COURSE 4

ANYONS

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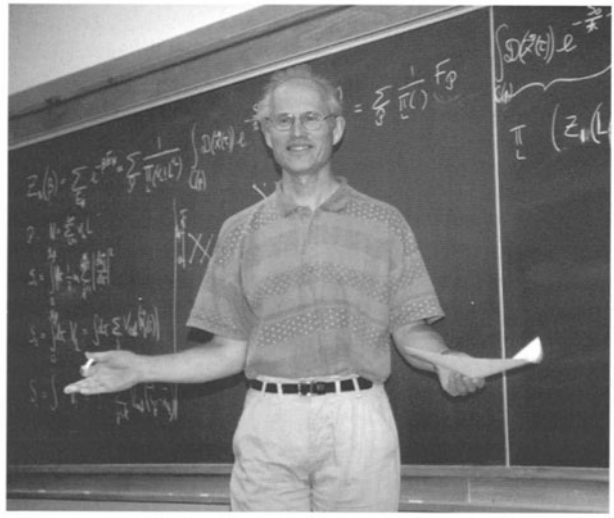

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Abstract

Fractional statistics of identical particles is a theoretical possibility both in one and two dimensions. Two-dimensional particles of this kind are called anyons. The most important application so far is in the theory of the fractionally quantized Hall effect, where the quasiparticle excitations can be described as anyons. The theory of identical particles, in particular the theory of anyons, is discussed here from the points of view of Schrödinger and Heisenberg quantization, as well as the Feynman path integral quantization. Two topics discussed in some detail are the equation of state of a gas of anyons, and the relation between particle interchange phases and geometric phases (Berry phases).

1 Introduction

The subject of these notes is the non-relativistic quantum theory of identical particles, and in particular the fractional statistics allowed in one- or two-dimensional systems. The concept of fractional statistics has now both theoretical and experimental interest, and may serve as an example, among many others, to illustrate that quantum mechanics is still a very active field of research, one hundred years after Planck's constant.

At least three different formulations of the quantum theory exist, mostly but not entirely equivalent, to be identified here by the names of Heisenberg, Schrödinger and Feynman quantization. Most attention is given to the last two, and to the conclusion drawn from both that "anyons" may exist in two dimensions having "any" statistics in between Bose–Einstein and Fermi–Dirac. Fractional statistics in one dimension is discussed in less detail here but is also the subject of other lectures.

It is a pleasure and a great honour to talk at Les Houches about Feynman path integrals for systems of identical particles, since the founder of the School, DeWitt, is also the founder of this particular branch of quantum theory [1,2]. By means of path integrals, Laidlaw and DeWitt gave the first topological proof of the symmetrization postulate in the quantum theory of

identical particles, and at the same time showed the fundamental difference between two- and three-dimensional space.

As an application of the general theory, the gas of anyons is discussed, in the two special cases of free anyons and of anyons in a magnetic field. Especially the magnetic field case should be of interest for applications in the fractionally quantized Hall effect. The final topic is the connection between the statistics phases related to the topology of the configuration space of a system of identical particles, and the geometric phases related to the geometry of the quantum mechanical state space. This connection has been used as a tool for investigating the statistics of quasi-particle excitations in the fractional quantum Hall system.

I hope that the list of references is useful and representative. It is certainly incomplete, and I want to apologize for omissions. Some review articles and books are *e.g.* the references [3–20]. Several of the books are reprint collections, containing reprints of many articles cited here, and many more.

1.1 The concept of particle statistics

Immediately after Heisenberg and Schrödinger formulated quantum mechanics as it is known today, Heisenberg and Dirac extended the theory to systems of identical particles [21–23]. They noted that the operators representing observables in such a system must be symmetric under any interchange of particle labels, since non-symmetric observables would allow an observer to distinguish between particles. This rather obvious statement was the key to the correct quantum theory, because symmetric operators preserve the symmetry properties of the wave functions. For example, if the operator A and the wave function ψ are both totally symmetric, then the wave function $A\psi$ is also totally symmetric. And similarly, if A is symmetric but ψ is totally antisymmetric, then $A\psi$ is totally antisymmetric. Consequently, there exists a complete quantum theory of identical particles using only the totally symmetric wave functions, and there exists a *different* complete theory using only the totally antisymmetric wave functions.

The symmetry or antisymmetry of the allowed wave functions is a characteristic property of a given system of identical particles, called the *statistics* of the particles. Particles described by symmetric wave functions satisfy *Bose–Einstein* statistics and are called *bosons.* Particles described by antisymmetric wave functions satisfy *Fermi–Dirac* statistics, they are *fermions,* and because of the antisymmetry they obey the *Pauli exclusion principle,* that two particles can not occupy the same quantum state. The symmetry or antisymmetry results in an effective attraction between bosons and an effective repulsion between fermions, both of a purely quantum mechanical nature. We may refer to this kind of attraction or repulsion as a *statistics interaction.* The mutual repulsion between fermions is quite literally a

tangible fact: we can walk on the earth because matter consists of a small number of different species of fermions. In fact, the stability of matter (at least the proof of stability) depends on the fermionic nature of matter [24, 25].

Since the theory of Heisenberg and Dirac predicted that identical particles had to be either bosons or fermions, and since this prediction was verified experimentally, there was not much need for a better theory. However, the theory could be questioned on philosophical rather than experimental grounds. One possible objection is the vagueness of the concept of particle interchange. The most obvious way to make it precise is perhaps to define it simply as an interchange of particle labels. Then it becomes a purely mathematical operation with no physical content, meaningful in the mathematical description of a system of identical particles, but with no counterpart in the physical reality. It simply reflects the fact that the correspondence between physics and mathematics is not one to one.

One may argue, however, that such an interpretation is too superficial. Let us consider particles that are so far apart that they can not be physically interchanged. Then it is intuitively obvious, and indeed true, that it does not matter whether we symmetrize or antisymmetrize our wave functions, or do neither of the two. This example suggests that the symmetrization or antisymmetrization postulate is not truly fundamental, but is rather a consequence of some more fundamental principle. It also indicates that this new fundamental principle must somehow give meaning to the concept of physical interchange of particles.

Regardless of whether an interchange of identical particles is regarded as a mathematical or a physical operation, it is obviously an identity transformation from the physical point of view. In quantum mechanics it is not unusual that a *physical* identity transformation is represented *mathematically* by a phase factor, since two wave functions represent the same physical state if they differ only by an overall phase factor. Any permutation of bosons is represented by the trivial phase factor $+1$, whereas even and odd permutations of fermions are represented by $+1$ and -1, respectively. A natural question is then, why only ± 1 and not more general phase factors?

Laidlaw and DeWitt answered this question in the context of nonrelativistic quantum mechanics when they applied the Feynman path integral formalism to systems of identical particles [1]. In their formalism the interchange of identical particles has a clear physical meaning as a continuous process in which each particle moves along a continuous path. The path dependence of the interchange is all important, since it relates the quantum mechanical concept of particle statistics to the topology of the classical configuration space. The phase factors associated with different interchange paths must define a representation of the *first homotopy group*

(the *fundamental group*) of the configuration space [26]. This requirement leads to the conclusion that only bosons and fermions can exist in Euclidean space of dimension three or higher, whereas more general possibilities open up in the two-dimensional case. The formalism does not apply in one dimension.

Using a more traditional approach to quantization, Leinaas and I derived the same relation between particle statistics and topology [27]. Our approach was based on the geometrical interpretation of wave functions which is the basis of gauge theories, and which goes back to Weyl and Dirac [28–32]. We studied in some detail the more general kinds of statistics allowed in one- and two-dimensional systems. In either case there exists a continuously variable parameter defining the statistics, interpolating continuously between Bose–Einstein and Fermi–Dirac statistics. In one dimension the parameter may be interpreted as the strength of a δ-function potential between bosons, and when the strength becomes infinite, the bosons become fermions [33–36]. In two dimensions the parameter may be chosen as a phase angle θ which is 0 for bosons and π for fermions, and we showed by the example of the two-dimensional harmonic oscillator that the continuous variation of the phase angle gives a continuous interpolation between the boson and fermion energy spectra. The *intermediate statistics*, as we called it, is now usually called *fractional statistics*. In the two-dimensional case, the word "fractional" refers to interchange phases that are arbitrary rational or irrational fractions of π.

A third approach leading to the same results is that of Goldin *et al.* [6, 37–41]. They studied the representations of the commutator algebra of particle density and current operators. This algebra has commutation relations that are independent of the particle statistics, but has inequivalent representations corresponding to the different statistics.

Wilczek arrived at the concept of fractional statistics by considering the fact that the spin of two-dimensional particles is theoretically allowed to take arbitrary values, not just integer or half-integer multiples of $\hbar$. The relation between spin and statistics would require particles of fractional spin to have fractional statistics as well [42, 43]. He introduced the name *anyons* for two-dimensional identical particles having an interchange phase of "any" fixed value, not necessarily 0 or π, and also proposed a model for them as particles carrying both electric charge and magnetic flux, so that the interchange phase could be understood as an Aharonov–Bohm effect [44–46].

The fundamental group of the configuration space of identical particles in the plane plays a fundamental role in the theory of anyons. This group is called the *braid group* [47, 48], and its role was emphasized especially by Wu [49, 50]. It is interesting that mathematicians have arrived at exactly the same configuration space concept from the opposite direction, namely

as a useful tool for studying the braid group [51–54]. For reviews of braids and knots and some applications in physics, see *e.g.* [55, 56].

The concept of the geometric phase, discovered by Berry [15, 57–61], was immediately applied to the calculation of interchange phases by Arovas *et al.* [62–64]. We will return to the question of how these phases are related, in Section 10 below.

In quantum field theory the symmetry or antisymmetry of many-particle wave functions results from the canonical commutation or anticommutation relations of the field operators [65]. It is not obvious how to interpolate continuously between commutation and anticommutation relations so as to get a quantum field theory for anyons, but a solution to this problem is to use either a "boson" or a "fermion" gauge and then describe the deviation from Bose–Einstein or Fermi–Dirac statistics as due to a "statistics" field, which is then a vector potential analoguous to the electromagentic vector potential [3, 4, 66–69]. The statistics vector potential is an example of a Chern–Simons field [70–76].

There are many other developments in the theory of fractional statistics about which little, or nothing, will be said here. Among those are statistics in one dimension [5, 77], or on two dimensional surfaces of a more complicated topology than the Euclidean plane. Thouless and Wu considered identical particles on the sphere, and found restrictions on the statistics angle dependent on the number of particles [78, 79]. Einarsson showed how to implement fractional statistics on a torus [80, 81], and more general discussions can be found *e.g.* in references [82, 83].

Certainly one of the most interesting topics is the connection between spin and statistics. The spin of the statistics field plays an essential part in establishing a connection [42, 43, 84–86]. However, it seems impossible to exclude for example the possibility that non-relativistic spin zero particles could be fermions, unless some extra assumptions are introduced [82, 87]. General topological arguments have been put forward, in which the existence of antiparticles is a crucial assumption [5, 88–92].

Particles and antiparticles are just one example of interacting anyons of different kinds. This is closely related to the possibility of interactions between *distinguishable* particles resembling the statistics interaction of identical particles [41, 93–95].

1.2 Statistical mechanics and the many-body problem

The statistical mechanics of bosons and fermions, *i.e.* the Bose–Einstein and Fermi–Dirac statistics, existed even before quantum mechanics received its final form [22, 96–100]. The theory is no more difficult than the corresponding theory of distinguishable particles, since the only effect of the indistinguishability of bosons or fermions is to forbid wave functions of the wrong symmetry type, thereby reducing the degeneracy of each energy level.

The statistics interaction between bosons or between fermions does not change the energies of individual levels.

The ideal gas, *i.e.* a gas of particles with no interaction apart from the statistics interaction, is a simple model which is useful for many purposes. In order to calculate the energy levels for a system of many non-interacting particles that are either bosons or fermions, one need only distribute the particles among the one-particle energy levels, counting degeneracies according to the Bose–Einstein or Fermi–Dirac statistics. Knowing the many-particle energy levels and their degeneracies, one may proceed to calculate the equation of state.

The statistical mechanics of anyons is more difficult. It has to be, simply because the bosonic and fermionic energy spectra are different, and the bosonic spectrum is supposed to change continuously into the fermion spectrum when the statistics angle θ changes from the boson value 0 to the fermion value π. The only way this can happen is that the energy levels move, either upwards or downwards. Thus, the statistics interaction of anyons affects not only the state counting, but also the energy eigenvalues.

The harmonic oscillator problem is the standard exercise in quantum mechanics, but even this is difficult for more than two anyons. The centre of mass motion in an external harmonic oscillator potential is separable, even for anyons, and the relative motion is governed by a two-body harmonic oscillator interaction potential. A slightly more general class of quadratic Hamiltonians, including that of a constant magnetic field, can be treated just as easily. The two-anyon harmonic oscillator problem was solved in reference [27]. Wu made the first attempt to solve the three-anyon problem, and found a class of exact solutions [101]. However, the ground state close to Fermi statistics was not among his exact solutions, and it is still not exactly known. More general exact solutions in harmonic oscillator potentials and magnetic fields, alone or together, have been found, but all have energies that depend linearly on the statistics angle [11, 102–115].

In the three-anyon problem, approximations to the wave functions corresponding to non-linear variation of energy have been found [116], and an almost complete separation of variables has been achieved [117]. The lowest part of the energy spectrum of three or four anyons in a harmonic oscillator potential has been calculated numerically [118–122]. Another line of attack is to use perturbation theory, starting from the known boson and fermion spectra [123–125]. The Hartree–Fock approximation has also been used [126, 127].

Arovas *et al.* made the first step towards determining the equation of state for a gas of non-interacting anyons when they calculated the second virial coefficient [66, 128]. Their result is exact, since it is obtained from the exactly soluble two-anyon problem. To have a finite density with only two particles, they put them in a box with hard walls. Comtet *et al.*, and also

Johnson and Canright, simplified the calculation by confining the particles in an external harmonic oscillator potential [102,129–131], in the same way as Fermi did for fermions [98].

The calculation of the third virial coefficient involves the three-anyon problem, which is not yet completely solved for any potential. Some exact results are nevertheless known. In particular, Sen has shown that the third virial coefficient is symmetric under a "supersymmetry" transformation which transforms bosons into fermions and vice versa, and more generally transforms θ into $\pi - \theta$ [132,133]. Other exact results are the first and second order perturbation expansions about the boson and fermion values $\theta = 0$ and $\theta = \pi$, not only for the third virial coefficient, but for the full cluster expansion [134–141]. The equation of state for anyons in a magnetic field can be computed exactly, in the strong field limit where all particles are in the ground state [142]. Numerical results exist for the third and fourth virial coefficients at general values of θ [122,143–146]. See also [147] for a summary and general discussion.

1.3 Experimental physics in two dimensions

There are three examples of physical systems that are studied experimentally, where it has been suggested that the theory of fractional statistics may be relevant. One of these applications, in the fractional quantum Hall effect, seems rather well established, whereas the other two, in high temperature superconductivity and in superfluid helium, are doubtful, at best. The last example, vortex motion in superfluid helium, will not be discussed any further here [148–153]. The statistics of vortices is discussed in more general contexts *e.g.* in [154–156].

It is a surprising fact that zero-, one- and two-dimensional experimental physics is possible in our three-dimensional world [157–163] (Ref. [157] is a review with nearly 2000 references). The strict confinement of electrons to surfaces, or even to lines or points, is possible thanks to the third law of thermodynamics, which states that all degrees of freedom freeze out in the limit of zero temperature. Thus, in a strongly confining potential at low enough temperature it may happen that the excitation energy in one or more directions is much higher than the average thermal energy of the particles, so that those dimensions are effectively frozen out.

Fowler, Fang, Howard and Stiles performed the first experiment with a two-dimensional electron gas in 1966, and later experiments use essentially the same technique [164]. The electrons are confined to the surface of a semiconductor by a strong electric field, and they move freely along the surface, whereas the energy ΔE needed to excite motion in the direction perpendicular to the surface is typically several millielectronvolt [165]. At a temperature of, for example, $T = 1\,\mathrm{K}$, the thermal energy is $k_B T \approx 0.1\,\mathrm{meV}$, where k_B is Boltzmann's constant. Hence, assuming for example

a transverse excitation energy of $\Delta E = 10\,\mathrm{meV}$, the fraction of electrons in the lowest excited transverse energy level is given by the Boltzmann factor

$$\mathrm{e}^{-\frac{\Delta E}{k_B T}} = \mathrm{e}^{-100} \approx 10^{-44}, \tag{1}$$

which is zero for all practical purposes. Thus the electron gas is truly two-dimensional.

Two-dimensional physics is no longer an exotic field since von Klitzing *et al.* discovered the quantized Hall effect (QHE) in 1980 [166, 167]. The discovery was totally unanticipated, and a new surprise was the discovery of the fractionally quantized Hall effect (FQHE) by Tsui *et al.* in 1982 [168, 169]. The effect observed is that, under certain conditions, the Hall resistance for a two-dimensional electron gas in a magnetic field is quantized as

$$R_{\mathrm{H}} = \frac{h}{\nu e^2} = \frac{25\,812.807\,\Omega}{\nu}, \tag{2}$$

where h is Planck's constant and e is the elementary charge. ν is either an integer or a rational fraction, which can be interpreted as the filling fraction, *i.e.* the number of degenerate energy levels (Landau levels) filled by conduction electrons, in the simple picture of a two-dimensional gas of free electrons. Thus, the fact that ν is not just inversely proportional to the magnetic field, but may stay constant while the field is changed by a finite amount, means that the number of conduction electrons varies with the field within certain limits.

The universality of the quantized Hall effect has been tested to a precision of 10^{-10} in an experiment comparing two different integer quantization levels in two different materials [170]. Thus, in spite of the fact that it involves an extremely complicated many-body problem, the integer quantum Hall effect seems to provide a precise method for measuring the fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ (in MKSA units), where $2\pi\hbar = h$, and c is the speed of light. It is independent of other methods, such as the measurement of the anomalous magnetic moments of electrons and muons, and gives a comparable precision. The same effect also provides a very accurate and stable standard resistor, easily realizable in the laboratory, and of a convenient magnitude. The conventional value of $25\,812.807\,\Omega$ is fixed by international agreement from January 1, 1990.

Laughlin proposed to explain the observed fractional quantization of the Hall resistance as the manifestation of a new state of matter, the incompressible quantum fluid, with elementary excitations that could be described as quasiparticles, or quasiholes, with fractional electric charge [171–174]. Halperin suggested that the fractional charge was associated with fractional statistics as well, and Arovas *et al.* verified by calculation the fractional values for both the charge and statistics phase angle of the quasiparticles

in Laughlin's theory [62, 175]. Jain *et al.* have tried to treat the integer and fractional quantum Hall effects in a more unified way [176–181]. Furthermore, Lütken and Ross have emphasized the universal character of the transition between different quantum Hall plateaux, and have suggested that the complete structure of the phase diagram, including the plateaux and the transition regions, can be understood as resulting from a discrete $SL(2, \mathbf{Z})$ symmetry [182–186]. But none of these theories change the prediction of quasiparticle excitations having fractional charge and statistics.

Different experiments seem to confirm the existence of fractionally charged excitations [187–193]. Thus, if fractional charge can be taken as a signature of fractional statistics [194], anyons may be said to have been directly observed in the fractional quantum Hall system.

Other examples of two-dimensional systems experimentally available are the high temperature superconductors, discovered by Bednorz and Müller [195–197]. The conduction takes place in two-dimensional layers, and Laughlin suggested a connection with the fractional quantum Hall effect [198, 199]. This idea raises two questions, discussed *e.g.* in references [14, 200]. First, whether systems of anyons show superfluidity and superconductivity, and second, whether such effects have anything to do with the observed high temperature superconductivity.

