{\rm space})=\xi\sim|B-B_c|^{-x} , \tag{27a}$$

$$k_B T_{\rm time}=\hbar\omega_0\sim\xi^{-z} . \tag{27b}$$

As long as $zp<2$ (which is believed to be the case), the crossover for $|B-B_c|$ small is always determined by $T_{\rm space}$. Thus, where Tsui and co-workers[29,30] observed that $|B-B_c|\sim(T^*)^{0.42}$ for small enough $|B-B_c|$, they sensibly interpreted this as a measurement of the exponent $p/2x$. (This is consistent with $x=\frac{7}{3}$ and $p=2$.) It is this exponent that they found is universal for transitions between integer and fractional quantum Hall liquids. The behavior of the system is also interesting when we do not focus on the region of parameter space close to B_c. In this case, it should be possible to explore the crossover from low- to high-temperature behavior due to the dynamics, rather than the inelastic scattering.

Quantum Hall liquid to insulator transition. The observed reentrant transition in the neighborhood of $\nu=\frac{1}{5}$ between an insulating and a quantum Hall liquid state is consistent with the proposed phase diagram. (See, also, Ref. 21.) Moreover, this transition should be in the same universality class as the transition between plateaus, and so should be described by the same critical exponents. In particular, the activation energy for conduction Δ_σ should vanish, according to dynamic scaling, as $|\Delta_\sigma|\propto\hbar\omega_0\sim\xi^{-z}$. This scaling behavior has not yet been critically tested experimentally. Preliminary results by Jiang *et al.*[33] suggest that $|\Delta_\sigma|\propto|B-B_c|^b$ with $b\sim0.5$.

We would expect that in less clean samples, similar reentrant behavior should be observed in the neighborhood of $\nu=\frac{1}{3}$ and, in still less clean samples, in the vicinity of $\nu=1$. To get a feeling for the sort of realistic values of parameters needed to observe this phenomenon, let us focus on the simple and dramatic case of $\nu=1$. We expect to observe the $s_{xy}=1$ quantum Hall state in samples that are sufficiently clean that for magnetic fields such that $\nu=1$, $\omega_c\tau>1$, where $\omega_c=eB/m^*c$, m^* is the band effective mass of the electrons, and τ is the elastic scattering time. Since $\hbar\omega_c$ at $\nu=1$ is equal to the Fermi energy E_F for $B=0$, this condition is equivalent to the condition $\hbar/\tau<E_F$. Thus, we must restrict ourselves to samples with high enough mobility that this condition is satisfied, or, equivalently, with a microscopic resistance $\rho_{xx}^0<h/e^2$. On the other hand, the cleaner the sample, the lower the temperatures necessary to observe the true low-temperature behavior. In the zero-field limit, we can deduce from weak localization theory and finite-size scaling that a crossover from metallic behavior (controlled by flows in the neighborhood of the unstable metallic fixed point) to insulating behavior (controlled by flows toward the stable insulating fixed point) occur at a length scale

$$L^*=L_0\exp[\pi^2(h/e^2\rho_{xx}^0)],$$

where L_0 is a microscopic length scale (ultraviolet cutoff) and the exponent π^2 is derived from the one-loop β function.[34] As usual, we relate this crossover length to a crossover temperature by finding the temperature T^* at which the inelastic scattering length $L_{\rm in}$ is equal to L^*. Again, as usual, we imagine that $L_{\rm in}=a(T/\Theta_0)^{p/2}$, where a is a micropscopic length, $k_B\Theta_0$ is a microscopic energy characterizing the scattering mechanism, and p is an exponent which, for phonon scattering in two dimensions with $B=0$, is thought[32] to be $p=1$. Combining these relations, we find a crossover temperature

$$T^*=T_0\exp[(2\pi^2/p)(h/e^2\rho_{xx}^0)], \tag{28}$$

where T_0 is a microscopic temperature scale. The exponent in this expression is approximately 20 times the dimensionless conductivity, so for clean samples in which ρ_{xx}^0 is much less than the quantum of resistive, the crossover temperature is astronomically low and insulating behavior is never seen. However, for ρ_{xx}^0 near h/e^2, insulating behavior is observable at accessible temperatures. We therefore predict that in samples with a microscopic resistance which is less than, but not much less than, the quantum of resistance, dramatic reentrant behavior should be observable at low temperatures: For $B=0$, the system should exhibit insulating behavior. This behavior should persist up to a critical magnetic field B_1, above which $s_{xy}=1$ quantum Hall liquid behavior should be observed. Finally, for B greater than a second critical field, B_2, insulating behavior again should be observed. Critical exponents and qualitative behavior should be similar to those observed in the neighborhood of $\nu=\frac{1}{5}$ in clean samples.