The second question must be answered experimentally, and some attempts have been made. The experiments are based on the general, but not very quantitative, prediction that anyons violate both time reversal and parity invariance, and that these effects are likely to arise because of local magnetic fields. The fields in adjacent layers might point in opposite directions, so as to cancel, or they might add up to a global field. Three experimental groups have tried to measure the effects of such global fields on transmitted or reflected polarized light, but with conflicting results [201–204]. A fourth group has probed the local magnetic field by means of muons, and set a rather small upper limit of 0.8 G [205]. Since no effect is seen either in this experiment or in the most sensitive of the optical experiments [203], the experimental evidence is clearly against the anyon theory for high temperature superconductivity.

1.4 The algebraic approach: Heisenberg quantization

The various approaches to the quantum theory for systems of identical particles mentioned so far, are closely related and may be grouped together under the heading of *Schrödinger quantization.* There exists an alternative approach, which we may call *Heisenberg quantization*, leading to somewhat different results, especially in one and two dimensions [152, 206–208]. Note that Schrödinger and Heisenberg quantization are not unique and detailed prescriptions for how to quantize, but rather two different general strategies. Schrödinger quantization is a *configuration space* approach, emphasizing the

role of the wave functions defined on the configuration space. Heisenberg quantization is a *phase space* approach, emphasizing the algebraic relations between observables, which in the classical theory are real valued functions defined on the phase space.

For example, the most general classical observable for one point particle on a line is a function $A = A(x,p)$ of the coordinate x and the momentum p. From two such observables A and B we may form the linear combination $C = \alpha A + \beta B$, where α and β are arbitrary real numbers, as well as two different bilinear products, the pointwise product $D = AB = BA$, and the Poisson bracket $E = \{A,B\} = -\{B,A\}$. By definition,

$$C(x,p) = \alpha A(x,p) + \beta B(x,p), \qquad D(x,p) = A(x,p)\,B(x,p), \tag{3}$$

and

$$E = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial p} \ . \tag{4}$$

In Heisenberg quantization one tries to represent the classical observables as linear, Hermitean operators on some complex Hilbert space, preserving as many as possible of the algebraic relations. The pointwise product is replaced by the operator product, and the Poisson bracket by the commutator product,

$$E = \frac{1}{i\hbar}[A,B] = \frac{1}{i\hbar}(AB - BA). \tag{5}$$

Since it is impossible to preserve all the algebraic relations exactly, one has to select some relations to be treated as more fundamental than the rest. Thus, in the example with one particle on a line, the relation

$$\{x,p\} = 1 \tag{6}$$

is considered fundamental, and is replaced by the canonical commutation relation

$$[x,p] = i\hbar. \tag{7}$$

However, for two or more identical particles this simple prescription does not work, and one has to find alternatives.

We will return to this point of view, although our main concern here is with the Schrödinger quantization. Briefly stated, the results are as follows, when the Heisenberg quantization is performed so as to respect the full symmetry between position and momentum variables. In one dimension fractional statistics is possible, described by one continuously variable statistics parameter. It is different from the fractional statistics obtained

by Schrödinger quantization in one dimension, and resembles more the two-dimensional fractional statistics of Schrödinger quantization. In two dimensions only the standard Bose–Einstein and Fermi–Dirac statistics are obtained. Thus, anyons are not included in this maximally symmetric version of Heisenberg quantization. In fact, anyons respect the rotational symmetry which involves coordinates only or momenta only, but break the phase space symmetry between coordinate and momentum.

1.5 More general quantizations

The basic philosophy behind both Schrödinger and Heisenberg quantization, as discussed above, is that the quantum theory of indistinguishable particles should resemble as much as possible the theory of distinguishable particles, that only such modifications are permitted as are necessary because the particles are indistinguishable. A number of different theories have been proposed departing more radically from the standard theory. They may allow interpolation between Bose–Einstein and Fermi–Dirac statistics independent of the configuration space dimension.

One possibility is to consider quantum field theories with fields that do not commute according to the canonical commutation or anticommutation relations. An example is the so-called *parastatistics* proposed by Green [209–211]. It allows not only the completely symmetric or antisymmetric representations of the symmetric group, but also more general symmetry classes [23,212]. Thus, parastatistics of order p allows Young tableaux of up to p rows in the para-Bose case, or up to p columns in the para-Fermi case, while infinite order parastatistics allows all symmetry classes. Doplicher *et al.* deduced precisely these three possibilities in local relativistic quantum theory without long range forces [213,214].

A number of proposed generalizations of the canonical commutation or anticommutation relations, starting with Wigner [215], are summarized in reference [216]. A simple example, leading to infinite statistics, is the so-called "q-mutation relations",

$$a_j a_k^\dagger - q a_k^\dagger a_j = \delta_{jk}, \tag{8}$$

where a and $a^\dagger$ are annihilation and creation operators, j,k label the degrees of freedom of the field, and q is a number [217–225]. See also [226]. A vacuum state $|0\rangle$ is postulated with the property that $a_j|0\rangle = 0$ for all j, and the Fock space is generated from it by repeated applications of creation operators. The scalar product in the Fock space is uniquely defined by the q-mutation relations, and Fivel and others have shown that the condition $-1 \le q \le 1$ is necessary and sufficient to ensure that the scalar product is positive definite [220,223,224]. No rules exist relating the products $a_j a_k$ and $a_k a_j$, or $a_j^\dagger a_k^\dagger$ and $a_k^\dagger a_j^\dagger$, except that it is possible to *prove* commutativity

in the boson case $q = 1$ and anticommutativity in the fermion case $q = -1$. The simple form $N_j = a_j^\dagger a_j$ for the number operators is no longer valid in general. In the special case of one degree of freedom the number operator is [221]

$$N = \sum_{n=1}^{\infty} \frac{(1-q)^n}{1-q^n} a^{\dagger n} a^n \,. \tag{9}$$

An entirely different approach, suggested by Haldane, is to modify directly the Pauli exclusion principle [227–237]. Johnson and Canright have applied this so called fractional exclusion statistics in the fractional quantum Hall system [238,239].

2 The configuration space

We will now discuss in more detail the quantum theory of identical particles. Our basic principle here is that an interchange of identical particles gives rise to a phase factor depending on the type of particles and on a continuous interchange path.

The path dependence of the phase factor suggests immediately a path integral approach [1], but we will discuss first the description by means of wave functions, which is usually more suitable for calculations [27]. There are two steps in our quantization scheme. The first step, discussed in the present section, is to identify the configuration space of the system of identical particles, and the different classes of possible interchange paths. The second step, discussed in the Sections 3 and 5, is to introduce wave functions on the configuration space. In two or higher dimensions the wave functions must be treated as geometrical objects.

Let $\mathbf{X}$ be the configuration space of a system of one particle. The configuration space of a system of N distinguishable particles moving in $\mathbf{X}$ is the Cartesian product space $\mathbf{X}^N$, defined as the set of all ordered N-tuples of the form

$$\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) \quad \text{with} \quad \boldsymbol{x}_j \in \mathbf{X} \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{10}$$

If p is a permutation of the particle labels $1, 2, \ldots, N$, then we define

$$p(\boldsymbol{x}) = (\boldsymbol{x}_{p^{-1}(1)}, \boldsymbol{x}_{p^{-1}(2)}, \ldots, \boldsymbol{x}_{p^{-1}(N)}). \tag{11}$$

The set of all permutations of N objects is the symmetric group S_N. It acts as a group of transformations on $\mathbf{X}^N$, by the above definition.

If the particles are indistinguishable, then a configuration of N particles is simply a set of N points in $\mathbf{X}$,

$$\boldsymbol{x} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\} \subset \mathbf{X}. \tag{12}$$

The order in which we list the particle positions is now arbitrary, so that the two points $\boldsymbol{x}$ and $p(\boldsymbol{x})$ in $\mathbf{X}^N$ represent the same configuration of the N-particle system. Thus, the configuration space of the system of N identical particles is the Cartesian product $\mathbf{X}^N$ with the identification $p(\boldsymbol{x}) \equiv \boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbf{X}^N$ and any $p \in S_N$. A natural name for this identification space is $\mathbf{X}^N/S_N$.

We will consider here only the Euclidean one-particle spaces $\mathbf{X} = \mathbf{R}^d$, of dimension $d = 1, 2, 3$, since these are the simplest examples and the most useful for applications. An important simplification in the Euclidean case is that the centre of mass position splits off in a trivial way, so that

$$\mathbf{X}^N/S_N = \mathbf{X} \times (\mathbf{X}^{N-1}/S_N), \tag{13}$$

where the factor $\mathbf{X}$ in the Cartesian product represents the centre of mass position, and $\mathbf{X}^{N-1}/S_N$ represents the relative positions of the particles. The same factorization is not possible when the one-particle space is, *e.g.*, a circle [27], a torus [80] or a sphere [78].

2.1 The Euclidean relative space for two particles

In the Euclidean case the interesting part of the configuration space is the relative space $\mathbf{R}^{d(N-1)}/S_N$. Let us examine the simplest case, $N = 2$. We have to label the particles arbitrarily as 1 and 2, in order to define the relative position as

$$\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2. \tag{14}$$

If the two particles are at the positions $\boldsymbol{a}$ and $\boldsymbol{b}$, then we get either $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{b}$ or $\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{a}$, depending on which one of the two possible labellings we choose. Thus, because the particles are identical, the relative positions $\boldsymbol{x}$ and $-\boldsymbol{x}$ describe the same configuration, and we see that the two-particle relative space $\mathbf{R}^d/S_2$ is $\mathbf{R}^d$ with the identification $\boldsymbol{x} \equiv -\boldsymbol{x}$.

An immediate consequence of the identification $\boldsymbol{x} \equiv -\boldsymbol{x}$ in $\mathbf{R}^d$ is that any time dependent curve $\boldsymbol{x}(t)$ is identified with $\boldsymbol{y}(t) = -\boldsymbol{x}(t)$. Hence the tangent (or velocity) vector $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ at $\boldsymbol{x}$ is identified with $\boldsymbol{w} = \mathrm{d}\boldsymbol{y}/\mathrm{d}t = -\boldsymbol{v}$ at $\boldsymbol{y} = -\boldsymbol{x}$.

The two spaces $\mathbf{R}^d/S_2$ and $\mathbf{R}^d$ are locally isometric, in fact the identification $\boldsymbol{x} \equiv -\boldsymbol{x}$ is clearly irrelevant whenever we look at a small region $\Omega \subset \mathbf{R}^d$ such that $-\boldsymbol{x} \notin \Omega$ for every $\boldsymbol{x} \in \Omega$. However, this isometry does not hold at the origin, because any open region in $\mathbf{R}^d$ containing the origin must contain at least one pair of points $\boldsymbol{x}$ and $-\boldsymbol{x}$. In other words, the origin is a singular point for the identification $\boldsymbol{x} \equiv -\boldsymbol{x}$. This local difference between $\mathbf{R}^d/S_2$ and $\mathbf{R}^d$ at the origin results also in a global difference.

Perhaps the most dramatic manifestation of the global difference is the fact that $\mathbf{R}^d$ is flat, whereas $\mathbf{R}^d/S_2$ is globally curved when $d \geq 2$.

To measure the global curvature one need not even approach the singularity at the origin. Curvature is defined in terms of the parallel transport of tangent vectors around closed curves, and the interesting curves are those starting at any given point $\boldsymbol{x} \in \mathbf{R}^d$ and ending at $-\boldsymbol{x}$. By definition, they are closed curves in $\mathbf{R}^d/S_2$. Start with some vector $\boldsymbol{v}$ at $\boldsymbol{x}$, and parallel transport it along any curve to $-\boldsymbol{x}$. Because $\mathbf{R}^d$ is flat, the vector is moved unchanged, but, as we have seen, the vector $\boldsymbol{v}$ at $-\boldsymbol{x}$ is identified with the vector $-\boldsymbol{v}$ at $\boldsymbol{x}$. Hence, the effect of the parallel transport around this kind of closed curve in $\mathbf{R}^d/S_2$ is to reverse the direction of every tangent vector.

This reversion by parallel transport may lead to confusion as to whether or not a given vector field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ on $\mathbf{R}^d/S_2$ is single-valued. Let us write

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{j=1}^{d} v_j(\boldsymbol{x})\, \boldsymbol{e}_{\boldsymbol{x}j}, \tag{15}$$

where each $\boldsymbol{e}_{\boldsymbol{x}j}$ is a basis vector at $\boldsymbol{x}$, and $v_j(\boldsymbol{x})$ is the j-th component of the vector $\boldsymbol{v}(\boldsymbol{x})$ located at $\boldsymbol{x}$. In Euclidean space we are used to basis vectors that are parallel everywhere, so that $\boldsymbol{e}_{\boldsymbol{x}j}$ is parallel to $\boldsymbol{e}_{\boldsymbol{y}j}$ for any two points $\boldsymbol{x}$ and $\boldsymbol{y}$. Unfortunately, as we have seen, it is impossible to introduce parallel basis vectors in $\mathbf{R}^d/S_2$, unless we place two sets of basis vectors, $\boldsymbol{e}_{\boldsymbol{x}1}, \boldsymbol{e}_{\boldsymbol{x}2}, \ldots, \boldsymbol{e}_{\boldsymbol{x}d}$ and $-\boldsymbol{e}_{\boldsymbol{x}1}, -\boldsymbol{e}_{\boldsymbol{x}2}, \ldots, -\boldsymbol{e}_{\boldsymbol{x}d}$, at the same point $\boldsymbol{x}$. It follows that if $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ is a single-valued vector field on $\mathbf{R}^d/S_2$, its components $v_j = v_j(\boldsymbol{x})$ with respect to parallel basis vectors are double-valued functions on $\mathbf{R}^d/S_2$. We may of course introduce basis vectors that are single-valued functions of position, so that the components of a single-valued vector field are also single-valued, but such basis vectors can not be parallel. As we shall see, similar problems arise when we introduce wave functions.

The generalization to N identical particles, with $N > 2$, is straightforward. Let us ignore those configurations where two or more particle positions coincide. Then each point in the full configuration space $\mathbf{R}^{dN}/S_N$, or in the relative space $\mathbf{R}^{d(N-1)}/S_N$, corresponds to $N!$ points either in $\mathbf{R}^{dN}$ or in $\mathbf{R}^{d(N-1)}$. In general, a closed curve in $\mathbf{R}^{d(N-1)}/S_N$ connects a point $\boldsymbol{x} \in \mathbf{R}^{d(N-1)}$ to the point $p(\boldsymbol{x})$, where p is any one of the $N!$ permutations in the symmetric group S_N. Parallel transport moves a vector $\boldsymbol{v}$ unchanged from $\boldsymbol{x}$ to $p(\boldsymbol{x})$. However, the vector $\boldsymbol{v}$ at $p(\boldsymbol{x})$ is not the same as $\boldsymbol{v}$ at $\boldsymbol{x}$. Rather, $\boldsymbol{v}$ at $\boldsymbol{x}$ is identified with $p(\boldsymbol{v})$ at $p(\boldsymbol{x})$, hence $\boldsymbol{v}$ at $p(\boldsymbol{x})$ is identified with $p^{-1}(\boldsymbol{v})$ at $\boldsymbol{x}$. Thus we see that the effect of the parallel transport of $\boldsymbol{v}$ is to transform it into $p^{-1}(\boldsymbol{v})$. Given one vector $\boldsymbol{v}$ at $\boldsymbol{x} \in \mathbf{R}^{d(N-1)}/S_N$, there are altogether $N!$ vectors at $\boldsymbol{x}$ that are parallel to it, by parallel transport around different closed curves in $\mathbf{R}^{d(N-1)}/S_N$.

2.2 *Dimensions* $d = 1, 2, 3$

Let us consider a little more explicitly the simplest examples with two identical particles in Euclidean space. In one dimension the relative space $\mathbf{R}/S_2$ is the half-line $x \geq 0$, where $x = x_1 - x_2$ is the single relative coordinate. Choosing $x \geq 0$ instead of $x \leq 0$ is of course pure convention, it simply means that we always label the rightmost particle as number 1.

In two dimensions the identification $\boldsymbol{x} \equiv -\boldsymbol{x}$ can be pictured as a folding of the plane into a cone of opening half-angle 30°. The points $\boldsymbol{x}$ and $-\boldsymbol{x}$ in the plane are folded onto the same point on the cone, and the origin of the plane becomes the top of the cone. Equivalently, writing the relative position as $\boldsymbol{x} = (x, y)$, we may define $\mathbf{R}^2/S_2$ as the upper half-plane $y \geq 0$, but with the boundary points $(x, 0)$ and $(-x, 0)$ identified. The cone is locally flat everywhere except at the top point, since it is locally isometric to the plane. But it is globally curved, with infinite curvature at the top, so that parallel transport of a tangent vector once around the top point reverses its direction.

In three dimensions, if we write the relative position as $\boldsymbol{x} = (x, y, z)$, then we may define $\mathbf{R}^3/S_2$ as the upper half-space $z \geq 0$, with the boundary points $(x, y, 0)$ and $(-x, -y, 0)$ identified. Again the origin is a singular point of the identification space, and the space is locally flat everywhere except at the origin, since it is locally isometric to $\mathbf{R}^3$. And again there is a global curvature, located at the origin, such that parallel transport of a tangent vector once around the origin reverses its direction.

2.3 *Homotopy*

In order to classify the interchange paths, we have to examine the path connectivity of the configuration space. Again we consider only the Euclidean case, so that it is enough to examine the relative space $\mathbf{R}^{d(N-1)}/S_N$.

Two curves from a point $\boldsymbol{x}$ to a point $\boldsymbol{y}$ are said to be *homotopic* if they can be continuously deformed one into the other [26]. A *homotopy class* consists of all the curves that are homotopic to one given curve. Concatenation of curves defines a natural product: two curves C_1 and C_2 can be spliced into one curve C_2C_1 if C_2 starts at the point where C_1 ends. That is, if C_1 goes from $\boldsymbol{x}$ to $\boldsymbol{y}$ and C_2 from $\boldsymbol{y}$ to $\boldsymbol{z}$, then C_2C_1 is a curve going from $\boldsymbol{x}$ to $\boldsymbol{z}$. This multiplication of curves is also a multiplication of homotopy classes.

If we consider only the closed curves, or *loops*, starting and ending at one given point $\boldsymbol{x}$, then the product of any two such loops is well-defined. The homotopy classes of loops at $\boldsymbol{x}$ form a group, called the *first homotopy group*, or *fundamental group*, of our space. In a connected space this definition does not depend on the point $\boldsymbol{x}$, in the sense that groups defined at different points are isomorphic. The single point $\boldsymbol{x}$ is a degenerate kind of loop, the

corresponding homotopy class consists of all the loops from $\boldsymbol{x}$ back to $\boldsymbol{x}$ that can be continuously deformed into a point, and this class is the unit element of the group. The inverse of a loop is the same loop traversed in the opposite direction.

By definition, a space is *simply connected* if every loop can be continuously deformed into a single point, or equivalently, if the fundamental group is the trivial group consisting of one element only. Similarly, it is *doubly connected* if the fundamental group has exactly two elements, and it is *infinitely connected* if the fundamental group is infinite, *e.g.* isomorphic to $\mathbf{Z}$, the addition group of integers.

The Euclidean space of any dimension is simply connected, and in particular the configuration space $\mathbf{R}^{dN}$ for N distinguishable particles in d dimensions is simply connected. The path connectivity of the configuration space $\mathbf{R}^{dN}/S_N$ for N identical particles is a matter of definition. In the literal sense $\mathbf{R}^{dN}/S_N$ is simply connected, but we want to argue that a more natural definition of homotopy is such that $\mathbf{R}^{dN}/S_N$ is not simply connected when $N \geq 2$.

Note that the fundamental group is the same for $\mathbf{R}^{dN}/S_N$ as for the relative configuration space $\mathbf{R}^{d(N-1)}/S_N$, because the centre of mass position splits off as in equation (13). We have seen in the example with $N = 2$ that there exist two classes of loops in $\mathbf{R}^d/S_2$ with respect to the parallel transport of relative tangent vectors, transporting a vector $\boldsymbol{v}$ into $+\boldsymbol{v}$ or $-\boldsymbol{v}$, respectively, and it is natural to define that a "+" and a "−" loop are not homotopic. If we want to deform a "+" loop continuously into a "−" loop, or vice versa, then one stage in the process must be a loop going through the singular point where the two particles collide. Such a loop is itself singular in the sense that the parallel transport of a vector is ambiguous. The natural solution is to simply exclude such singular paths, or equivalently, to exclude the singular point from the relative space, making it multiply connected.

In the general N-particle case there will be at least $N!$ inequivalent classes of loops corresponding to the $N!$ possible permutations of particle labels in the local space of tangent vectors. This definition of homotopy means that we exclude all the singular points of the configuration space, *i.e.* all those configurations in which two or more particles are at the same position. This restriction implies that the one-dimensional case ($d = 1$) is uninteresting, because the relative space $\mathbf{R}^{N-1}/S_N$ without its singular points is connected, but has no continuous paths that interchange particle positions. It implies further that in dimension two or higher ($d \geq 2$) there is always a homomorphism from the fundamental group onto the symmetric group S_N. In dimension three or higher ($d \geq 3$) the homomorphism is in fact an isomorphism: the fundamental group is just the symmetric group

S_N. In two dimensions, however, the fundamental group is a non-trivial extension of the symmetric group, called the braid group.

2.4 The braid group

For two particles in the plane ($N = 2$, $d = 2$), *i.e.* for the relative configuration space $\mathbf{R}^2/S_2$, the fundamental group is $\mathbf{Z}$. This is so because every loop has an integer *winding number*, which is the number of times it encircles the origin, and the winding number is additive under concatenation of loops. By arbitrary convention, we count anticlockwise winding as positive and clockwise winding as negative. Two loops are homotopic if and only if they have the same winding number, in other words, the winding number labels uniquely a homotopy class. A curve in $\mathbf{R}^2/S_2$ can also be regarded as a curve in $\mathbf{R}^2$, and its winding number is even if the curve in $\mathbf{R}^2$ starts at $\boldsymbol{x}$ and returns to $\boldsymbol{x}$, or odd if the curve in $\mathbf{R}^2$ goes from $\boldsymbol{x}$ to $-\boldsymbol{x}$. Thus, parallel transport of a tangent vector $\boldsymbol{v}$ around a closed loop in $\mathbf{R}^2/S_2$ gives the vector $(-1)^Q\boldsymbol{v}$, where Q is the winding number of the loop.

For N particles in the plane ($d = 2$), *i.e.* for the relative configuration space $\mathbf{R}^{2(N-1)}/S_N$, the fundamental group is the braid group B_N [47,48,51-54]. We have seen that $B_2 = \mathbf{Z}$. In general, B_N can be generated from $N-1$ elements, in the following way. The j-th generating element T_j is the homotopy class of loops that do nothing more than interchange the particles j and $j+1$ in the anticlockwise direction. It can be represented graphically as in Figure 1. Obviously, two such generators commute if they do not interfere, that is,

$$T_jT_k = T_kT_j \qquad \text{if} \qquad |j-k| > 1. \tag{16}$$

Neighbouring generators do not commute, but satisfy the following relations,

$$T_jT_{j+1}T_j = T_{j+1}T_jT_{j+1} \qquad \text{for} \qquad j = 1, 2, \ldots, N-2, \tag{17}$$

which can be proved graphically as in Figure 2. Note that T_j and T_{j+1} are homotopy classes of loops, so that the equality sign here means homotopy of loops.

It is easy to see that every one-dimensional representation of B_N is given by one single number τ. In fact, if the generator T_j is represented by the number τ_j, then the relation $\tau_j\tau_{j+1}\tau_j = \tau_{j+1}\tau_j\tau_{j+1}$ means that $\tau_j = \tau_{j+1} = \tau$, independent of j. The general braid has the form

$$b = T_{j_1}^{n_1}T_{j_2}^{n_2}\cdots T_{j_K}^{n_K}, \tag{18}$$

where each index j_k is an integer from 1 to $N-1$, and each power n_k is a positive or negative integer. In the one-dimensional representation b is

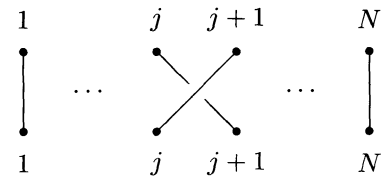


Fig. 1. The braid group generator T_j, an anticlockwise continuous interchange of the particles j and $j+1$. The horizontal axis represents space, $\mathbf{R}^2$, the vertical axis represents "time", *i.e.* the parameter of the curve.

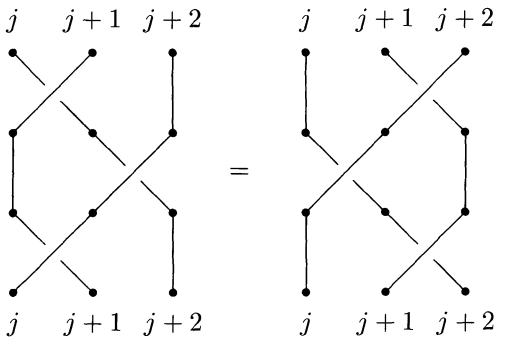


Fig. 2. Graphical proof of the relation $T_jT_{j+1}T_j = T_{j+1}T_jT_{j+1}$.

represented by τ^Q, where Q is the winding number, defined as

$$Q = \sum_{k=1}^{K} n_k. \tag{19}$$

The difference between the braid group B_N and the symmetric group S_N is that there is one more set of defining relations for the symmetric group,

$$T_j^{-1} = T_j \qquad \text{for} \qquad j = 1, 2, \ldots, N-1. \tag{20}$$

This implies for the one-dimensional representations of the symmetric group that $\tau^{-1} = \tau$. Hence there are exactly two such representations, one with $\tau = 1$ and one with $\tau = -1$.

In three or higher dimensions a clockwise continuous interchange of two particles is homotopic to an anticlockwise interchange. See Figure 3. Therefore equation (20) holds, so that the fundamental group of the configuration space $\mathbf{R}^{dN}/S_N$ in dimension $d \geq 3$ is S_N.

3 Schrödinger quantization in one dimension

The one-dimensional case is rather special, since particles on the line can not be continuously interchanged without colliding. The mathematical

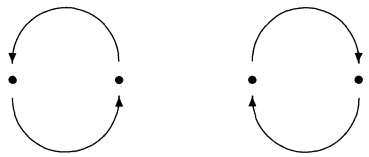


Fig. 3. Interchange of two particles, either anticlockwise (left) or clockwise (right). In three or higher dimensions these two interchange loops are homotopic, by rotation an angle π about the line joining the particles.

expression of this fact is that the configuration space of a system of identical particles on the line has a boundary, consisting of those configurations where two or more particle positions coincide. In particular, as we have seen, the relative space of the two-particle system is a half-line, with the origin as a boundary. Therefore the quantization problem reduces to the problem of specifying the proper boundary conditions on the wave functions.

The role of boundary conditions in quantum mechanics is to make certain operators Hermitean, and the most important operator is the Hamiltonian H. Hermiticity of H means that probability is conserved. Let us assume a standard two-particle Hamiltonian of the form

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V(x_1, x_2) \\ &= -\frac{\hbar^2}{4m}\frac{\partial^2}{\partial X^2} - \frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2} + V(X, x). \end{aligned} \tag{21}$$

Here m is the particle mass, $X = (x_1+x_2)/2$ the centre of mass position, and $x = x_1-x_2$ the relative position. For identical particles the potential V must be symmetric, $V(x_2, x_1) = V(x_1, x_2)$, or equivalently $V(X, -x) = V(X, x)$, which implies that H is symmetric under interchange of particle labels, as an observable should be. For simplicity we will further assume here that V is non-singular as a function of x, or at least is no more singular than $1/x$. We will discuss a $1/x^2$ potential below, in connection with Heisenberg quantization. More singular potentials lead to important complications.

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{22}$$

for the wave function $\psi = \psi(x_1, x_2, t) = \psi(X, x, t)$ implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_X}{\partial X} + \frac{\partial j_x}{\partial x} = 0, \tag{23}$$

where $\rho = |\psi|^2$ is the probability density, and

$$j_X = \mathrm{Re}\left(\psi^* \frac{\hbar}{4mi} \frac{\partial \psi}{\partial X}\right), \qquad j_x = \mathrm{Re}\left(\psi^* \frac{\hbar}{mi} \frac{\partial \psi}{\partial x}\right) \tag{24}$$

are the X and x components of the probability current density. ψ^* is the complex conjugate of ψ.

The physically acceptable way to impose conservation of probability is to require that the normal component of the probability current vanishes everywhere on the boundary. That is, in the two-particle case, $j_x(X, 0) = 0$ for every X. However, this is a quadratic boundary condition for the wave function, whereas the superposition principle demands a linear condition. We therefore postulate that

$$\begin{aligned} \left.\frac{\partial \psi}{\partial x}\right|_{X=\text{const.}} &= \frac{1}{2}\left(\left.\frac{\partial \psi}{\partial x_1}\right|_{x_2=\text{const.}} - \left.\frac{\partial \psi}{\partial x_2}\right|_{x_1=\text{const.}}\right) \\ &= \eta\psi \quad \text{at} \quad x = x_1 - x_2 = 0, \end{aligned} \tag{25}$$

with η a real parameter, independent of ψ. This is a stronger condition, implying that $j_x = 0$ at $x = 0$, and it is linear. η could in principle be a function of X, but that would break translation invariance. The particles are bosons if $\eta = 0$ and fermions if $\eta = \pm\infty$, but in principle η is a continuous variable that could take any intermediate value.

Since the wave function $\psi = \psi(X, x)$ is defined only for $x \geq 0$, we are free to extend the domain of definition to $x < 0$, for example by imposing the bosonic symmetry $\psi(X, -x) = \psi(X, x)$. The symmetric extension will make the partial derivative at $x = 0$ discontinuous if equation (25) holds with $\eta \neq 0$. The discontinuity of the partial derivative is then equivalent to a statistics interaction described by the δ-function potential

$$V_s(x) = \frac{2\eta\hbar^2}{m}\,\delta(x). \tag{26}$$

As is well known, the δ-function potential has exactly one bound state if $\eta < 0$.

We may use the external harmonic oscillator potential

$$V = \frac{1}{2}\,m\omega^2\left(x_1^2 + x_2^2\right) = m\omega^2 X^2 + \frac{1}{4}\,m\omega^2 x^2 \tag{27}$$

as an example to illustrate how the parameter η defines a continuous interpolation between bosons and fermions. The Schrödinger equation is separable, so that we need only solve the energy eigenvalue equation

$$\left(-\frac{\hbar^2}{m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{4}m\omega^2x^2\right)\psi=E\psi \tag{28}$$

for the relative wave function $\psi=\psi(x)$. For a given relative energy E there exists a unique solution which is square integrable in the limit $x\to\infty$, and it can be expressed in terms of the confluent hypergeometric function $U=U(a,b,z)$ defined in Chapter 13 of reference [240],

$$\psi(x)=c_0\,U\left(\frac{1}{4}-\frac{E}{2\hbar\omega}\,,\,\frac{1}{2}\,,\,\frac{x^2}{a_0^2}\right)\exp\left(-\frac{x^2}{2a_0^2}\right), \tag{29}$$

where c_0 is a normalization constant, and a_0 is a characteristic length,

$$a_0=\sqrt{\frac{2\hbar}{m\omega}}\,. \tag{30}$$

The boundary condition at $x=0$, equation (25), gives the following energy quantization condition, involving the Euler Γ-function [240],

$$\frac{\psi'(0)}{\psi(0)}=-\frac{2\,\Gamma\left(\frac{3}{4}-\frac{E}{2\hbar\omega}\right)}{a_0\,\Gamma\left(\frac{1}{4}-\frac{E}{2\hbar\omega}\right)}=\eta. \tag{31}$$

In particular, with $\eta=0$ we get the boson spectrum

$$E=\left(2n+\frac{1}{2}\right)\hbar\omega,\qquad n=0,1,2,\ \ldots\,, \tag{32}$$

and with $\eta=\pm\infty$ we get the fermion spectrum

$$E=\left(2n+\frac{3}{2}\right)\hbar\omega,\qquad n=0,1,2,\ \ldots \tag{33}$$

The level spacing is constant for bosons and fermions, but not for intermediate values of η. Figure 4 shows how the lowest energies vary with η.

The obvious generalization to the N-particle case is the convention that the general wave function $\psi=\psi(x_1,x_2,\ \ldots\,,x_N)$ is defined for $x_1\geq x_2\geq\ \ldots\geq x_N$ and satisfies the boundary conditions

$$\frac{\partial\psi}{\partial x_j}-\frac{\partial\psi}{\partial x_{j+1}}=2\eta\psi\quad\text{at}\quad x_j=x_{j+1}\quad(j=1,2,\ \ldots\,,N-1). \tag{34}$$

Lieb and Liniger have solved this particular N-particle problem in the case when $\eta>0$ and there is no other external or interaction potential [34,35].

4 Heisenberg quantization in one dimension

The Schrödinger quantization, as presented above, is not the only way to get intermediate statistics of particles in one dimension. In fact, the Heisenberg quantization leads just as naturally to a different type of intermediate statistics, equivalent to an inverse square statistics potential rather than a δ-function potential [152, 206]. The one-dimensional case is special in this respect. In higher dimensions only bosons and fermions emerge if we apply Heisenberg quantization in the most straightforward way.

The indistinguishability of the particles implies extra freedom in the quantization for a system of two or more particles, because it restricts the class of observables. To see how, it is again convenient to discuss the two-particle case as an example. The centre of mass position $X = (x_1 + x_2)/2$ and the total momentum $P = p_1 + p_2$ are observables, since they are symmetric under interchange, but the relative position $x = x_1 - x_2$ and momentum $p = (p_1 - p_2)/2$ are antisymmetric and therefore not observables. Thus, the canonical commutation relation, equation (7), between relative position and momentum is meaningless in a minimal theory which includes only such operators as represent observable quantities.

If we can not use x and p as basic observables, then the next simplest choice are the quadratic polynomials x^2, p^2 and xp, which are symmetric and therefore observables, at least in the classical theory. It is convenient to introduce an arbitrary length scale a_0 and define the dimensionless observables

$$A = \frac{a_0^2}{4\hbar^2}\, p^2 + \frac{1}{4a_0^2}\, x^2 \,, B = \frac{a_0^2}{4\hbar^2}\, p^2 - \frac{1}{4a_0^2}\, x^2 \,, C = \frac{1}{4\hbar}\, (xp + px) \,. \tag{35}$$

In the quantum theory they should satisfy the following commutation relations, which follow either from the Poisson brackets in the classical theory, or from the canonical commutation relation in the quantum theory,

$$[A, B] = iC, \qquad [A, C] = -iB, \qquad [B, C] = -iA. \tag{36}$$

It is natural to adopt equation (36) as the basic set of commutation relations defining the quantum theory of two identical particles on the line. They define the Lie algebra $sp(1, \mathbf{R}) = sl(2, \mathbf{R})$ of the real symplectic group $Sp(1, \mathbf{R}) = SL(1, \mathbf{R})$, consisting of the area-preserving linear transformations in the plane[1].

There exists a quadratic Casimir operator,

$$\Gamma = A^2 - B^2 - C^2, \tag{37}$$

[1] Unfortunately, different conventions exist, and this group is often called $Sp(2, \mathbf{R})$.

commuting with all operators in the Lie algebra. It must take a constant value if we require the linear representation of the Lie algebra to be irreducible, implying that only two of the three observables A, B and C are independent. Clearly two independent variables are just what we need to describe the two-dimensional relative phase space. In the classical case, $\Gamma = 0$ identically, wheras equation (35) together with equation (7) imply that $\Gamma = -3/16$. However, if x and p do not exist as operators, then we have to give up both equation (35) and equation (7), and there is no obvious reason any more to require that either $\Gamma = 0$ or $\Gamma = -3/16$.

There exists in fact a family of physically acceptable irreducible representations of $sp(1, \mathbf{R})$, depending on one continuously variable parameter $\alpha_0 > 0$. If we denote the basis vectors of one such representation by $|\alpha_0, n\rangle$, with $n = 0, 1, 2, \ldots$, then

$$\begin{aligned} \Gamma\,|\alpha_0, n\rangle &= \alpha_0(\alpha_0 - 1)\,|\alpha_0, n\rangle, \\ A\,|\alpha_0, n\rangle &= (\alpha_0 + n)\,|\alpha_0, n\rangle, \\ (B + iC)\,|\alpha_0, n\rangle &= \sqrt{(n+1)(n+2\alpha_0)}\;|\alpha_0, n+1\rangle, \\ (B - iC)\,|\alpha_0, n\rangle &= \sqrt{n(n-1+2\alpha_0)}\;|\alpha_0, n-1\rangle. \end{aligned} \tag{38}$$

Note that if a_0 is given by equation (30), then

$$2\hbar\omega A = \frac{p^2}{m} + \frac{1}{4}\,m\omega^2 x^2 \tag{39}$$

is just the harmonic oscillator relative Hamiltonian encountered earlier in equation (28). Thus, $\alpha_0 = 1/4$ corresponds to bosons and $\alpha_0 = 3/4$ to fermions, and the parameter α_0 provides a continuous interpolation between these two special cases. When α_0 changes, the whole harmonic oscillator spectrum is rigidly shifted with all level spacings constant, which proves that Schrödinger and Heisenberg quantization lead to inequivalent types of intermediate statistics. Figure 4 shows the bottom part of the harmonic oscillator energy spectrum as a function of the statistics parameter, both for Schrödinger and Heisenberg quantization.