The Hall insulator. Goldman, Shayegan, and Tsui[35] have performed experiments to look for evidence for a $s_{xy}=\frac{1}{7}$ quantum Hall liquid state. In this study, the focus was on a small dip in the resistance as a function of magnetic field observed in the neighborhood of $\nu=\frac{1}{7}$. However, the most salient feature of the experiment is that the resistance was found to be a diverging function of temperature for all magnetic fields in the neighborhood; it simply was diverging a little less fast at $\nu=\frac{1}{7}$. This behavior we interpret as evidence that the system is in the insulating state, but that it passes "near" the phase boundary of the $\nu=\frac{1}{7}$ quantum Hall liquid state in the sense that if the sample were a little less disordered or the electron-electron interactions were a little shorter ranged, the system would have exhibited a $\nu=\frac{1}{7}$ Hall liquid phase. In other words, the dip in the resistance is a precursor effect due to critical fluctuations on the insulating side of the Hall liquid phase boundary.

Here, we would like to draw attention to another aspect of these data which relates directly to the properties of the insulating phase and, presumably, has nothing to do with the quantum Hall liquid state. By the time the temperature is 140 mK or lower, the system is already exhibiting strongly insulating behavior in the sense that $\rho_{xx}>20h/e^2$. Despite this, the measured ρ_{xy} is observed to be roughly temperature independent and to obey approximately the classical relation $\rho_{xy}\approx B/nec$. It is, of course, difficult to obtain a completely reliable measurement of ρ_{xy} when $\rho_{xy}\ll\rho_{xx}$, so it would probably be useful for the experiment to be repeated. However, taken at face value, we feel that this observation is a direct confirmation of the existence of the Hall insulator.

Quasi-particle Hall insulator in the quantum Hall liquid phase. The law of corresponding states implies that there should appear a signature of the Hall insulator behavior in the quantum Hall liquid regime. In particular, it is important to the occurrence of the quantum Hall effect that the quasiparticle gas is in an insulating state, and we infer that this insulating state must be a Hall insulator. It therefore follows from Eq. (3) that for finite temperatures (or finite-size systems) there exists a simple relation between the deviation $\delta\rho_{xy}$ of the Hall resistance from its zero-temperature value ($\delta\rho_{xy}=\rho_{xy}-h/s_{xy}e^2$) and the deviation of the longitudinal resistance $\delta\rho_{xx}$ from its zero-temperature value ($\delta\rho_{xx}=\rho_{xx}$):

$$\delta\rho_{xy}\sim(\rho_{xx})^2 . \tag{29}$$

Hall insulator versus Wigner crystal. As we have stressed before for a disordered system, there is no sharp distinction between the Wigner crystal and a disorder-induced insulator, since disorder necessarily destroys the long-range order of the Wigner crystal.[6] Nonetheless, there is a clear qualitative question that determines how one should think about any given set of experiments on an insulating state of a quantum Hall device: (1) It is best to think about the state in terms of large but finite pinned Wigner-crystal "grains" if there is sufficiently long short-range Wigner-crystal order. (2) If, on the other hand, the Wigner-crystal order is of shorter range than the localization length, then it is better to think of the insulating state as being basically an effect of the disorder. Above, we interpreted the reentrant insulator–quantum Hall liquid–insulator transition observed in the neighborhood of $\nu=\frac{1}{5}$ in the ultrahigh mobility GaAs heterojunctions as being evidence of the validity of the law of corresponding states.