4.1 The coordinate representation

We may change basis from the harmonic oscillator eigenstates $|\alpha_0, n\rangle$ to the eigenstates $|x\rangle$ of the relative position x, restricted to $x \geq 0$. In this coordinate representation x^2 is diagonal, whereas p^2 is a differential operator containing the parameter α_0,

$$p^2 = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\lambda\hbar^2}{x^2}\,, \qquad \text{with} \qquad \lambda = 4\left(\alpha_0 - \frac{1}{4}\right)\left(\alpha_0 - \frac{3}{4}\right). \tag{40}$$

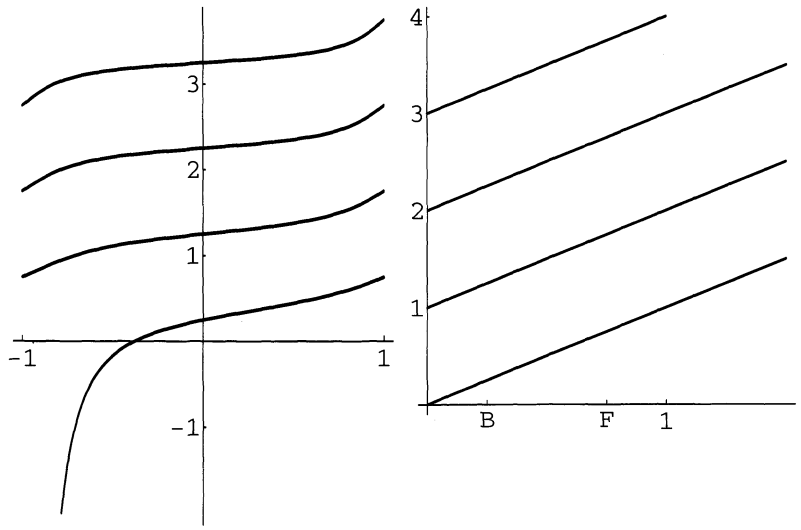


Fig. 4. $E/(2\hbar\omega)$, where E is the energy of relative motion of two identical particles with harmonic oscillator interaction. The lowest energies are shown as functions of the statistics parameter $\eta' = (2/\pi)\arctan\eta$ (Schrödinger quantization, left), or α_0 (Heisenberg quantization, right). Bosons have $\eta' = 0$ and $\alpha_0 = 1/4$, fermions have $\eta' = \pm 1$ and $\alpha_0 = 3/4$. From [208], reprinted with permission.

x^2 and p^2 define the operators A and B, whereas C is given by the commutation relation,

$$C = -i[A, B] = \frac{i}{8\hbar^2}[p^2, x^2] = \frac{1}{4i}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}x\right). \tag{41}$$

When the above definition of p^2 is inserted into the harmonic oscillator Hamiltonian, equation (39), the result is an extra inverse square statistics potential,

$$V_{\mathrm{s}}(x) = \frac{\lambda\hbar^2}{mx^2}\,, \qquad \text{with} \qquad \lambda = 4\left(\alpha_0 - \frac{1}{4}\right)\left(\alpha_0 - \frac{3}{4}\right), \tag{42}$$

in the Schrödinger equation, vanishing precisely in the boson case $\alpha_0 = 1/4$ and the fermion case $\alpha_0 = 3/4$. The modified eigenfunctions are of the form

$$\psi(x) = c_0\, x^{(2\alpha_0 - \frac{1}{2})}\, M\left(\alpha_0 - \frac{E}{2\hbar\omega}\,,\ 2\alpha_0\,,\ \frac{x^2}{a_0^2}\right)\exp\left(-\frac{x^2}{2a_0^2}\right). \tag{43}$$

The main difference from equation (29) is that the confluent hypergeometric function $M = M(a, b, z)$ replaces U, and the energy quantization condition

is now that M reduces to a polynomial, which happens when [240]

$$E = 2(n + \alpha_0)\hbar\omega, \qquad n = 0, 1, 2, \ldots \tag{44}$$

The choice of eigenfunction in equation (43) is dictated by the boundary condition at $x \to 0+$, and there is an argument behind the choice of boundary condition. The eigenvalue equation $H\psi = E\psi$, regarded as a second order ordinary differential equation,

$$\left(-\frac{\hbar^2}{m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\lambda\hbar^2}{mx^2} + \frac{1}{4}m\omega^2x^2\right)\psi = E\psi, \tag{45}$$

has two independent solutions behaving asymptotically as x^ν in the limit $x \to 0+$, where

$$\nu = \nu_\pm = \frac{1}{2} \pm (2\alpha_0 - 1). \tag{46}$$

(The case $\alpha_0 = 1/2$ is special, then the asymptotic form is either $\sqrt{x}$ or $\sqrt{x}\log x$.) The general solution $\psi = \psi(x)$ has the asymptotic form (for $\alpha_0 \neq 1/2$),

$$\psi(x) \sim c_+x^{\nu_+} + c_-x^{\nu_-}, \tag{47}$$

for some constants $c_\pm$, implying the asymptotic form of the probability current density,

$$j_x = \mathrm{Re}\left(\psi^*\frac{\hbar}{mi}\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) \sim \frac{2\hbar(2\alpha_0 - 1)}{m}\,\mathrm{Im}\left(c_-^*c_+\right). \tag{48}$$

The condition that the wave function must be square integrable requires that $c_- = 0$ for $\nu_- \leq -1/2$, *i.e.* for $\alpha_0 \geq 1$, but puts no restriction on the coefficients $c_\pm$ when $0 < \alpha_0 < 1$. One possible linear condition which will make $j_x \to 0$ as $x \to 0+$, is that

$$c_- = \eta c_+, \tag{49}$$

with η a real parameter. The superposition principle requires that $\eta = c_-/c_+$ must be the same for all wave functions. The parameter η here is of course related to the one introduced earlier, equation (25) is in fact just the special case $\alpha_0 = 1/4$.

In the present case η can not vary continuously, however. The point is that we want all three operators A, B and C to have a common domain of definition, but C changes the asymptotic form of the eigenfunctions of A in the limit $x \to 0+$, unless we impose one of the two conditions $c_+ = 0$ or $c_- = 0$. In fact,

$$C\left(c_+x^{\nu_+} + c_-x^{\nu_-}\right) = \frac{1}{4i}\left((2\nu_+ + 1)c_+x^{\nu_+} + (2\nu_- + 1)c_-x^{\nu_-}\right), \tag{50}$$

which means that C transforms η into

$$\widetilde{\eta} = \frac{(2\nu_- + 1)c_-}{(2\nu_+ + 1)c_+} = \frac{2\nu_- + 1}{2\nu_+ + 1}\,\eta = \frac{1-\alpha_0}{\alpha_0}\,\eta. \tag{51}$$

Which of the two conditions $c_+ = 0$ or $c_- = 0$ we impose, is only a matter of convention, since we may interchange ν_+ and ν_- by replacing α_0 with $1 - \alpha_0$. We choose the condition $c_- = 0$, so that wave functions have the asymptotic form $x^{(2\alpha_0 - \frac{1}{2})}$ as $x \to 0+$, and this convention selects the particular solution in equation (43).

There is a somewhat more physical way to understand why only the values $\eta = 0$ or $\eta = \pm\infty$ are left invariant by the operator C. The reason is that C is the infinitesimal generator of scaling transformations, it scales x and hence η, since η has the same dimension as $x^{(\nu_+ - \nu_-)} = x^{2(2\alpha_0 - 1)}$. To see that C generates scaling of x, consider the transformed wave function $\widetilde{\psi} = (I - 2i\epsilon C)\psi$, where ψ is a general wave function, I is the identity operator and ϵ is an infinitesimal parameter. The functions $\widetilde{\psi}$ and ψ have the same shape, but $\widetilde{\psi}$ is expanded by the factor $1 + \epsilon = 1/(1 - \epsilon)$ as compared to ψ, since

$$\widetilde{\psi}(x) = \psi(x) - \epsilon\left(x\psi'(x) + \frac{1}{2}\psi(x)\right) = \left(1 - \frac{\epsilon}{2}\right)\psi((1-\epsilon)x)\,. \tag{52}$$

The Heisenberg quantization for systems of more than two identical particles is an unsolved problem. However, if the two-particle Heisenberg quantization in one dimension is regarded as a special kind of Schrödinger quantization, involving an inverse square statistics potential, then it can be immediately generalized to the N-particle case [206]. The statistics potential becomes

$$V_{\mathrm{s}}(x_1, x_2, \ldots, x_N) = \sum_{1 \le j < k \le N} \frac{\lambda\hbar^2}{m(x_j - x_k)^2}\,,$$

$$\text{with} \quad \lambda = 4\left(\alpha_0 - \frac{1}{4}\right)\left(\alpha_0 - \frac{3}{4}\right). \tag{53}$$

The general wave function $\psi = \psi(x_1, x_2, \ldots, x_N)$ is defined for $x_1 \ge x_2 \ge \ldots \ge x_N$ and satisfies the boundary conditions

$$\psi(x_1, x_2, \ldots, x_N) \sim (x_j - x_{j+1})^{(2\alpha_0 - \frac{1}{2})} \quad \text{as}$$
$$x_j - x_{j+1} \to 0+ \quad (j = 1, 2, \ldots, N-1). \tag{54}$$

The N-particle problem of this kind is again exactly soluble, when there is no external or interaction potential besides the statistics potential, or when there is a harmonic oscillator potential which is either external or defines a two-particle interaction [241–250].

5 Schrödinger quantization in dimension $d \geq 2$

The geometrical interpretation of the wave function is too trivial to be meaningful in one dimension, but is non-trivial in higher dimensions, and indeed necessary when we want a theory that can describe fermions as well as bosons. Bosons are easy to describe, because the wave functions of a system of N bosons are symmetric functions on $\mathbf{R}^{dN}$ and hence are single-valued functions on the configuration space $\mathbf{R}^{dN}/S_N$. The fermionic wave functions are also single-valued on $\mathbf{R}^{dN}$, but their antisymmetry implies that they are double-valued on $\mathbf{R}^{dN}/S_N$. Many-valued wave functions find a natural place in the geometrical picture introduced by Weyl and Dirac around 1930 [28,29]. The mathematical structures involved are called *fibre bundles* in modern terminology [26,251–256].

To be more precise, in our case the fibre bundle is a *vector bundle*. It has the configuration space $\mathbf{R}^{dN}/S_N$ as its *base space*, and at every point $\boldsymbol{x} \in \mathbf{R}^{dN}/S_N$ there is located a *fibre*, which is a finite dimensional complex Hilbert space $h_{\boldsymbol{x}}$. A wave function $\boldsymbol{\Psi}$ is a *cross-section* of the fibre bundle, that is, the function value $\boldsymbol{\Psi}(\boldsymbol{x})$ at the point $\boldsymbol{x} \in \mathbf{R}^{dN}/S_N$ is a vector in the local Hilbert space $h_{\boldsymbol{x}}$. Let us assume that the complex vector space $h_{\boldsymbol{x}}$ has dimension r, independent of $\boldsymbol{x}$. Then we may choose, for every $\boldsymbol{x} \in \mathbf{R}^{dN}/S_N$, a set of basis vectors $\boldsymbol{\chi}_{\boldsymbol{x}1}, \boldsymbol{\chi}_{\boldsymbol{x}2}, \ldots, \boldsymbol{\chi}_{\boldsymbol{x}r} \in h_{\boldsymbol{x}}$, so that we may write

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \sum_{k=1}^{r} \psi_k(\boldsymbol{x}) \boldsymbol{\chi}_{\boldsymbol{x}k} \,, \tag{55}$$

where each component $\psi_k(\boldsymbol{x})$ is a complex number. Each ψ_k is a complex valued function defined on the configuration space $\mathbf{R}^{dN}/S_N$, it is one component of an r-component wave function

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \end{pmatrix} . \tag{56}$$

Note that we use the term "wave function" here for two different, related objects. One is the cross-section $\boldsymbol{\Psi}$ of the fibre bundle, and the other is the column matrix ψ of r complex valued functions. The set of basis vectors

$$\{\boldsymbol{\chi}_{\boldsymbol{x}k} \mid \boldsymbol{x} \in \mathbf{R}^{dN}/S_N,\, k = 1, 2, \ldots, r\} \tag{57}$$

may be called a *gauge*. It defines a translation between the language of fibre bundles and the language of multi-component complex valued wave functions.

The wave function $\boldsymbol{\Psi}$ is assumed to be single-valued, in the sense that the function value $\boldsymbol{\Psi}(\boldsymbol{x}) \in h_{\boldsymbol{x}}$ at any given point $\boldsymbol{x}$ is unique. It is a geometrical object which exists without any reference to local basis vectors, whereas the r-component complex wave function ψ is undefined until we have "chosen a gauge", *i.e.* defined a set of local basis vectors. Clearly ψ is also single-valued as long as we introduce a unique set of basis vectors at each point $\boldsymbol{x}$. However, we may sometimes want to use simultaneously two or more different sets of local basis vectors, with the result that ψ becomes many-valued. This is the natural way to introduce the double-valued fermion wave functions, as we shall see.

5.1 Scalar wave functions

Let us examine the simplest case, when the particles have no spin or other internal degrees of freedom. Then there is only one basis vector $\boldsymbol{\chi}_{\boldsymbol{x}} \in h_{\boldsymbol{x}}$, and equation (55) simplifies to

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \psi(\boldsymbol{x})\boldsymbol{\chi}_{\boldsymbol{x}} \,. \tag{58}$$

A vector bundle with one-dimensional fibres is called a *line bundle.*

It is natural to impose the normalization condition $|\boldsymbol{\chi}_{\boldsymbol{x}}| = 1$. This still leaves us with the freedom to make a change of basis of the form

$$\boldsymbol{\chi}_{\boldsymbol{x}} \mapsto \widetilde{\boldsymbol{\chi}}_{\boldsymbol{x}} = \mathrm{e}^{i\alpha(\boldsymbol{x})}\, \boldsymbol{\chi}_{\boldsymbol{x}} \,, \tag{59}$$

where $\alpha = \alpha(\boldsymbol{x})$ is an $\boldsymbol{x}$-dependent real phase (in addition it could be time dependent). Such a local change of basis is called a *local gauge transformation.* The complex wave function ψ must transform as follows,

$$\psi(\boldsymbol{x}) \mapsto \widetilde{\psi}(\boldsymbol{x}) = \mathrm{e}^{-i\alpha(\boldsymbol{x})}\, \psi(\boldsymbol{x}) \,, \tag{60}$$

since $\boldsymbol{\Psi}$ is gauge independent,

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \psi(\boldsymbol{x})\boldsymbol{\chi}_{\boldsymbol{x}} = \widetilde{\psi}(\boldsymbol{x})\widetilde{\boldsymbol{\chi}}_{\boldsymbol{x}} \,. \tag{61}$$

In order to define gauge invariant differentiation we need a *connection* on the fibre bundle, *i.e.* a rule for parallel transport between the fibres along continuous curves in the base space. The parallel transport along some curve C from a point $\boldsymbol{x}$ to a point $\boldsymbol{y}$ must define a linear and unitary operator $P_C(\boldsymbol{y}, \boldsymbol{x})$: $h_{\boldsymbol{x}} \to h_{\boldsymbol{y}}$. It is actually sufficient to define the infinitesimal parallel displacement from $\boldsymbol{x}$ to $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$, and we postulate the following rule,

$$P(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}, \boldsymbol{x})\, \boldsymbol{\chi}_{\boldsymbol{x}} = \big(1 + i\, \mathrm{d}x^j\, a_j(\boldsymbol{x})\big)\, \boldsymbol{\chi}_{\boldsymbol{x}+\mathrm{d}\boldsymbol{x}} \,. \tag{62}$$

Here we denote the local coordinates in $\mathbf{R}^{dN}/S_N$ by x^j, $j = 1, 2, \ldots, dN$, and we use the summation convention for repeated indices. $a_1, a_2, \ldots, a_{dN}$

are the components of a covariant vector field, a *gauge potential*, which is real in order to make $P(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},\boldsymbol{x})$ unitary. The infinitesimal rule implies the following rule for a finite curve C,

$$P_C(\boldsymbol{y},\boldsymbol{x})\,\boldsymbol{\chi}_{\boldsymbol{x}} = \exp\left(i\int_C \mathrm{d}x^j\, a_j(\boldsymbol{x})\right)\boldsymbol{\chi}_{\boldsymbol{y}}\,. \tag{63}$$

The gauge invariant differentiation D_j with respect to x^j is defined by the relation

$$\boldsymbol{\Psi}(\boldsymbol{x}+\mathrm{d}\boldsymbol{x}) = P(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},\boldsymbol{x})\left(\boldsymbol{\Psi}(\boldsymbol{x}) + \mathrm{d}x^j\,[D_j\boldsymbol{\Psi}](\boldsymbol{x})\right). \tag{64}$$

By definition, both $\boldsymbol{\Psi}(\boldsymbol{x})$ and $[D_j\boldsymbol{\Psi}](\boldsymbol{x})$ are vectors in the local Hilbert space $h_{\boldsymbol{x}}$, whereas $\boldsymbol{\Psi}(\boldsymbol{x}+\mathrm{d}\boldsymbol{x})$ belongs to the neighbouring space $h_{\boldsymbol{x}+\mathrm{d}\boldsymbol{x}}$. We need the parallel displacement operator in order to compare the local vectors at two different, neighbouring points. Note that this definition is explicitly gauge independent, since it does not involve the local basis vectors.

Once we have chosen a gauge, which defines ψ as the component of $\boldsymbol{\Psi}$, it is very natural to define $D_j\psi$ as the component of $D_j\boldsymbol{\Psi}$,

$$[D_j\boldsymbol{\Psi}](\boldsymbol{x}) = [D_j\psi](\boldsymbol{x})\boldsymbol{\chi}_{\boldsymbol{x}}\,. \tag{65}$$

Comparing equation (64) with another formula,

$$\boldsymbol{\Psi}(\boldsymbol{x}+\mathrm{d}\boldsymbol{x}) = \psi(\boldsymbol{x}+\mathrm{d}\boldsymbol{x})\,\boldsymbol{\chi}_{\boldsymbol{x}+\mathrm{d}\boldsymbol{x}} = \left(\psi(\boldsymbol{x}) + \mathrm{d}x^j\,[\partial_j\psi](\boldsymbol{x})\right)\boldsymbol{\chi}_{\boldsymbol{x}+\mathrm{d}\boldsymbol{x}}\,, \tag{66}$$

in which $\partial_j = \partial/\partial x^j$ is the ordinary partial derivative, we see that

$$D_j\psi = \partial_j\psi - ia_j\psi, \tag{67}$$

which we write simply as

$$D_j = \partial_j - ia_j\,. \tag{68}$$

By definition, the local gauge transformation in equation (59) transforms a_j into $\widetilde{a}_j$, such that

$$P(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},\boldsymbol{x})\,\widetilde{\boldsymbol{\chi}}_{\boldsymbol{x}} = \left(1 + i\,\mathrm{d}x^j\,\widetilde{a}_j(\boldsymbol{x})\right)\widetilde{\boldsymbol{\chi}}_{\boldsymbol{x}+\mathrm{d}\boldsymbol{x}}\,. \tag{69}$$

It follows that

$$\widetilde{a}_j = a_j - \partial_j\alpha, \tag{70}$$

and hence that

$$\widetilde{D}_j = \partial_j - i\,\widetilde{a}_j = D_j + i\partial_j\alpha = \mathrm{e}^{-i\alpha(\boldsymbol{x})}\,D_j\,\mathrm{e}^{i\alpha(\boldsymbol{x})}. \tag{71}$$

This formula for the gauge transformation of the differentiation operator D_j implies that $D_j\psi$ transforms in the same way as ψ under a gauge transformation,

$$D_j\psi \mapsto \widetilde{D}_j\widetilde{\psi} = \mathrm{e}^{-i\alpha(\boldsymbol{x})}\,D_j\psi. \tag{72}$$

5.2 Homotopy

The commutator

$$f_{jk} = i[D_j, D_k] = \partial_j a_k - \partial_k a_j \tag{73}$$

is invariant under local gauge transformations, and measures the non-triviality, or *curvature*, of the connection. It is the *field strength* corresponding to the gauge potential a_j, thus it corresponds to the magnetic flux density in electromagnetism (the electric field is included if we add time components), or to the Riemann curvature tensor in geometry.

Here we want to discuss only the special case when $f_{jk} = 0$ identically. One way to justify this restriction might be to say that we do not want to discuss the kind of interactions represented by a non-vanishing field strength. Another way might be to say that we want to study the ambiguity in the gauge potential a_j for a given field strength f_{jk}. In fact, if $f_{jk}^{(1)} = f_{jk}^{(2)}$, with

$$f_{jk}^{(i)} = \partial_j a_k^{(i)} - \partial_k a_j^{(i)}\,, \tag{74}$$

then the difference $a_j = a_j^{(1)} - a_j^{(2)}$ is a gauge potential having $f_{jk} = 0$. See however reference [257] for an example where the non-vanishing part of f_{jk} is also important.

The curvature, or field strength, f_{jk} vanishes identically if and only if a_j is the gradient of some function α, in other words, if and only if there exists a gauge transformation such that $\widetilde{a}_j = a_j - \partial_j \alpha = 0$ identically. The "if" part of this statement is trivial, that $f_{jk} = 0$ when $a_j = \partial_j \alpha$, The "only if" part is true with the important reservation that the function α is guaranteed to be single-valued only when the space is simply connected. Let us see how the relation between a_j and α depends on the path connectivity.

If $a_j = \partial_j \alpha$, then α is obtained from a_j by a line integral,

$$\alpha(\boldsymbol{y}) = \alpha(\boldsymbol{x}) + \int_C \mathrm{d}z^j\, a_j(\boldsymbol{z})\,, \tag{75}$$

where C is an arbitrary curve from $\boldsymbol{x}$ to $\boldsymbol{y}$. This equation proves that the line integral is independent of C whenever a_j is the gradient of a single-valued function α.

Conversely, when a_j is given, we may always try to solve the equation $\partial_j \alpha = a_j$ for α by choosing one point $\boldsymbol{x}$, fixing $\alpha(\boldsymbol{x})$ arbitrarily, and using equation (75) to define $\alpha(\boldsymbol{y})$ for general $\boldsymbol{y}$. If $f_{jk} = 0$ identically, then the value of the line integral is unchanged by a *continuous* deformation of the curve C, and the function α defined by equation (75) has the desired gradient a_j. The invariance of the line integral under continuous deformation means that equation (75) defines a unique value of $\alpha(\boldsymbol{y})$ for every homotopy class of curves from $\boldsymbol{x}$ to $\boldsymbol{y}$. In particular, if we restrict ourselves to a simply

connected region Ω, then by definition there exists exactly one homotopy class of curves inside Ω from $\boldsymbol{x}$ to $\boldsymbol{y}$, and equation (75) defines a function α which is single-valued on Ω. In a doubly connected region where $f_{jk} = 0$, α defined by equation (75) may become double-valued, and so on.

Let us summarize our discussion so far. We assume that the wave function $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\boldsymbol{x})$ is single-valued, and that there exists a set of local basis vectors $\{\boldsymbol{\chi}_{\boldsymbol{x}}\}$ which is also single-valued (*i.e.*, contains only one basis vector at each point $\boldsymbol{x}$), such that the connection is given by equation (62). We assume further that the gauge potential a_j is such that the curvature, or field strength, f_{jk} vanishes. With a single-valued basis the complex wave function $\psi = \psi(\boldsymbol{x})$ is also single-valued. We have shown how to introduce a local gauge transformation, as defined in equation (59), such that the transformed gauge potential vanishes, *i.e.* $\widetilde{a}_j = a_j - \partial_j \alpha = 0$. Such a gauge transformation is always single-valued in a simply connected region, otherwise it may be many-valued. For example, it may be double-valued in a doubly connected region. A many-valued gauge transformation will transform the single-valued complex wave function ψ into a many-valued complex wave function $\widetilde{\psi} = \mathrm{e}^{-i\alpha}\psi$.

Thus, if the field strength vanishes, $f_{jk} = 0$, we may choose a gauge such that the vector potential vanishes, $a_j = 0$, but in a multiply connected space this may imply that the complex valued wave function becomes many-valued. On the other hand, we may always work with single-valued wave function, but then, if the space is multiply connected, we may have to live with a vector potential which is not zero.

5.3 *Interchange phases*

If we put $\boldsymbol{y} = \boldsymbol{x}$ in equation (63), then the basis vector $\boldsymbol{\chi}_{\boldsymbol{x}}$ is the same on the two sides of the equality sign, so that the parallel transport operator becomes just a gauge independent phase factor,

$$P_C(\boldsymbol{x}, \boldsymbol{x}) = \exp\left(i \oint_C \mathrm{d}x^j \, a_j(\boldsymbol{x}) \right). \tag{76}$$

Obviously, the parallel transport around first one loop C_1 and then another loop C_2 gives a phase factor which is the product of the two individual phase factors,

$$P_{C_2 C_1}(\boldsymbol{x}, \boldsymbol{x}) = P_{C_2}(\boldsymbol{x}, \boldsymbol{x}) P_{C_1}(\boldsymbol{x}, \boldsymbol{x}). \tag{77}$$

Our previous assumption that $f_{jk} = 0$ implies that the line integral in equation (76), and therefore the whole phase factor, is invariant under any continuous deformation of the loop C. We may deform the loop continuously without moving the point $\boldsymbol{x}$ where it starts and ends, and such deformations produce loops in the same homotopy class as the original loop C. Thus, all

loops belonging to the same homotopy class have the same phase factor. However, the phase factor $P_C(\boldsymbol{x}, \boldsymbol{x})$ is unchanged even if we deform the loop C continuously in such a way that the point $\boldsymbol{x}$ moves. In summary, we have derived the following important result.

- *A connection with zero curvature on a complex line bundle is uniquely characterized by a one-dimensional unitary representation of the fundamental group.*

We have seen that the fundamental group for the configuration space $\mathbf{R}^{dN}/S_N$ of N identical particles in dimension d is S_N if $d \geq 3$. The symmetric group S_N with $N > 1$ has exactly two one-dimensional representations: the completely symmetric representation defining bosons, and the completely antisymmetric representation defining fermions.

We have also seen that the fundamental group in two dimensions is the braid group B_N. Since S_N is a homomorphic image of B_N, any representation of S_N defines a representation of B_N, but B_N has more general one-dimensional representations in addition to the symmetric and antisymmetric representations.

In particular, the braid group B_2 for two particles is isomorphic to $\mathbf{Z}$, and its general representation by phase factors is characterized by one real number, a phase angle θ, such that

$$Q \mapsto \mathrm{e}^{-iQ\theta}, \tag{78}$$

where Q is the winding number of the given homotopy class of loops. Obviously, this relation defines θ only up to an arbitrary multiple of 2π, since Q is an integer. Two-dimensional identical particles characterized by a general *statistics angle* θ are called *anyons*. Special cases are bosons, with $\theta = 0$, and fermions, with $\theta = \pi$.

We see that for two anyons there is a phase factor $\mathrm{e}^{-i\theta}$ associated with a loop of winding number $Q = 1$. Let us introduce Cartesian coordinates x, y and polar coordinates r, ϕ such that the relative position of the two particles is

$$\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2 = (x, y) = (r\cos\phi, r\sin\phi). \tag{79}$$

The relative angle ϕ increases by π when we go through a loop of winding number one, and we symbolize this by saying that we go from the point (r, ϕ) to $(r, \phi + \pi)$. The parallel transport around this loop takes the basis vector $\boldsymbol{\chi}_{r,\phi}$ at (r, ϕ) into

$$\boldsymbol{\chi}_{r,\phi+\pi} = \mathrm{e}^{-i\theta}\boldsymbol{\chi}_{r,\phi}\,. \tag{80}$$

This corresponds to the following condition on the complex wave function ψ, in polar coordinates,

$$\psi(r, \phi + \pi) = \mathrm{e}^{i\theta}\psi(r, \phi). \tag{81}$$

It is important to read this formula correctly. The implicit convention when we write $\phi + \pi$, is that the angle ϕ is increased continuously by π.

Note also that the phase factor $e^{i\theta}$ appears when we use parallel basis vectors, so that the gauge potential a_j vanishes. In every case, except for bosons, local basis vectors that are parallel, have to be many-valued on the relative configuration space $\mathbf{R}^2/S_2$. For example, in the fermionic case $e^{-i\theta} = -1$, parallel basis vectors are double-valued.

5.4 *The statistics vector potential*

As we have seen, it is not necessary to use parallel many-valued basis vectors and the corresponding many-valued complex wave functions. One alternative is to use single-valued basis vectors, with a non-vanishing gauge potential. The most general possibility, however, is to use many-valued basis vectors and at the same time a non-vanishing gauge potential. Thus, let ψ denote the many-valued wave function relative to a parallel basis, satisfying equation (81), and let the wave function ψ_ν be defined by

$$\psi_\nu(r,\phi) = e^{-i\nu\phi}\psi(r,\phi), \tag{82}$$

where ν is some arbitrary constant. The new wave function satisfies the following symmetry condition,

$$\psi_\nu(r,\phi+\pi) = e^{i(\theta-\nu\pi)}\psi_\nu(r,\phi). \tag{83}$$

The gauge invariant derivative is trivial in the "parallel" gauge, $D_r = \partial_r$, $D_\phi = \partial_\phi$, but is non-trivial in the "ν" gauge,

$$D_r = \partial_r, \qquad D_\phi = \partial_\phi + i\nu. \tag{84}$$

The corresponding formulae in Cartesian coordinates are,

$$\begin{aligned} D_x &= \cos\phi\, D_r - \frac{\sin\phi}{r} D_\phi = \partial_x - i\nu\frac{y}{r^2}, \\ D_y &= \sin\phi\, D_r + \frac{\cos\phi}{r} D_\phi = \partial_y + i\nu\frac{x}{r^2}. \end{aligned} \tag{85}$$

Note that the gauge potential in the general gauge,

$$a_x = \nu\frac{y}{r^2}, \qquad a_y = -\nu\frac{x}{r^2}, \tag{86}$$

is singular at $r = 0$ when $\nu \neq 0$. The reason is that the gauge transformation in equation (82) is singular, in the sense that the factor $e^{-i\nu\phi}$ is discontinuous at $r = 0$.

We see that the definition in equation (82), with the special choice

$$\nu = \nu_B \equiv \frac{\theta}{\pi}, \tag{87}$$

gives a wave function $\psi_\nu = \psi_B$ which is single-valued, since it has the bosonic symmetry

$$\psi_B(r, \phi + \pi) = \psi_B(r, \phi). \tag{88}$$

We may call this the *bosonic* gauge. The next simplest choice is the *fermionic* gauge, with

$$\nu = \nu_{\rm F} \equiv \nu_B - 1 = \frac{\theta - \pi}{\pi}\,, \tag{89}$$

in which the wave function $\psi_\nu = \psi_{\rm F}$ is double-valued, having the fermionic symmetry

$$\psi_{\rm F}(r, \phi + \pi) = -\psi_{\rm F}(r, \phi). \tag{90}$$

Remember that a_x and a_y given here are only the relative components of the gauge potential, *i.e.* the components in the relative space $\mathbf{R}^2/S_2$. If we transform from the centre of mass and relative coordinates

$$\begin{aligned} X &= \frac{x_1 + x_2}{2}\,, \qquad x = x_1 - x_2, \\ Y &= \frac{y_1 + y_2}{2}\,, \qquad y = y_1 - y_2, \end{aligned} \tag{91}$$

back to the particle coordinates x_1, y_1, x_2, y_2, the gauge potential must transform in the same way as the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{1}{2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\,, \qquad \frac{\partial}{\partial x_2} = \frac{1}{2}\frac{\partial}{\partial X} - \frac{\partial}{\partial x}\,, \\ \frac{\partial}{\partial y_1} &= \frac{1}{2}\frac{\partial}{\partial Y} + \frac{\partial}{\partial y}\,, \qquad \frac{\partial}{\partial y_2} = \frac{1}{2}\frac{\partial}{\partial Y} - \frac{\partial}{\partial y}\,. \end{aligned} \tag{92}$$

By assumption, the centre of mass components of the gauge potential vanish. Hence, the gauge potential expressed in particle coordinates is, with $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$,

$$\begin{aligned} a_{1x} = a_x &= \nu\frac{y_1 - y_2}{r^2}\,, \qquad a_{2x} = -a_x = \nu\frac{y_2 - y_1}{r^2}\,, \\ a_{1y} = a_y &= -\nu\frac{x_1 - x_2}{r^2}\,, \qquad a_{2y} = -a_y = -\nu\frac{x_2 - x_1}{r^2}\,. \end{aligned} \tag{93}$$

A natural way to interpret equation (93) is that a particle at the position $\boldsymbol{x} \in \mathbf{R}^2$ experiences a certain vector potential $\mathbf{A} = \mathbf{A}(\boldsymbol{x})$. It does not experience its own field, only the one generated by the other particle. Thus, particle 1 at $\boldsymbol{x}_1 = (x_1, y_1)$ generates a vector potential at $\boldsymbol{x} = (x, y)$ with components

$$\begin{aligned} A_x(\boldsymbol{x}) &= \nu\frac{y - y_1}{(x - x_1)^2 + (y - y_1)^2}\,, \\ A_y(\boldsymbol{x}) &= -\nu\frac{x - x_1}{(x - x_1)^2 + (y - y_1)^2}\,, \end{aligned} \tag{94}$$

and particle 2 at $\boldsymbol{x}_2$ experiences the vector potential $\boldsymbol{a}_2 = \mathbf{A}(\boldsymbol{x}_2)$.

To the vector potential $\mathbf{A}$ corresponds the field strength, or flux density,

$$B = \partial_x A_y - \partial_y A_x. \tag{95}$$

Green's theorem gives the relation between B and $\mathbf{A}$ in integral form,

$$\int_\Omega \mathrm{d}^2\boldsymbol{x}\, B = \oint_{\partial\Omega} \mathrm{d}\boldsymbol{x}\cdot\mathbf{A} = \begin{cases} -2\pi\nu & \text{if } \boldsymbol{x}_1 \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{96}$$

where Ω is a region with boundary $\partial\Omega$, and where the direction of the line integral is anti-clockwise. This shows that the flux is located exactly at the position of particle 1,

$$B(\boldsymbol{x}) = -2\pi\nu\,\delta(\boldsymbol{x} - \boldsymbol{x}_1). \tag{97}$$

5.5 *The N-particle case*

The generalization to N particles is quite straightforward. As always, we use a notation identical to or similar to the one introduced in equation (10) and equation (11).

A closed loop in the configuration space $\mathbf{R}^{dN}/S_N$, or in the relative space $\mathbf{R}^{d(N-1)}/S_N$, induces a permutation $p \in S_N$ of the N identical particles, and is characterized by a winding number Q. If we work in the many-valued parallel gauge, where the gauge invariant differentiation is trivial, $D_j = \partial_j$, then an interchange path of winding number Q is accompanied by a phase factor $\mathrm{e}^{iQ\theta}$ in the wave function.

If we work instead in the single-valued bosonic gauge, in the double-valued fermionic gauge, or more generally in some many-valued "ν" gauge with a non-vanishing gauge potential, then the interchange phase factor in the wave function is $\mathrm{e}^{iQ(\theta-\nu\pi)}$. In addition there is a gauge potential which has the following components, as we can see by generalizing equation (93),

$$\begin{aligned} a_{jx} &= \nu \sum_{k\neq j} \frac{y_j - y_k}{(x_j - x_k)^2 + (y_j - y_k)^2}, \\ a_{jy} &= -\nu \sum_{k\neq j} \frac{x_j - x_k}{(x_j - x_k)^2 + (y_j - y_k)^2}. \end{aligned} \tag{98}$$

It is worth noting that the *components* as defined in equation (98) are many-valued, but they define a single-valued *vector field* on $\mathbf{R}^{dN}/S_N$. Like in the two-particle case, the special choice $\nu = \nu_B = \theta/\pi$ gives symmetric (*i.e.* bosonic) wave functions, whereas $\nu = \nu_\mathrm{F} = \nu_B - 1$ gives antisymmetric (fermionic) wave functions.

The N-particle Hamiltonian is

$$H = \frac{1}{2m}(\boldsymbol{p} - \hbar\boldsymbol{a})^2 + V = \frac{1}{2m}\sum_{j=1}^{N}(\boldsymbol{p}_j - \hbar\boldsymbol{a}_j)^2 + V. \tag{99}$$

In our notation particle j has the position $\boldsymbol{x}_j = (x_j, y_j)$ and the canonical momentum

$$\boldsymbol{p}_j = \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{x}_j} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right). \tag{100}$$

The statistics vector potential $\boldsymbol{a}_j = (a_{jx}, a_{jy})$ is given by equation (98). The potential $V = V(\boldsymbol{x}) = V(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$ may be the sum $V = V_{\mathrm{E}} + V_{\mathrm{I}}$ of an external, one-particle part

$$V_{\mathrm{E}} = \sum_{j=1}^{N} V_1(\boldsymbol{x}_j), \tag{101}$$

and an internal or interaction, two-particle part

$$V_{\mathrm{I}} = \sum_{j<k} V_2(\boldsymbol{x}_j, \boldsymbol{x}_k), \tag{102}$$

where V_2 is a symmetric function, $V_2(\boldsymbol{y}, \boldsymbol{x}) = V_2(\boldsymbol{x}, \boldsymbol{y})$. There is usually no need to include more complicated interactions involving three or more particles, although it could of course easily be done. Note however that the square of the statistics vector potential, $\boldsymbol{a}^2 = \sum_j \boldsymbol{a}_j^2$, contains three-body terms.

Note that the N-anyon problem for $N \geq 2$ actually contains another continuously variable parameter, in addition to the phase angle θ [258–261]. The reason is that the eigenfunctions of the above Hamiltonian depend on the boundary conditions in the limits $|\boldsymbol{x}_j - \boldsymbol{x}_k| \to 0$ for $1 \leq j < k \leq N$. Usually the condition of quadratic integrability forbids singular solutions and thereby fixes uniquely the asymptotic form of the wave function for small distances, but there is some exception to this rule.

5.6 *Chern–Simons theory*

An equivalent point of view in the N-particle case is that particle j, at $\boldsymbol{x}_j \in \mathbf{R}^2$, experiences a vector potential $\boldsymbol{a}_j = \mathbf{A}(\boldsymbol{x}_j)$ which is generated by all the other particles,

$$\begin{aligned} A_x(\boldsymbol{x}) &= \nu\sum_{k\neq j}\frac{y - y_k}{(x - x_k)^2 + (y - y_k)^2}, \\ A_y(\boldsymbol{x}) &= -\nu\sum_{k\neq j}\frac{x - x_k}{(x - x_k)^2 + (y - y_k)^2}. \end{aligned} \tag{103}$$

It is easily seen by comparison with the equations (94, 95) and (97) that the statistics gauge potential $\mathbf{A}$ defined here satisfies the field equation

$$\partial_x A_y - \partial_y A_x = -2\pi\nu\rho, \tag{104}$$

where $\rho = \rho(\boldsymbol{x})$ is the particle density,

$$\rho(\boldsymbol{x}) = \sum_{k\neq j} \delta(\boldsymbol{x} - \boldsymbol{x}_k). \tag{105}$$

In a field theory we want to modify the source term ρ so that the particle j itself is included, even though this leads to the usual problems with self-interactions.

The field equation can be derived from a Lagrangian if we allow the field components A_x and A_y to become explicitly time dependent, and add a third field component A^0. It is convenient to introduce the relativistic notation

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2) = (ct, x, y), \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \\ A^\mu &= (A^0, \mathbf{A}) = (A^0, A_x, A_y), \quad A_\mu = (A^0, -\mathbf{A}), \quad j^\mu = (c\rho, \boldsymbol{j}). \end{aligned} \tag{106}$$

c is a constant velocity (the speed of light). Then we may add two field equations to obtain the following relativistic form, where ϵ is antisymmetric and $\epsilon^{012} = 1$,

$$\epsilon^{\mu\kappa\lambda}\partial_\kappa A_\lambda = \frac{2\pi\nu}{c}\, j^\mu. \tag{107}$$

A necessary consistency condition is that the current j^μ is conserved, $\partial_\mu j^\mu = 0$. These equations are gauge invariant, so that if A_μ is a solution, then the gauge transformed field

$$\widetilde{A}_\mu = A_\mu - \partial_\mu \alpha \tag{108}$$

is also a solution. There always exists a "radiation gauge" with $\widetilde{A}_0 = 0$, therefore the introduction of A_0 does not change the physics.

The field equations for A_μ follow from the Lagrangian density

$$\mathcal{L} = \frac{\hbar c}{4\pi\nu}\,\epsilon^{\mu\kappa\lambda} A_\mu \partial_\kappa A_\lambda - \hbar j^\mu A_\mu\,. \tag{109}$$

The first term is the Chern–Simons term, which exists only when space-time is three-dimensional. $\mathcal{L}$ is gauge invariant when the current j^μ is conserved, in the sense that it changes by a divergence under the gauge transformation of equation (108),

$$\widetilde{\mathcal{L}} = \mathcal{L} - \partial_\mu \left(\frac{\hbar c}{4\pi\nu} \alpha \epsilon^{\mu\kappa\lambda} \partial_\kappa A_\lambda - \hbar\alpha j^\mu \right). \tag{110}$$

It is well known that the 2 + 1-dimensional Maxwell theory with an additional Chern–Simons term describes massive photons. If an ordinary Maxwell term is added to the Lagrangian density $\mathcal{L}$ above, then the modified theory describes anyons of finite size [262,263].

When the matter fields that define the current j^μ are quantized, either as bosons or fermions, relativistically or non-relativistically, the result is a field theory of anyons [67–69]. We will not discuss these theories any further here.

6 The Feynman path integral for anyons

Laidlaw and DeWitt worked out the Feynman path integral treatment of systems of identical particles [1,49,66], inspired by earlier work of Schulman on the path integral for configuration spaces that are not simply connected [264,265]. We derive it here from the Schrödinger formalism, to show that the two are equivalent. Closely related is the work by Wiegel and by and Inomata and Singh on the entanglement of polymers, and by Gerry and Singh on the path integral treatment of the Aharonov–Bohm effect [266–269]. See also [211,237] for the path integral treatment of exclusion statistics and parastatistics.