It is important to note that this interpretation is somewhat at odds with the most prevalent interpretation of the same results, which are widely believed to be evidence[36–39] of the existence of a Wigner crystal, with the reentrance of the phase boundary being evidence of the first-order nature of the liquid-crystal transition. Other experimental evidence that is taken as corroborating evidence for the existence of the Wigner crystal are the following: (1) the observation[36–39] of a fairly well-defined (magnetic-field-dependent) temperature ($T^*\sim 100$ mK) below which the characteristic "Wigner-crystal behavior" is observed; (2) the observation[32,36–39] of a fairly well-defined, and strikingly small, threshold electric field ($E_T\sim 1$ mV/cm) above which non-Ohmic behavior and broadband noise[38] are observed (in Ref. 36, rather larger threshold fields are reported); and (3) the observation[36,39] of a moderately well-defined resonance at fixed wave number as a function of frequency with a strikingly low frequency scale ($\omega\sim 1$ GHz). All of these observations find a natural explanation if one assumes that in these samples (which are certainly among the least disordered systems in existence) there is substantial Wigner-crystal order.

We wish to show that these experimental results have another rather straightforward interpretation in terms of the properties of the Hall insulator in the neighborhood of the critical point using more or less standard notions of scaling. To understand the nature of the electromagnetic response of a Hall insulator in the scaling regime, let us first start (as a warmup) by considering a simpler problem, namely, the transition between the insulating state and the $s_{xy}=1$ quantum Hall liquid, and let us for now ignore electron-electron interactions. We have already shown that as a result of the "floating up" of the delocalized states, the reentrant phase diagram is a natural, indeed a necessary, feature of this theory. Let us consider now the finite frequency and finite electric-field response of the system just on the insulating side of the transition. Nonlinear I-V relations arise in this system either through well-known modifications of the variable-range-hopping conductivity, or through electric-field-induced transitions between the localized and the delocalized states. (This is analogous to Zener tunneling.) Since in the temperature range probed by the experiments, variable range hopping is never observed, we ignore this first contribution and concentrate on the second. The energy that an electron can easily acquire from the electric field is $eE\xi_0$, where ξ_0 is the localization length. The energy required to promote an electron to the delocalized states, Δ_σ, is the difference between the Fermi energy and the energy of the delocalized states. Thus, at zero temperature we expect significant nonlinear conductance to occur above a threshold field

$$E_T\sim \Delta_\sigma/e^*\xi_0\ , \tag{30a}$$

where in this case $e^*=e$. This simple single-electron picture is, of course, invalid near the transition. However, using the law of corresponding states to relate the properties of the integer and fractional quantum Hall regimes, and reinterpreting the above results in terms of scaling variables, we can obtain a more generally valid expression in the Hall insulator near the critical point. First, we must reinterpret the frequency scale as the characteristic frequency determined by dynamic scaling, $\Delta_\sigma\sim(\xi_0)^{-z}$, as discussed above. (In the single-particle theory, $\Delta_\sigma\sim|B-B_c|$, which implies $z=1/x\approx\frac{3}{7}$, which is unlikely to be the correct value of the exponent in the fully interacting theory.) Furthermore, the proper quasiparticle charge e^* must be used in Eq. (30a), where e^* is the charge of the quasiparticle in the nearby Hall liquid state (e.g., near $\nu=\frac{1}{5}$, $e^*=e/5$. This is, of course, a nontrivial interaction effect). In the neighborhood of the transition to the quantum Hall liquid, E_T is small due to the smallness of Δ_σ and the largeness of ξ_0; $E_T\sim|B-B_c|^{x(1+z)}$. For the same reason, we expect broadband noise (microscopic shot noise) when E exceeds E_T. Similar reasoning leads to the conclusion that there should be a peak in the absorptive response of the system at a characteristic frequency (which we expect to be independent of wave number k so long as $k\xi_0<1$) determined by dynamic scaling, $\hbar\omega\sim\Delta_\sigma$. Finally, we can easily use finite-size scaling to relate the zero-temperature properties to finite-temperature properties. For example, the threshold field at finite temperature should be determined by a scaling function of the form

$$E_T\sim(\Delta_\sigma/e^*\xi_0)f_T(L_{\rm in}/\xi_0,\Delta_\sigma/k_BT)\ , \tag{30b}$$

where $f_T(x,y)$ is another scaling function which tends to 0 when either of its arguments is small compared to 1 and approaches a constant as its two arguments approach infinity. The characteristic temperature $T^*(B)$ should also be determined by scaling, as above. According to this interpretation, a characteristic temperature at which the behavior of the system changes qualitatively (although, of course, there is no phase transition) is determined by either the equation $k_BT_{\rm WC}\sim\Delta_\sigma$, or by $l_{\rm in}(T^*)\sim\xi_0$.