What we will derive is the path integral formula for the partition function of the N-particle system,

$$Z_N(\beta) = \mathrm{Tr}\, \mathrm{e}^{-\beta \widehat{H}_N} = \sum_{k=0}^{\infty} \mathrm{e}^{-\beta E_{Nk}} \,. \tag{111}$$

We have had little need so far to distinguish explicitly between operators and numbers, but in the present section we will use a "hat" to denote operators, in order to prevent unnecessary confusion.

We assume that the particles are confined by an external potential in such a way that the Hamiltonian operator $\widehat{H}_N$ has a discrete eigenvalue spectrum, and the trace is well-defined. At least in principle, Z_N as a function of β determines the energy eigenvalues $E_{N0} \leq E_{N1} \leq \ldots \leq E_{Nk} \leq \ldots$. In statistical mechanics $\beta = 1/k_B T$, where T is the temperature and k_B is Boltzmann's constant. More formally, β may be thought of as imaginary time, since the time evolution operator for a time interval $t = -i\hbar\beta$ is, with $\widehat{H} = \widehat{H}_N$,

$$\mathrm{e}^{-\frac{i}{\hbar} t \widehat{H}} = \mathrm{e}^{-\beta \widehat{H}}. \tag{112}$$

The simplest derivation starts from the "bosonic" description, in which the N-particle wave functions are symmetric complex valued functions on $\mathbf{R}^{dN}$. The Hamiltonian operator is given by equation (99), and any deviation from Bose–Einstein statistics is described by a vector potential $\boldsymbol{a}$ on $\mathbf{R}^{dN}$, as

given in equation (98), with $\nu = \nu_B = \theta/\pi$. The "fermionic" description, where the wave functions are antisymmetric, leads to the same end result, but the derivation is complicated by an extra minus sign associated with every odd permutation of the N particles. Therefore we prefer the bosonic description.

6.1 *Eigenstates for position and momentum*

For distinguishable particles the general position eigenstate is $|\boldsymbol{x}\rangle$, where $\boldsymbol{x} \in \mathbf{R}^{dN}$. We have the *orthonormality relation*

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{113}$$

and the *completeness relation*, also called the *resolution of the identity operator*,

$$\widehat{I} = \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, |\boldsymbol{x}\rangle\langle \boldsymbol{x}| \,. \tag{114}$$

The bosonic position eigenstates are symmetric under all permutations of particle labels, and with the proper normalization they are

$$|\boldsymbol{x}\rangle_B = \frac{1}{\sqrt{N!}} \sum_{p\in S_N} |p(\boldsymbol{x})\rangle \,. \tag{115}$$

This definition implies that

$${}_B\langle \boldsymbol{x}|\boldsymbol{y}\rangle_B = \frac{1}{N!} \sum_{p,q\in S_N} \delta(p(\boldsymbol{x}) - q(\boldsymbol{y})) = \sum_{r\in S_N} \delta(r(\boldsymbol{x})-\boldsymbol{y}), \tag{116}$$

where $r = q^{-1}p$. Note that the permutations p and q here act on $\mathbf{R}^{dN}$ as linear operators of determinant ± 1, and therefore we may change variables in the Dirac δ-function without introducing an extra Jacobi determinant. It follows that if we restrict both $\boldsymbol{x}$ and $\boldsymbol{y}$ to lie in the true configuration space $\mathbf{R}^{dN}/S_N$, then we have the standard orthonormality relation also for the bosonic position eigenstates,

$${}_B\langle \boldsymbol{x}|\boldsymbol{y}\rangle_B = \delta(\boldsymbol{x}-\boldsymbol{y}). \tag{117}$$

The identity operator in the space of bosonic states is

$$\widehat{I}_B = \int_{\mathbf{R}^{dN}/S_N} \mathrm{d}^{dN}\boldsymbol{x}\, |\boldsymbol{x}\rangle_B\, {}_B\langle \boldsymbol{x}| = \frac{1}{N!} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, |\boldsymbol{x}\rangle_B\, {}_B\langle \boldsymbol{x}| \,. \tag{118}$$

When regarded as an operator on the full Hilbert space that includes states of all symmetry classes, $\widehat{I}_B$ is the projection operator onto the subspace of symmetric states. The above definition of $|\boldsymbol{x}\rangle_B$ gives that

$$\widehat{I}_B = \frac{1}{(N!)^2} \sum_{p,q\in S_N} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, |p(\boldsymbol{x})\rangle\langle q(\boldsymbol{x})| \,. \tag{119}$$

If we define $\boldsymbol{y} = p(\boldsymbol{x})$, $\boldsymbol{z} = q(\boldsymbol{x})$, $r = qp^{-1}$ and $s = pq^{-1}$, then we get two more forms of the above resolution of the bosonic identity operator,

$$\widehat{I}_B = \frac{1}{N!} \sum_{r \in S_N} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{y}\, |\boldsymbol{y}\rangle\langle r(\boldsymbol{y})| = \frac{1}{N!} \sum_{s \in S_N} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{z}\, |s(\boldsymbol{z})\rangle\langle \boldsymbol{z}| \ . \quad (120)$$

The momentum eigenstates for distinguishable particles may be defined as

$$|\boldsymbol{p}\rangle = \left(\frac{1}{2\pi\hbar}\right)^{\frac{dN}{2}} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\ \mathrm{e}^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}} |\boldsymbol{x}\rangle \ . \quad (121)$$

They satisfy similar orthonormality and completeness relations as the position eigenstates,

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = \delta(\boldsymbol{p}-\boldsymbol{q}), \qquad \widehat{I} = \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{p}\, |\boldsymbol{p}\rangle\langle \boldsymbol{p}| \ . \quad (122)$$

6.2 *The path integral*

The trace in the definition of the partition function, equation (111), is the trace within the subspace of bosonic states, which we may obtain by inserting the projection operator $\widehat{I}_B$. Thus [66],

$$Z_N(\beta) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\widehat{H}}\widehat{I}_B\right) = \frac{1}{N!} \sum_{p \in S_N} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, \langle p(\boldsymbol{x})|\mathrm{e}^{-\beta\widehat{H}}|\boldsymbol{x}\rangle \ , \quad (123)$$

where we have used equation (120) and the general formula $\mathrm{Tr}(\widehat{A}|\psi\rangle\langle\chi|) = \langle\chi|\widehat{A}|\psi\rangle$. As remarked earlier, the formula would be the same in the fermionic picture, except for a minus sign for every odd permutation, compensated for by the use of $\nu = \nu_{\mathrm{F}} = \nu_B - 1$ instead of $\nu = \nu_B$ in the statistics vector potential.

We now expand as follows,

$$\mathrm{e}^{-\beta\widehat{H}}\widehat{I}_B = \mathrm{e}^{-\frac{\beta}{n}\widehat{H}}\widehat{I}\,\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}\widehat{I}\cdots\widehat{I}\,\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}\widehat{I}_B \ . \quad (124)$$

It is simpler to use the identity operator $\widehat{I}$ of the full Hilbert space here instead of the identity operator $\widehat{I}_B$ of the bosonic subspace, which would seem more logical. Actually the above formula is equally valid if we replace every $\widehat{I}$ by $\widehat{I}_B$, because the Hamiltonian $\widehat{H}$ and hence the exponential operators in the formula are symmetric under interchange of particle labels, so that they commute with $\widehat{I}_B$. It is essential to include $\widehat{I}_B$ once, but once is enough. Using the resolution of the identity, equation (114), we get the following formula,

$$Z_N(\beta) = \frac{1}{N!} \sum_{p \in S_N} \int_{\mathbf{R}^{dNn}} \mathrm{d}^{dN}\boldsymbol{x}\,\mathrm{d}^{dN}\boldsymbol{u}\cdots\mathrm{d}^{dN}\boldsymbol{z}\,\mathrm{d}^{dN}\boldsymbol{y}\ \langle p(\boldsymbol{x})|\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}|\boldsymbol{u}\rangle$$
$$\langle \boldsymbol{u}|\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}\cdots|\boldsymbol{z}\rangle\langle\boldsymbol{z}|\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}|\boldsymbol{y}\rangle\langle\boldsymbol{y}|\mathrm{e}^{-\frac{\beta}{n}\widehat{H}}|\boldsymbol{x}\rangle \ . \quad (125)$$

The only manifestation of the fact that the particles are identical, is that we integrate over all paths $\boldsymbol{x} \to \boldsymbol{y} \to \boldsymbol{z} \to \ \ldots \to \boldsymbol{u} \to p(\boldsymbol{x})$ starting at $\boldsymbol{x}$ and ending at $p(\boldsymbol{x})$, and we average over all permutations p. These are precisely the closed paths in the configuration space $\mathbf{R}^{dN}/S_N$.

The desired path integral formula for the partition function is obtained in the limit $n \to \infty$, when a suitable approximation is used for the propagator

$$G(\boldsymbol{y}, \boldsymbol{x}; \tau) = \langle \boldsymbol{y} | \mathrm{e}^{-\frac{\tau}{\hbar}\widehat{H}} | \boldsymbol{x} \rangle \cdot \tag{126}$$

Here the real variable $\tau = \hbar\beta/n$ corresponds to the imaginary time interval $t = -i\tau$. The approximation must be sufficiently accurate in the limit $\tau \to 0$.

By complex conjugation of the definition of G follows the Hermitean symmetry

$$G(\boldsymbol{y}, \boldsymbol{x}; \tau)^* = G(\boldsymbol{x}, \boldsymbol{y}; \tau). \tag{127}$$

An approximation which respects this symmetry is the following,

$$\begin{aligned} G(\boldsymbol{y}, \boldsymbol{x}; \tau) \approx \langle \boldsymbol{y} | \exp\Big(-\frac{\tau}{2\hbar}\,\widehat{V} - \frac{\tau\hbar}{4m}\,\widehat{\boldsymbol{a}}^2\Big) \exp\Big(\frac{\tau}{2m}\,\widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{p}}\Big) \exp\Big(-\frac{\tau}{2\hbar m}\,\widehat{\boldsymbol{p}}^2\Big) \\ \times \exp\Big(\frac{\tau}{2m}\,\widehat{\boldsymbol{p}} \cdot \widehat{\boldsymbol{a}}\Big) \exp\Big(-\frac{\tau}{2\hbar}\,\widehat{V} - \frac{\tau\hbar}{4m}\,\widehat{\boldsymbol{a}}^2\Big) | \boldsymbol{x} \rangle \qquad (128) \\ = \exp\Big(-\frac{\tau}{2\hbar}\,(V(\boldsymbol{y}) + V(\boldsymbol{x})) \\ -\frac{\tau\hbar}{4m}\,\big(|\boldsymbol{a}(\boldsymbol{y})|^2 + |\boldsymbol{a}(\boldsymbol{x})|^2\big)\Big)\, G_1(\boldsymbol{y}, \boldsymbol{x}; \tau), \end{aligned}$$

where

$$G_1(\boldsymbol{y}, \boldsymbol{x}; \tau) = \langle \boldsymbol{y} | \exp\Big(\frac{\tau}{2m}\,\widehat{\boldsymbol{a}} \cdot \widehat{\boldsymbol{p}}\Big) \exp\Big(-\frac{\tau}{2\hbar m}\,\widehat{\boldsymbol{p}}^2\Big)\, \widehat{I} \exp\Big(\frac{\tau}{2m}\,\widehat{\boldsymbol{p}} \cdot \widehat{\boldsymbol{a}}\Big) | \boldsymbol{x} \rangle \cdot \tag{129}$$

Here we have inserted the identity operator between two of the exponentials, and when we use equation (122), we get that

$$G_1(\boldsymbol{y}, \boldsymbol{x}; \tau) = \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{p} \exp\Big(-\frac{\tau}{2\hbar m}\,\boldsymbol{p}^2\Big)\, G_2(\boldsymbol{p}, \boldsymbol{y}; \tau)^*\, G_2(\boldsymbol{p}, \boldsymbol{x}; \tau), \tag{130}$$

with

$$G_2(\boldsymbol{p}, \boldsymbol{x}; \tau) = \langle \boldsymbol{p} | \exp\Big(\frac{\tau}{2m}\,\widehat{\boldsymbol{p}} \cdot \widehat{\boldsymbol{a}}\Big) | \boldsymbol{x} \rangle \cdot \tag{131}$$

Without further justification we now introduce the following approximation for G_2, which is formally just a first order approximation for the exponential

function,

$$G_2(\boldsymbol{p},\boldsymbol{x};\tau) \approx \langle \boldsymbol{p}|\Big(\widehat{I}+\frac{\tau}{2m}\,\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{a}}\Big)|\boldsymbol{x}\rangle = \Big(1+\frac{\tau}{2m}\,\boldsymbol{p}\cdot\boldsymbol{a}(\boldsymbol{x})\Big)\,\langle \boldsymbol{p}|\boldsymbol{x}\rangle$$
$$\approx \exp\!\Big(\frac{\tau}{2m}\,\boldsymbol{p}\cdot\boldsymbol{a}(\boldsymbol{x})-\frac{i}{\hbar}\,\boldsymbol{p}\cdot\boldsymbol{x}\Big)\,. \tag{132}$$

It leads to the following approximation for G_1,

$$G_1(\boldsymbol{y},\boldsymbol{x};\tau) \approx \left(\frac{2\pi\hbar m}{\tau}\right)^{\frac{dN}{2}} \exp\!\Big(\frac{\tau\hbar}{8m}\,(\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{a}(\boldsymbol{x}))^2-\frac{m}{2\tau\hbar}\,(\boldsymbol{y}-\boldsymbol{x})^2$$
$$+\frac{i}{2}\,(\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{a}(\boldsymbol{x}))\cdot(\boldsymbol{y}-\boldsymbol{x})\Big)\,, \tag{133}$$

where the normalization factor comes from the Gaussian integral

$$\int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{p}\,\exp\!\left(-\frac{\tau}{2\hbar m}\,\left(\boldsymbol{p}-\frac{\hbar}{2}\,(\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{a}(\boldsymbol{x}))-\frac{im}{\tau}\,(\boldsymbol{y}-\boldsymbol{x})\right)^2\right)$$
$$=\left(\frac{2\pi\hbar m}{\tau}\right)^{\frac{dN}{2}}\,. \tag{134}$$

And finally the desired approximation for G,

$$G(\boldsymbol{y},\boldsymbol{x};\tau) \approx \left(\frac{2\pi\hbar m}{\tau}\right)^{\frac{dN}{2}} \exp\!\Big(-\frac{m}{2\tau\hbar}\,(\boldsymbol{y}-\boldsymbol{x})^2-\frac{\tau}{2\hbar}\,(V(\boldsymbol{y})+V(\boldsymbol{x}))$$
$$+\frac{i}{2}\,(\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{a}(\boldsymbol{x}))\cdot(\boldsymbol{y}-\boldsymbol{x})\Big)\,. \tag{135}$$

We have neglected the following terms in the exponent,

$$-\frac{\tau\hbar}{4m}\left(|\boldsymbol{a}(\boldsymbol{y})|^2+|\boldsymbol{a}(\boldsymbol{x})|^2\right)+\frac{\tau\hbar}{8m}\,(\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{a}(\boldsymbol{x}))^2$$
$$=-\frac{\tau\hbar}{8m}\,(\boldsymbol{a}(\boldsymbol{y})-\boldsymbol{a}(\boldsymbol{x}))^2\,, \tag{136}$$

because the Gaussian weight factor $\exp\!\big(-m(\boldsymbol{y}-\boldsymbol{x})^2/2\tau\hbar\big)$ makes the distance $|\boldsymbol{y}-\boldsymbol{x}|$ small, of order $\sqrt{\tau}$.

In the limit $n\to\infty$ the discrete path $\boldsymbol{x}\to\boldsymbol{y}\to\boldsymbol{z}\to\ \ldots\to\boldsymbol{u}\to p(\boldsymbol{x})$ approaches a continuous path $\boldsymbol{x}(\tau)$, with $0\le\tau\le\hbar\beta$, while the product of propagators diverges as follows,

$$G\Big(p(\boldsymbol{x}),\boldsymbol{u};\frac{\hbar\beta}{n}\Big)\cdots G\Big(\boldsymbol{z},\boldsymbol{y};\frac{\hbar\beta}{n}\Big)\,G\Big(\boldsymbol{y},\boldsymbol{x};\frac{\hbar\beta}{n}\Big)$$
$$\to\left(\frac{2\pi mn}{\beta}\right)^{\frac{dNn}{2}}\mathrm{e}^{-\frac{S}{\hbar}-i\theta Q}\,. \tag{137}$$

S is the action in imaginary time, which we define here as

$$S = \int_0^{\hbar\beta} \mathrm{d}\tau \left(\frac{m}{2} \left| \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} \right|^2 + V(\boldsymbol{x}) \right), \tag{138}$$

and Q is the winding number of the N-particle path, defined by the line integral of the statistics vector potential along the path $\boldsymbol{x}(\tau)$,

$$\int \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{a}(\boldsymbol{x}) = \sum_{j=1}^{N} \int \mathrm{d}\boldsymbol{x}_j \cdot \boldsymbol{a}_j(\boldsymbol{x}) = -\pi\nu Q = -\theta Q. \tag{139}$$

We define the path integral measure as

$$\mathcal{D}(\boldsymbol{x}(\tau)) = \lim_{n\to\infty} \left(\frac{2\pi mn}{\beta} \right)^{\frac{dNn}{2}} \mathrm{d}^{dN}\boldsymbol{x}\, \mathrm{d}^{dN}\boldsymbol{u} \cdots \mathrm{d}^{dN}\boldsymbol{z}\, \mathrm{d}^{dN}\boldsymbol{y}\,, \tag{140}$$

including the divergent normalization factor from the product of propagators. With these definitions,

$$Z_N(\beta) = \frac{1}{N!} \sum_{p\in S_N} \int_{\mathcal{C}(p)} \mathcal{D}(\boldsymbol{x}(\tau))\ \mathrm{e}^{-\frac{S}{\hbar} - i\theta Q}\,. \tag{141}$$

The domain of integration $\mathcal{C}(p)$ for a given permutation $p \in S_N$ consists of all continuous paths $\boldsymbol{x}(\tau)$ with $0 \le \tau \le \hbar\beta$ and with $\boldsymbol{x}(\hbar\beta) = p(\boldsymbol{x}(0))$.

Let $\mathcal{C}(p,Q)$ consist of those paths in $\mathcal{C}(p)$ that have the winding number Q, and define

$$P_p(Q,\beta) = \frac{\int_{\mathcal{C}(p,Q)} \mathcal{D}(\boldsymbol{x}(\tau))\ \mathrm{e}^{-\frac{S}{\hbar}}}{\int_{\mathcal{C}(p)} \mathcal{D}(\boldsymbol{x}(\tau))\ \mathrm{e}^{-\frac{S}{\hbar}}}\,. \tag{142}$$

Thus, $P_p(Q,\beta)$ is the probability of the winding number Q, given the permutation p, and given that the particles are bosons. Define also the Fourier transform of the probability distribution of winding numbers, which is called the probability generating function,

$$F_p(\theta,\beta) = \sum_{Q=-\infty}^{\infty} P_p(Q,\beta)\, \mathrm{e}^{-i\theta Q}\,. \tag{143}$$

Below we will usually write simply $P_p(Q)$ and $F_p(\theta)$ instead of $P_p(Q,\beta)$ and $F_p(\theta,\beta)$, but one should remember that these quantities are temperature dependent. These definitions enable us to isolate the dependence of the partition function Z_N on the statistics angle θ,

$$Z_N(\beta) = \frac{1}{N!} \sum_{p\in S_N} F_p(\theta) \int_{\mathcal{C}(p)} \mathcal{D}(\boldsymbol{x}(\tau))\ \mathrm{e}^{-\frac{S}{\hbar}}\,. \tag{144}$$

The fact that the probability $P_p(Q)$ is real implies that $F_p(\theta)^* = F_p(-\theta)$. Furthermore, in the definition of $P_p(Q)$, both the action S and the path integral measure $\mathcal{D}(\boldsymbol{x}(\tau))$ are invariant under the time reversal transformation

$$\boldsymbol{x}(\tau) \mapsto \widetilde{\boldsymbol{x}}(\tau) = \boldsymbol{x}(\hbar\beta - \tau). \tag{145}$$

If the path $\boldsymbol{x}(\tau)$ has the winding number Q, then the time reversed path $\widetilde{\boldsymbol{x}}(\tau)$ has the winding number $-Q$. Hence time reversal invariance implies that $P_p(-Q) = P_p(Q)$, which in turn implies that the probability generating function is real,

$$F_p(\theta)^* = F_p(-\theta) = F_p(\theta). \tag{146}$$

The fact that $F_p(\theta)$ is real, is consistent with our path integral formula for the partition function Z_N, equation (144), and with the fact that Z_N is real by definition.