At present, we feel[40] that both the present interpretation and the interpretation in terms of the disordered Wigner crystal are consistent with the experiments. Further tests of the theories can be made by measuring the variation of the various characteristic energies as a function of $|B-B_c|$ to test the predictions of scaling. (Preliminary indications from the measurements in Ref. 38, which show that E_T increases as the critical point is approached, would appear to be inconsistent with the scaling assumption. However, in Ref. 36(b), the critical field is found to vanish as the critical point is approached, which would be consistent with the scaling assumption.) A simpler test is to study more disordered samples (which are correspondingly less likely to support substantial Wigner-crystal order). If similar reentrant insulating behavior and similar nonlinear responses of the system can be observed in a more disordered sample in the neighborhood of $\nu=\frac{1}{3}$ or 1, this will be exceedingly strong evidence in support of the present theory. Even more importantly, if Hall insulating behavior is observed in the insulating regime with $\rho_{xy} \sim B/nec$, this will constitute overwhelming evidence that the insulating state is not a Wigner crystal. Conversely, if $\rho_{xy} \gg B/nec$, then it is probably better to think of the insulating state as a disordered Wigner crystal.

Finally, we wish to emphasize that where ξ_0 is large, substantial Wigner-crystal order could exist without it affecting any of our conclusions, so long as the Wigner-crystal correlation length $\xi_{WC} \leq \xi_0$. However, the existence of such short-range Wigner-crystal order could have important consequences on other properties of the system which depend on shorter-range effects. Energetic considerations, such as determine the relative stability of the various phases, could be quite sensitive to the presence of short-range Wigner-crystal order. In particular, this could strongly influence the shape (although not the topology) of the phase diagram in Fig. 1(c) and produce a variety of Wigner-crystal-like features in the intermediate energy response of the system.

The critical conductance. Equation (26) contains a series of predictions concerning the critical conductance at various transitions under the assumption that the transition is governed by a single fixed point. We are unaware of any *direct* experimental tests of these predictions (but see Ref. 26 for a discussion of a related, *indirect* experimental verification of these relations). In particular, care must be taken when studying the properties of small samples,[41] since finite-size corrections (especially the contribution of edge states[42]) can affect the results strongly.

The $\nu=\frac{1}{2}$ anomaly. At this stage, we cannot say anything concrete about the $\nu=\frac{1}{2}$ anomaly itself. However, from the law of corresponding states we can conclude that similar behavior should be observed at $\nu=\frac{1}{4},\frac{3}{4}$, etc. We happen to know that this behavior has, in fact, been observed.[43]

Low-field phenomena. It is exceedingly dangerous to apply any of the present results in the low-field regime where $l_{el} < l(B)$. However, we have recently become aware of results on the superconductor-to-insulator transition in thin-metal films that are tantalizingly suggestive of the existence of a Hall insulating state in these materials as well. Specifically, Hebard *et al.*[44] have studied a series of samples as a function of disorder and magnetic field. For any given sample which is superconducting at $T=0$ in zero applied field, they have identified a critical magnetic field B_c such that for $B<B_c$, the sample is superconducting at $T=0$ and for $B>B_c$, the sample is insulating in the sense that ρ_{xx} diverges. However, there appears in the same experiments to be a second critical field $B_c'>B_c$, such that for $B<B_c'$, ρ_{xy} remains very small at $T=0$, while for $B>B_c'$, ρ_{xy} diverges. (Because of the high density of electrons in these materials, it is not clear to us whether for $B_c<B<B_c'$, $\rho_{xy}\sim B/nec$ or whether $\rho_{xy}\to 0$.) Despite the fact that these experiments are clearly well out of the regime of validity of our theory, since $l_{el}\sim$ a lattice constant $\ll l(B)$, it is tempting to identify the intermediate zero-temperature phase between as being a Hall insulator, so that as a function of magnetic field, there are two transitions: a superconductor-to-Hall insulator transition followed by a Hall-insulator-to-Anderson-insulator transition. We are currently trying to extend our results to treat this problem.