6.3 *Conjugation classes in S_N*

By definition, a *cycle* of length L is a cyclic permutation $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_L \mapsto i_1$ of the L integers $i_1, i_2, \ldots, i_L$, and it is denoted by $(i_1 i_2 \ldots i_L)$. It follows directly from the definition that, for example,

$$(i_1 i_2 \ldots i_L) = (i_2 \ldots i_L i_1), \qquad (i_1 i_2 \ldots i_L)^{-1} = (i_L \ldots i_2 i_1). \tag{147}$$

A *transposition* is a cycle $(i_1 i_2)$ of length 2, and the generators of S_N introduced in Subsection 2.4 are the transpositions $T_j = (j, j+1)$. A cycle of length L may be written as a product of transpositions, *e.g.* in the following way,

$$(i_1 i_2 \ldots i_L) = (i_1 i_2)(i_2 i_3) \cdots (i_{L-1} i_L). \tag{148}$$

Two cycles commute if they are disjoint. Every permutation $p \in S_N$ can be factored into a product of disjoint cycles, and the factorization is unique apart from the order of the factors. Let ν_L be the number of cycles of length L in the factorization of p. Then the sequence of non-negative integers $\nu_1, \nu_2, \ldots, \nu_L, \ldots$ is called a *partition* of N, because $\sum_L L\nu_L = N$. Let $\nu = \sum_L \nu_L$ be the number of cycles in the factorization of p, then the *sign* of p is

$$\operatorname{sgn}(p) = \prod_{j<k} \frac{p(j) - p(k)}{j - k} = \prod_L (-1)^{(L-1)\nu_L} = (-1)^{N-\nu}. \tag{149}$$

Two permutations $p, q \in S_N$ are *conjugate* if $q = rpr^{-1}$ for some $r \in S_N$. The conjugation class of p consists of all permutations that are conjugate

to p. The mapping $p \mapsto rpr^{-1}$ preserves the group structure of S_N, *i.e.*, it is an *automorphism* of S_N.

The conjugate of a cycle is

$$r\,(i_1 i_2\ \ldots i_L)\,r^{-1} = (r(i_1), r(i_2),\ \ldots\ , r(i_L)). \tag{150}$$

It follows that two cycles are conjugate if and only if they have the same length. And in general, two elements of S_N are conjugate if and only if they have the same cycle structure, in the sense that they define the same partition of N when factorized into disjoint cycles. Thus the conjugation classes in S_N are in one to one correspondence with the partitions of N. Let $\mathcal{P}$ be the conjugation class in S_N corresponding to the partition $\sum_L L\nu_L = N$. Then the number of elements in $\mathcal{P}$ is

$$N_{\mathcal{P}} = \frac{N!}{\prod_L (\nu_L!\, L^{\nu_L})}\,. \tag{151}$$

Let us now go back to equation (123). The integral there depends on the permutation p, but it is the same for any permutation $q = rpr^{-1}$ conjugate to p. That is,

$$\int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{y}\, \langle q(\boldsymbol{y})|\mathrm{e}^{-\beta\widehat{H}}|\boldsymbol{y}\rangle = \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, \langle p(\boldsymbol{x})|\mathrm{e}^{-\beta\widehat{H}}|\boldsymbol{x}\rangle\,. \tag{152}$$

To prove this equality, note that the operator $\mathrm{e}^{-\beta\widehat{H}}$ is permutation invariant, which implies that

$$\begin{aligned}\langle q(\boldsymbol{y})|\mathrm{e}^{-\beta\widehat{H}}|\boldsymbol{y}\rangle &= \langle r^{-1}(q(\boldsymbol{y}))|\mathrm{e}^{-\beta\widehat{H}}|r^{-1}(\boldsymbol{y})\rangle \\ &= \langle p(r^{-1}(\boldsymbol{y}))|\mathrm{e}^{-\beta\widehat{H}}|r^{-1}(\boldsymbol{y})\rangle\,.\end{aligned} \tag{153}$$

The substitution $\boldsymbol{x} = r^{-1}(\boldsymbol{y})$ completes the proof.

Therefore we need not sum over all permutations $p \in S_N$ in equation (123), it is enough to pick one arbitrary permutation p from each conjugation class $\mathcal{P}$ in S_N. The sum reduces to a sum over all conjugation classes, or equivalently, a sum over all partitions of N,

$$Z_N(\beta) = \sum_{\mathcal{P}} \frac{1}{\prod_L (\nu_L!\, L^{\nu_L})} \int_{\mathbf{R}^{dN}} \mathrm{d}^{dN}\boldsymbol{x}\, \langle p(\boldsymbol{x})|\mathrm{e}^{-\beta\widehat{H}}|\boldsymbol{x}\rangle\,. \tag{154}$$

The path integral formula, equation (144), is modified accordingly,

$$Z_N(\beta) = \sum_{\mathcal{P}} \frac{1}{\prod_L (\nu_L!\, L^{\nu_L})}\, F_{\mathcal{P}}(\theta) \int_{\mathcal{C}(\mathcal{P})} \mathcal{D}(\boldsymbol{x}(\tau))\ \mathrm{e}^{-\frac{S}{\hbar}}\,. \tag{155}$$

The class $\mathcal{C}(\mathcal{P})$ of paths consists of all continuous paths inducing one arbitrary, but fixed, permutation $p \in \mathcal{P}$.

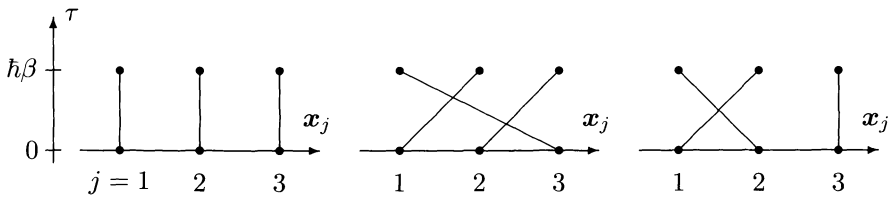


Fig. 5. Schematic representations of the three classes of closed three-anyon paths $\boldsymbol{x}(\tau)$. The classes $1+1+1$ (left) and 3 (middle) are even, whereas $2+1$ (right) is odd.

The probability generating function $F_{\mathcal{P}}(\theta)$ depends implicitly both on β and on the potential V. The values at the boson point $\theta = 0$ and at the fermion point $\theta = \pi$ are known,

$$F_{\mathcal{P}}(0) = \sum_{Q} P_{\mathcal{P}}(Q) = 1,$$
$$F_{\mathcal{P}}(\pi) = \sum_{Q} P_{\mathcal{P}}(Q)\,(-1)^{Q} = \operatorname{sgn}(\mathcal{P}) = (-1)^{N-\nu}. \tag{156}$$

The fermion value follows because the winding number Q is always even for all even permutations and always odd for all odd permutations. More generally, it follows that

$$F_{\mathcal{P}}(\theta \pm \pi) = \operatorname{sgn}(\mathcal{P})\, F_{\mathcal{P}}(\theta). \tag{157}$$

Take $N = 3$ as an example. The $3! = 6$ permutations in S_3 fall into three conjugation classes, illustrated in Figure 5. The two classes, or partitions, $1+1+1$ and 3 are even, and the class $2+1$ is odd.

6.4 The non-interacting case

The N-boson path integral occurring in equation (155) can be simplified if there is no interaction potential so that the particles are only influenced by an external potential,

$$V(\boldsymbol{x}) = \sum_{j=1}^{N} V_1(\boldsymbol{x}_j). \tag{158}$$

Then the factorization of the permutation p into disjoint cycles implies a similar factorization of the path integral, so that

$$Z_N(\beta) = \sum_{\mathcal{P}} F_{\mathcal{P}}(\theta) \prod_{L} \frac{1}{\nu_L!} \left(\frac{1}{L} \int_{\mathcal{C}_L} \mathcal{D}(\boldsymbol{x}(\tau))\; \mathrm{e}^{-\frac{S}{\hbar}} \right)^{\nu_L}, \tag{159}$$

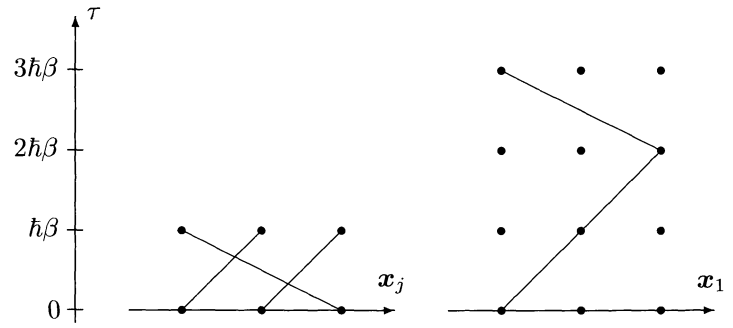


Fig. 6. A three-particle path $\boldsymbol{x}(\tau)$ inducing a cyclic permutation, and the same path represented as a closed one-particle path over three times the time interval.

where $\mathcal{C}_L$ is a class of L-particle paths which induce a fixed cyclic permutation of the L particles. But an L-particle path inducing a cyclic permutation is equivalent to a one-particle path over a τ-interval which is L times as long, as illustrated for a three-cycle in Figure 6. Therefore the cyclic L-particle path integral is related to the one-particle partition function, and it follows that

$$Z_N(\beta) = \sum_{\mathcal{P}} F_{\mathcal{P}}(\theta,\beta) \prod_L \frac{1}{\nu_L!} \left(\frac{Z_1(L\beta)}{L} \right)^{\nu_L} . \tag{160}$$

6.5 *Duality of Feynman and Schrödinger quantization*

It is worthwhile observing that the expansion of the partition function given in equation (160) is actually valid much more generally, if we interpret it in a suitable way. The present discussion is partly based on reference [270].

In equation (160) we considered noninteracting anyons in two dimensions, treating them as bosons with no other interactions than a statistics interaction turning them into anyons. But we may consider more generally bosons or fermions in any dimension, or even particles described by wave functions of a more general symmetry class Y, with quite general external potentials and interactions between the particles. We may always write

$$Z_N^Y(\beta) = \sum_{\mathcal{P}} F_{\mathcal{P}}^Y(\beta) \prod_L \frac{1}{\nu_L!} \left(\frac{Z_1(L\beta)}{L} \right)^{\nu_L} , \tag{161}$$

interpreting the coefficient $F_{\mathcal{P}}^Y(\beta)$ as a general correction factor describing the effect of all external and internal interactions, including statistics interactions that might transmute the particle identity. For example, bosons

($Y = B$) might turn into anyons, as in equation (160), and in particular bosons might turn into fermions. Or fermions ($Y = F$) might turn into anyons, and in this connection bosons count as anyons as well.

In the general expansion, equation (161), the superscript Y identifies the symmetry class of the wave functions used in computing the N-particle partition function $Z_N^Y(\beta)$, and we sum over conjugation classes in the symmetric group S_N, because the conjugation classes classify the paths going into the Feynman path integral. Thus the formula relates two approaches that are dual, in a certain sense: either solving the Schrödinger equation for identical particles and classifying the solutions according to the symmetry of the wave functions, or expanding the Feynman path integral as a sum over permutations.

A symmetry class is the same as an irreducible representation of the symmetric group S_N, which is identified by its Young tableau Y. Consider the Hilbert space $\mathcal{H}$ of all wave functions, of arbitrary symmetry. Every permutation $p \in S_N$ acts as a linear operator $\widehat{p}$ on $\mathcal{H}$, and so does every conjugation class $\mathcal{P} \subset S_N$, if we define

$$\widehat{\mathcal{P}} = \sum_{p \in \mathcal{P}} \widehat{p} \,. \tag{162}$$

The projection operator projecting out the subspace $\mathcal{H}_Y$ of the symmetry class Y, is

$$\widehat{I}_Y = \frac{d(Y)}{N!} \sum_{\mathcal{P}} \chi(\mathcal{P}; Y) \widehat{\mathcal{P}} \,, \tag{163}$$

where $d(Y)$ is the dimension of the irreducible representation Y, and $\chi(\mathcal{P}; Y)$ is the character in the representation Y of the permutations belonging to the conjugation class $\mathcal{P}$. The inverse relation is

$$\widehat{\mathcal{P}} = N_{\mathcal{P}} \sum_{Y} \frac{\chi(\mathcal{P}; Y)}{d(Y)} \widehat{I}_Y \,. \tag{164}$$

$N_{\mathcal{P}}$ is the number of elements in the conjugation class $\mathcal{P}$, as given in equation (151).

By definition, the partition function $Z_N^Y(\beta)$ is the trace of the operator $\mathrm{e}^{-\beta \widehat{H}}$ restricted to the subspace $\mathcal{H}_Y$,

$$Z_N^Y(\beta) = \mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{H}} \widehat{I}_Y\right) = \frac{d(Y)}{N!} \sum_{\mathcal{P}} \chi(\mathcal{P}; \mathcal{Y}) \, \mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{\mathcal{H}}} \widehat{\mathcal{P}}\right) \,. \tag{165}$$

The last equality follows from equation (163). Comparing with equation (161), we see that we may define

$$F_{\mathcal{P}}^Y(\beta) = \left[\prod_L \nu_L! \left(\frac{L}{Z_1(L\beta)}\right)^{\nu_L}\right] \frac{d(Y)}{N!} \chi(\mathcal{P}; Y) \, \mathrm{Tr}\left(\mathrm{e}^{-\beta \widehat{H}} \widehat{\mathcal{P}}\right) \,. \tag{166}$$

On the other hand, equation (164) with equation (151) gives that

$$\mathrm{Tr}\left(\mathrm{e}^{-\beta\widehat{H}}\widehat{\mathcal{P}}\right) = \frac{N!}{\prod_L(\nu_L!\,L^{\nu_L})}\sum_Y \frac{\chi(\mathcal{P};Y)}{d(Y)}\,Z_N^Y(\beta)\,. \tag{167}$$

Thus we see that if we are able to calculate the partition function $Z_N^{Y'}(\beta)$ for every symmetry class Y', for example by solving the Schrödinger equation for wave functions of arbitrary symmetry, then we may calculate every coefficient $F_{\mathcal{P}}^Y$ from the formula

$$F_{\mathcal{P}}^Y(\beta) = d(Y)\chi(\mathcal{P};Y)\left(\prod_L \frac{1}{(Z_1(L\beta))^{\nu_L}}\right)\sum_{Y'}\frac{\chi(\mathcal{P};Y')}{d(Y')}\,Z_N^{Y'}(\beta)\,. \tag{168}$$

Note that $F_{\mathcal{P}}^Y$ depends on the symmetry class Y only through the representation dimension $d(Y)$ and the character $\chi(\mathcal{P};Y)$.

In particular, equation (160) refers to the case $Y = B$, for which $d(Y) = d(B) = 1$ and $\chi(\mathcal{P};Y) = \chi(\mathcal{P};B) = 1$.

7 The harmonic oscillator

We can gain some general insight into the properties of anyons, and even learn something about the gas of free anyons, by doing the harmonic oscillator problem.

For two anyons it can be solved analytically, and all the energy eigenvalues depend linearly on the statistics angle θ, with a slope of fixed absolute value, sometimes changing sign at the bosonic values $\theta = 2n\pi$ for integer n. For three anyons, however, only about one third of the energy eigenvalues have the simple linear θ dependence and are known analytically.

7.1 The two-dimensional harmonic oscillator

Let us treat the one-particle problem in some detail, in order to have some notation and results for later use. The Hamiltonian for one particle of mass m in a harmonic oscillator potential is

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{x}^2\,. \tag{169}$$

$\boldsymbol{x}$ is the postition, $\boldsymbol{p}$ the canonical momentum, and ω the angular frequency, which characterizes the strength of the potential. H is rotationally symmetric and commutes with the canonical angular momentum $L = xp_y - yp_x$.

In order to quantize H and L simultaneously, it is convenient to introduce the characteristic length

$$\lambda = \sqrt{\frac{\hbar}{m\omega}}\,, \tag{170}$$

the dimensionless complex coordinate

$$z = \frac{x + iy}{\lambda} , \tag{171}$$

and the differential operators

$$\partial = \frac{\partial}{\partial z} = \frac{\lambda}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) , \qquad \partial^* = \frac{\partial}{\partial z^*} = \frac{\lambda}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) , \tag{172}$$

such that $\partial z = \partial^* z^* = 1$, $\partial^* z = \partial z^* = 0$. In this complex formalism z and its complex conjugate z^* are treated as independent variables. We have that

$$\begin{aligned} L &= \hbar\,(z\partial - z^*\partial^*) , \\ H &= \hbar\omega\left(-2\,\partial\partial^* + \frac{|z|^2}{2}\right) . \end{aligned} \tag{173}$$

The following annihilation and creation operators,

$$\begin{aligned} a &= \partial + \frac{z^*}{2} , \qquad a^\dagger = -\partial^* + \frac{z}{2} , \\ b &= \partial^* + \frac{z}{2} , \qquad b^\dagger = -\partial + \frac{z^*}{2} , \end{aligned} \tag{174}$$

satisfy the canonical commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1 , \qquad [a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0 , \tag{175}$$

and allow us to write

$$\begin{aligned} L &= \hbar\left(a^\dagger a - b^\dagger b\right) , \\ H &= \hbar\omega\left(a^\dagger a + b^\dagger b + 1\right) . \end{aligned} \tag{176}$$

The non-normalized wave function

$$\psi_0 = \exp\left(-\frac{|z|^2}{2}\right) \tag{177}$$

is the unique solution of the equations $a\psi_0 = b\psi_0 = 0$, and describes the ground state of the Hamiltonian H. A complete orthonormal set of simultaneous eigenfunctions of L and H are

$$\psi_{j,k} = \frac{1}{\sqrt{\pi j!k!}}\,(a^\dagger)^j\,(b^\dagger)^k\,\psi_0 , \tag{178}$$

with $j, k = 0, 1, 2, \ldots$. The state $\psi_{j,k}$ has angular momentum $\ell\hbar = (j-k)\hbar$ and energy $E = (j + k + 1)\hbar\omega$.