Note added in proof. After we had completed this paper, we received two papers, one by Santos *et al.*[45] and one by D'Iorio *et al.*[46] In Santos *et al.*'s paper a two-dimensional hole gas in a magnetic field is studied and reentrant insulating behavior observed in the neighborhood of $\nu=\frac{1}{3}$ which is "strikingly similar" to that of the electron gas near $\nu=\frac{1}{5}$. Anomalies in ρ_{xy} are observed, but not discussed in any detail. Other than the fact that this system consists of holes rather than electrons, the only two other possibly significant differences are that the holes have a factor of 10 heavier effective mass and a factor of 20 lower zero-field mobility ($\mu\sim 4\times 10^5$ $\mathrm{cm^2/V\,sec}$). Santos *et al.* still attribute this behavior to Wigner crystallization, and attribute the fact that the transition is shifted to lower filling factors to the increased mixing between Landau levels. We would like to suggest that it is more likely that the difference in behavior is due to the lower mobility of the hole gas samples. In the paper of D'Iorio *et al.*, quantum Hall behavior is studied in Si inversion layers, which are still more disordered ($\mu\sim 4\times 10^4$ $\mathrm{cm^2/V\,sec}$). They observe reentrant insulating behavior in the neighborhood of $\nu=1$, and similar nonlinear *I-V* curves are obtained. It is absolutely imperative that careful measurements of ρ_{xy} be performed on both systems. Also, after completion of this work, it was pointed out to us by Z. Wang that theoretical evidence of the existence of a Hall insulator was obtained previously by Efetov.[47] Efetov studied the problem of *noninteracting* electrons in the strong magnetic-field limit and concluded that in the insulating state, ρ_{xy} remains finite.

It was also pointed out to us[48] that there are some existing data[49] which appear to show a direct quantum-Hall-liquid-to-insulator transition in the neighborhood of $\nu=\frac{2}{9}$, although no data that we are aware of show a plateau in ρ_{xy}. A direct, continuous transition from an $s_{xy}=\frac{2}{9}$ quantum Hall liquid to an insulator is forbidden according to the selection rules we have derived, so further experimental study of this point could be very important.

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APPENDIX: REVIEW OF THE ARGUMENT THAT THE CRITICAL CONDUCTANCE IS UNIVERSAL

The conductivity is simply related to a particular correlation function (e.g., the current-current correlation function). Since we are dealing with a quantum critical phenomenon, there should be no fundamental distinction between a dynamical correlation function and a static correlation function, as there would be in classical statistical mechanics. Near the critical point, the difference between the actual Hamiltonian of the system H and the fixed point Hamiltonian H^* is

$$H = H^* + \sum_j \widetilde{\Lambda}_j O_j \ , \tag{A1}$$

where $\{O_j\}$ are a complete set of scaling operators. We imagine, moreover, that H is a field theory (since we are only interested in long-wavelength phenomena), so we regularize it with an ultraviolet cutoff K. Then, the conductivity can be written as

$$\sigma_{ab} = f_{ab}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \ldots; K) \ . \tag{A2}$$

Simply rescaling the units of space and time so that $K = 1$, we find

$$\sigma_{ab} = K^{d_\sigma} f_{ab}(\Lambda_1, \Lambda_2, \Lambda_3, \ldots; 1) \ , \tag{A3}$$

where $d_\sigma = 0$ is the engineering dimension of the conductivity and $\Lambda_j = K^{d_j} \widetilde{\Lambda}_j$ are the dimensionless coupling constants, where d_j is the engineering dimension of O_j. This is just straight rescaling. We now imagine thinning degrees of freedom by "integrating out" degrees of freedom between K and λK, and then rescaling the cutoff back to K, i.e., by subjecting the system to a renormalization-group transformation. The result is

$$\sigma_{ab} = \lambda^{-\delta_\sigma} f_{ab}(\lambda^{-\delta_1}\Lambda_1, \lambda^{-\delta_2}\Lambda_2, \lambda^{-\delta_3}\Lambda_3, \ldots; 1) \ , \tag{A4}$$

where δ_σ is the scaling dimension of the conductivity and δ_j is the scaling dimension of O_j.

To arrive at the conclusion that the critical conductivity is universal, it is necessary at least to show that on the critical surface (i.e., the basin of attraction of H^*) in the neighborhood of H^*, σ_{ab} is independent of parameters. This can be shown under the following assumptions: (1) The scaling dimension $\delta_\sigma = 0$. That $\delta_\sigma = d_\sigma$ follows[50] from gauge invariance and the fact that σ_{ab} is a current-current correlation function. (2) There are no marginal variables in the neighborhood of H^*. That is, some of the operators have positive scaling dimension and others have negative scaling dimension but none have zero scaling dimension. (To be explicit, we consider the case in which there is only one relevant variable so $\delta_1 > 0$, but $\delta_j < 0$ for $j > 1$. Thus, the critical surface in the neighborhood of the fixed point is defined by $\Lambda_1 = 0$.) (3) There are certain restrictions on the functional form of f_{ab} (as discussed below). Now, at the critical point, we can take the limit as $\lambda \to 0$ to obtain the long-distance physics, and, in particular, to obtain the dc conductivity. As a result, we conclude that at the critical point, σ_{ab} takes on the universal value

$$\sigma_{ab}^{(c)} = \lim_{\lambda \to 0} f_{ab}(0, \lambda^{-\delta_2}\Lambda_2, \lambda^{-\delta_3}\Lambda_3, \ldots; 1)$$
$$= f_{ab}(0,0,0,\ldots;1) \ . \tag{A5}$$

As attractive as this conclusion is, there are several ways in which it could be misleading: (1) It is possible that the value of f could depend on the way in which the limit is approached. This is particularly true if there are two irrelevant operators with the same scaling dimension. For instance, suppose that $\delta_2 = \delta_3$. Then, f could depend on the ratio Λ_2/Λ_3, even in the limit in which both couplings tend to zero. This would produce a one-parameter family of critical values of σ_{ab}. (2) Even if this does not occur, there is no guarantee that $f_{ab}(0,0,0,\ldots;1)$ is a finite constant. It is possible that it is equal to zero or is infinite. While this is indeed a universal value, it is not what is typically meant by a universal value of the conductance. In the present case, neither of these possibilities has been ruled out. Thus, we must view the notation that the value of the conductivity at the critical point is universal as an interesting hypothesis, which remains to be tested. We have therefore not based any of our arguments on this hypothesis.

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[48]We thank J. K. Jain for pointing this out to us. A possible interpretation of this observation, which was also realized independently by Jain, is that the *shape* of the phase diagram in the neighborhood of $\nu=\frac{2}{9}$ reflects the presence of substantial short-range Wigner-crystal order. In the absence of disorder, there could be a direct first-order transition from the $s_{xy}=\frac{2}{9}$ quantum Hall liquid to a Wigner crystal, which in the presence of weak disorder would split into a continuous $s_{xy}=\frac{2}{9}$ to $s_{xy}=\frac{1}{5}$ transition followed by a second continuous $s_{xy}=\frac{1}{5}$ to insulator transition. If the disorder is weak, the strip of $s_{xy}=\frac{1}{5}$ phase that appears between the $s_{xy}=\frac{2}{9}$ phase and the insulator must (by continuity) occupy a very narrow strip of the phase diagram and might, consequently, be difficult to resolve at finite temperature. As emphasized by Jain, according to this interpretation, the observation of reentrant insulating behavior in the neighborhood $\nu=\frac{2}{9}$ is suggestive of the existence of substantial short-range Wigner-signal order. It would therefore be extremely useful to study the behavior of ρ_{xy} at low temperature in this regime.

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