These one-particle energy levels give the following partition function, with $\xi = \beta\hbar\omega$,

$$Z_1(\beta) = \sum_{j,k} \mathrm{e}^{-\beta E_{j,k}} = \frac{\mathrm{e}^{-\xi}}{(1-\mathrm{e}^{-\xi})^2} = \frac{1}{4\sinh^2\left(\frac{\xi}{2}\right)} \cdot \tag{179}$$

The repeated action of the creation operators $a^\dagger$ and $b^\dagger$ on ψ_0 produces wave functions that are polynomials in z and z^*, multiplied by ψ_0. It is natural to split off the Gaussian factor ψ_0 explicitly and write a general wave function ψ as

$$\psi = \widetilde{\psi}\psi_0 = \widetilde{\psi}\,\exp\left(-\frac{|z|^2}{2}\right) \cdot \tag{180}$$

In consequence, we replace a general operator A by $\widetilde{A}$, defined such that

$$A\psi = (\widetilde{A}\widetilde{\psi})\,\psi_0\,. \tag{181}$$

This gives in particular that

$$\begin{aligned} \widetilde{a} &= \partial\,, \qquad & \widetilde{a}^\dagger &= -\partial^* + z\,, \\ \widetilde{b} &= \partial^*\,, \qquad & \widetilde{b}^\dagger &= -\partial + z^*\,, \end{aligned} \tag{182}$$

and hence,

$$\begin{aligned} \widetilde{L} &= \hbar\left(\widetilde{a}^\dagger\widetilde{a} - \widetilde{b}^\dagger\widetilde{b}\right) = \hbar\,(z\partial - z^*\partial^*) = L, \\ \widetilde{H} &= \hbar\omega\left(\widetilde{a}^\dagger\widetilde{a} + \widetilde{b}^\dagger\widetilde{b} + 1\right) = \hbar\omega\,(-2\,\partial\partial^* + z\partial + z^*\partial^* + 1)\,. \end{aligned} \tag{183}$$

Thus, one possible approach to the simultaneous eigenvalue problem for L and H is to look for polynomials $\widetilde{\psi}$ that are eigenfunctions of $\widetilde{L} = L$ and of $\widetilde{H}$. The homogeneous polynomial $z^j(z^*)^k$, for example, is a solution of the eigenvalue equations

$$z\partial(z^j(z^*)^k) = jz^j(z^*)^k\,, \qquad z^*\partial^*(z^j(z^*)^k) = kz^j(z^*)^k\,. \tag{184}$$

It is not an eigenfunction of $\widetilde{H}$, because

$$\partial\partial^*(z^j(z^*)^k) = jkz^{j-1}(z^*)^{k-1}\,, \tag{185}$$

but we see that a unique eigenfunction of $\widetilde{H}$ and $\widetilde{L}$ can be constructed as a linear combination

$$\widetilde{\psi}_{j,k} = z^j(z^*)^k + \alpha_1 z^{j-1}(z^*)^{k-1} + \cdots + \alpha_n z^{j-n}(z^*)^{k-n}\,, \tag{186}$$

with $n = \min(j,k)$. This is of course the same state as in equation (178). In fact, starting with the ground state $\widetilde{\psi}_0 = 1$, we have

$$\begin{aligned} \widetilde{\psi}_{j,k} &= (\widetilde{a}^\dagger)^j (\widetilde{b}^\dagger)^k \widetilde{\psi}_0 = (-\partial^* + z)^j (-\partial + z^*)^k \widetilde{\psi}_0 \\ &= (-\partial^* + z)^j \, (z^*)^k = (-\partial + z^*)^k \, z^j \, . \end{aligned} \tag{187}$$

The lowest order term in equation (186) is either z^{j-k} or $(z^*)^{k-j}$, depending on whether $j \geq k$ or $j \leq k$. Thus the asymptotic form of the wave function $\psi = \widetilde{\psi}\psi_0$ as $|z| \to 0$ is $|z|^{|\ell|}$, where $\ell = j - k$, and $\ell\hbar$ is the angular momentum. The quantum number $n = \min(j,k)$ describes radial excitations.

A standard asymptotic analysis of the eigenvalue equations $H\psi = E\psi$ and $L\psi = \ell\hbar\psi$ gives the same result. Let r and ϕ be polar coordinates, that is, $x = r\cos\phi$, $y = r\sin\phi$, then we have

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) + \frac{1}{2}\, m\omega^2 r^2 \, . \tag{188}$$

An eigenfunction for $L = -i\hbar\partial/\partial\phi$ has the form

$$\psi(r,\phi) = f(r)\, \mathrm{e}^{i\ell\phi} \, . \tag{189}$$

Assuming that $f(r)$ has the asymptotic form $f(r) \propto r^\mu$ as $r \to 0$, and inserting into the equation $H\psi = E\psi$, we get asymptotically as $r \to 0$ the equation $\mu^2 - \ell^2 = 0$. The minimum requirement is that ψ must be quadratically integrable, implying the inequality $\mu > -1$, but since there is no reason for ψ to be singular at $r = 0$, we have to require that $\mu \geq 0$, and hence $\mu = |\ell|$.

7.2 Two anyons in a harmonic oscillator potential

The two-particle Hamiltonian is

$$H = \frac{1}{2m}\left(\boldsymbol{p}_1{}^2 + \boldsymbol{p}_2{}^2\right) + \frac{1}{2}\, m\omega^2 \left(\boldsymbol{x}_1{}^2 + \boldsymbol{x}_2{}^2\right) . \tag{190}$$

We introduce the anyon statistics by requiring an arbitrary wave function ψ to be multivalued, with

$$\psi(\boldsymbol{x}_2, \boldsymbol{x}_1) = \mathrm{e}^{i\theta}\, \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{191}$$

for an anticlockwise interchange of particle positions. The phase angle θ is defined modulo 2π, and we will assume here that $\theta = \nu\pi$ with $0 \leq \nu < 2$. Then $\nu = 0$ represents bosons, while $\nu = 1$ represents fermions.

The motion of the two particles can be decomposed into independent motions of the centre of mass position $\mathbf{X} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and the relative position $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. The canonically conjugate momenta are $\mathbf{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ and $\boldsymbol{p} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2$. This gives the following expression for the Hamiltonian,

$$H = \frac{\mathbf{P}^2}{4m} + \frac{\boldsymbol{p}^2}{m} + m\omega^2 \mathbf{X}^2 + \frac{1}{4} m\omega^2 \boldsymbol{x}^2 \,. \tag{192}$$

Thus the centre of mass is represented as a "particle" of mass $2m$, whereas the relative coordinate describes a "particle" of the "reduced mass" $m/2$.

The anyonic symmetry condition affects the relative motion only, and takes the following form, still with the anticlockwise convention for changing $\boldsymbol{x}$ into $-\boldsymbol{x}$,

$$\psi(\mathbf{X}, -\boldsymbol{x}) = \mathrm{e}^{i\theta}\, \psi(\mathbf{X}, \boldsymbol{x}) \,. \tag{193}$$

This condition is singular at $\boldsymbol{x} = 0$, whenever θ is not an integer multiple of π, and so forces the wave function to behave singularly there, in the sense that it goes to infinity or is not differentiable.

Let r and ϕ be the relative polar coordinates, then the relative motion part of the wave function must have the following asymptotic form as $r \to 0$,

$$\psi_{\mathrm{rel}}(r, \phi) = r^{\mu}\, \mathrm{e}^{i\ell\phi} \,, \tag{194}$$

with $\ell\pi = \theta + 2k\pi$ for some integer k. Like in the one-particle case we get from the energy eigenvalue equation, to leading order in r, that $\mu^2 - \ell^2 = 0$. We choose the solution $\mu = |\ell|$, to make ψ finite in the limit $r \to 0$, even if it should happen that $|\ell| < 1$ so that quadratic integrability allows the choice $\mu = -|\ell|$. We have assumed here that $\theta = \nu\pi$ and $0 \le \nu < 2$, hence there are two classes of energy eigenstates: class (I) having $\mu = \nu, \nu+2, \nu+4, \ldots$, and class (II) having $\mu = 2-\nu, 4-\nu, 6-\nu, \ldots$.

Let us introduce the complex coordinates z_1 and z_2 in the same way as before, and define $Z = (z_1 + z_2)/2$ and $z = z_1 - z_2$. Then the Hamiltonian is

$$\begin{aligned} H = &= \hbar\omega \left(-2\, \frac{\partial^2}{\partial z_1\, \partial z_1^*} - 2\, \frac{\partial^2}{\partial z_2\, \partial z_2^*} + \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2} \right) \\ &= \hbar\omega \left(-\frac{\partial^2}{\partial Z\, \partial Z^*} - 4\, \frac{\partial^2}{\partial z\, \partial z^*} + |Z|^2 + \frac{|z|^2}{4} \right) \end{aligned} \tag{195}$$

and the total angular momentum is

$$\begin{aligned} L &= \hbar \left(z_1 \frac{\partial}{\partial z_1} - z_1^* \frac{\partial}{\partial z_1^*} + z_2 \frac{\partial}{\partial z_2} - z_2^* \frac{\partial}{\partial z_2^*} \right) \\ &= \hbar \left(Z \frac{\partial}{\partial Z} - Z^* \frac{\partial}{\partial Z^*} + z \frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} \right) . \end{aligned} \tag{196}$$

Next we define annihilation and creation operators as follows,

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial Z^*} + \frac{Z}{\sqrt{2}}\,, & a^\dagger &= -\frac{1}{\sqrt{2}}\frac{\partial}{\partial Z} + \frac{Z^*}{\sqrt{2}}\,, \\ b &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial Z} + \frac{Z^*}{\sqrt{2}}\,, & b^\dagger &= -\frac{1}{\sqrt{2}}\frac{\partial}{\partial Z^*} + \frac{Z}{\sqrt{2}}\,, \\ c &= \sqrt{2}\frac{\partial}{\partial z^*} + \frac{z}{2\sqrt{2}}\,, & c^\dagger &= -\sqrt{2}\frac{\partial}{\partial z} + \frac{z^*}{2\sqrt{2}}\,, \\ d &= \sqrt{2}\frac{\partial}{\partial z} + \frac{z^*}{2\sqrt{2}}\,, & d^\dagger &= -\sqrt{2}\frac{\partial}{\partial z^*} + \frac{z}{2\sqrt{2}}\,. \end{aligned} \tag{197}$$

The non-vanishing commutators among these operators are

$$[a, a^\dagger] = [b, b^\dagger] = [c, c^\dagger] = [d, d^\dagger] = 1\,. \tag{198}$$

With these definitions we obtain the following form of the total angular momentum and Hamiltonian,

$$\begin{aligned} L &= \hbar\left(a^\dagger a - b^\dagger b + c^\dagger c - d^\dagger d\right), \\ H &= \hbar\omega\left(a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d + 2\right). \end{aligned} \tag{199}$$

Two energy eigenstates having the correct symmetry under particle interchange, and belonging to the classes (I) and (II) defined above, are

$$\psi_0^{(\mathrm{I})} = z^\nu\,\psi_0\,, \qquad \psi_0^{(\mathrm{II})} = (z^*)^{2-\nu}\,\psi_0\,, \tag{200}$$

with

$$\psi_0 = \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2}\right) = \exp\left(-|Z|^2 - \frac{|z|^2}{4}\right). \tag{201}$$

They have energies $E_0^{(\mathrm{I})} = (2+\nu)\hbar\omega$ and $E_0^{(\mathrm{II})} = (4-\nu)\hbar\omega$.

We may construct a complete set of energy eigenstates by starting from these "ground states" and acting with the creation operators $a^\dagger$, $b^\dagger$, $c^\dagger$ and $d^\dagger$, within certain restrictions. Because $c^\dagger$ and $d^\dagger$ are antisymmetric under interchange, we have to use either $(c^\dagger)^2$, $(d^\dagger)^2$ or $c^\dagger d^\dagger$ in order to preserve the interchange symmetry of the wave functions. There are two further restrictions, when $0 < \nu < 1$ or $1 < \nu < 2$, because the action of either $(c^\dagger)^2$ on $\psi_0^{(\mathrm{I})}$ or of $(d^\dagger)^2$ on $\psi_0^{(\mathrm{II})}$ produces a singular wave function. However, $c^\dagger d^\dagger$ is always a "good" operator. Hence the general eigenstates are, with j, k, l, m independent non-negativ integers,

$$\begin{aligned} \psi_{j,k,l,m}^{(\mathrm{I})} &= (a^\dagger)^j\,(b^\dagger)^k\,(c^\dagger d^\dagger)^l\,(d^\dagger)^{2m}\,\psi_0^{(\mathrm{I})}\,, \\ \psi_{j,k,l,m}^{(\mathrm{II})} &= (a^\dagger)^j\,(b^\dagger)^k\,(c^\dagger d^\dagger)^l\,(c^\dagger)^{2m}\,\psi_0^{(\mathrm{II})}\,. \end{aligned} \tag{202}$$

Here j,k are quantum numbers of centre of mass excitations, while l,m describe excitations of the relative degrees of freedom. The corresponding energy levels are

$$\begin{aligned} E^{(\mathrm{I})}_{j,k,l,m} &= (2+\nu+j+k+2l+2m)\,\hbar\omega\,, \\ E^{(\mathrm{II})}_{j,k,l,m} &= (4-\nu+j+k+2l+2m)\,\hbar\omega\,. \end{aligned} \tag{203}$$

This gives the two-particle partition function, again with $\xi=\beta\hbar\omega$,

$$\begin{aligned} Z_2(\beta) &= \sum_{j,k,l,m} \mathrm{e}^{-\beta E_{j,k,k,l,m}} = \frac{\mathrm{e}^{-(2+\nu)\xi}+\mathrm{e}^{-(4-\nu)\xi}}{(1-\mathrm{e}^{-\xi})^2(1-\mathrm{e}^{-2\xi})^2} \\ &= \frac{\cosh((1-\nu)\xi)}{8\sinh^2\left(\frac{\xi}{2}\right)\sinh^2\xi}\,. \end{aligned} \tag{204}$$

7.3 *More than two anyons*

Although the complete solution of the N-anyon problem in a harmonic potential can only be obtained numerically when $N>2$, it is still possible to find a number of exact energy eigenstates.

The N-particle Hamiltonian is a sum of N one-particle contributions,

$$H=\sum_{j=1}^{N}\left(\frac{1}{2m}\boldsymbol{p}_j{}^2+\frac{1}{2}m\omega^2\boldsymbol{x}_j{}^2\right). \tag{205}$$

The centre of mass motion can be separated from the relative motion, because the potential is separable by the identity

$$\sum_{j=1}^{N}\boldsymbol{x}_j{}^2=\frac{1}{N}\left(\sum_{j=1}^{N}\boldsymbol{x}_j\right)^2+\frac{1}{N}\sum_{k=2}^{N}\sum_{j=1}^{k-1}(\boldsymbol{x}_j-\boldsymbol{x}_k)^2\,. \tag{206}$$

The centre of mass energy spectrum is identical to the one-particle spectrum, that is, the energy levels are $n\hbar\omega$ with degeneracy n, for $n=1,2,\ldots$. Hence the centre of mass motion contributes to the energy but not to the degeneracy of the ground state.

The bosonic ground state has all N particles in the lowest one-particle level, hence it is non-degenerate and has energy $N\hbar\omega$, including the centre of mass contribution. In the fermionic ground state the N one-particle states of lowest energy are filled. Thus there exist "magic numbers" $N=n(n+1)/2=1,3,6,10,\ldots$, with $n=1,2,\ldots$, such that the one-particle levels up to and including $n\hbar\omega$ are completely filled. The total energy is then

$$E=(1+4+\cdots+n^2)\hbar\omega=\frac{n(n+1)(2n+1)}{6}\,\hbar\omega=\frac{N\sqrt{8N+1}}{3}\,\hbar\omega. \tag{207}$$

For these magic numbers the fermionic ground state is non-degenerate, but for other values of N it is degenerate. Thus the degeneracy is 2 for $N = 2$, 3 for $N = 4$ and $N = 5$, 4 for $N = 7$ and $N = 9$, 6 for $N = 8$, and so on.

Like in the one-particle case we may introduce complex coordinates z_j as well as annihilation and creation operators $a_j, b_j, a_j^\dagger, b_j^\dagger$, in order to write the Hamiltonian as

$$H = \hbar\omega \sum_{j=1}^{N} \left(-2\partial_j \partial_j^* + \frac{1}{2} |z_j|^2 \right) = \hbar\omega \sum_{j=1}^{N} \left(a_j^\dagger a_j + b_j^\dagger b_j + 1 \right) , \tag{208}$$

and the total angular momentum as

$$L = \hbar \sum_{j=1}^{N} \left(z_j \partial_j - z_j^* \partial_j^* \right) = \hbar \sum_{j=1}^{N} \left(a_j^\dagger a_j - b_j^\dagger b_j \right) . \tag{209}$$

Assuming that all the operators $a_j^\dagger a_j$ and $b_j^\dagger b_j$ are non-negative, we derive the following inequality relating the energy E and total angular momentum $\ell\hbar$,

$$E \geq (|\ell| + N)\hbar\omega . \tag{210}$$

The non-normalized wave function

$$\psi_0 = \exp\left(-\frac{1}{2} \sum_{j=1}^{N} |z_j|^2 \right) \tag{211}$$

is annihilated by all the operators a_j and b_j. Again we may split off the Gaussian factor ψ_0 and write any wave function ψ as $\psi = \widetilde{\psi}\psi_0$, at the same time as we replace any operator A by $\widetilde{A}$, such that $A\psi = (\widetilde{A}\widetilde{\psi})\,\psi_0$. Then we have, in the same way as before, that

$$\begin{aligned} \widetilde{a}_j &= \partial_j , \qquad & \widetilde{a}_j^\dagger &= -\partial_j^* + z_j , \\ \widetilde{b}_j &= \partial_j^* , \qquad & \widetilde{b}_j^\dagger &= -\partial_j + z_j^* , \end{aligned} \tag{212}$$

and,

$$\begin{aligned} \widetilde{L} &= \hbar \sum_{j=1}^{N} \left(\widetilde{a}_j^\dagger \widetilde{a}_j - \widetilde{b}_j^\dagger \widetilde{b}_j \right) = \hbar \sum_{j=1}^{N} \left(z_j \partial_j - z_j^* \partial_j^* \right) = L, \\ \widetilde{H} &= \hbar\omega \sum_{j=1}^{N} \left(\widetilde{a}_j^\dagger \widetilde{a}_j + \widetilde{b}_j^\dagger \widetilde{b}_j + 1 \right) \\ &= \hbar\omega \sum_{j=1}^{N} \left(-2\partial_j \partial_j^* + z_j \partial_j + z_j^* \partial_j^* + 1 \right) . \end{aligned} \tag{213}$$

The generalization of the exact two-anyon energy eigenstates of equation (200) are the states

$$\psi_0^{(\mathrm{I})} = \Delta^\nu \, \psi_0 \, , \qquad \psi_0^{(\mathrm{II})} = (\Delta^*)^{2-\nu} \, \psi_0 \, , \tag{214}$$

where Δ is the lowest degree totally antisymmetric polynomial in the variables $z_1, z_2, \ldots, z_N$,

$$\Delta = \prod_{k=2}^{N} \prod_{j=1}^{k-1} (z_j - z_k) = \prod_{j<k} (z_j - z_k) \, . \tag{215}$$

Up to a sign, this is the Vandermonde determinant,

$$\Delta = \pm \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ {z_1}^{N-1} & {z_2}^{N-1} & \ldots & {z_N}^{N-1} \end{vmatrix} . \tag{216}$$

Since Δ is a function of $z_1, z_2, \ldots, z_N$, but not of $z_1^*, z_2^*, \ldots, z_N^*$, it follows that Δ^* is a function only of $z_1^*, z_2^*, \ldots, z_N^*$. And since Δ is a homogeneous polynomial, Δ^ν and $(\Delta^*)^{2-\nu}$ are both homogeneous functions, of degrees $N(N-1)\nu/2$ and $N(N-1)(2-\nu)/2$, respectively. The homogeneity means by definition that

$$\begin{aligned} \left(\sum_{j=1}^{N} z_j \partial_j \right) \Delta^\nu &= \frac{N(N-1)}{2} \, \nu \, \Delta^\nu \, , \\ \left(\sum_{j=1}^{N} z_j^* \partial_j^* \right) (\Delta^*)^{2-\nu} &= \frac{N(N-1)}{2} \, (2-\nu) \, (\Delta^*)^{2-\nu} \, . \end{aligned} \tag{217}$$

Thus the total angular momentum of the states $\psi_0^{(\mathrm{I})}$ and $\psi_0^{(\mathrm{II})}$ is

$$\ell_0^{(\mathrm{I})} \hbar = \frac{N(N-1)}{2} \, \nu \hbar \, , \qquad \ell_0^{(\mathrm{II})} \hbar = \frac{N(N-1)}{2} \, (\nu - 2) \hbar \, , \tag{218}$$

respectively, and the total energy is

$$E_0^{(\mathrm{I})} = \left(\frac{N-1}{2} \, \nu + 1 \right) N \hbar\omega \, , E_0^{(\mathrm{II})} = \left(\frac{N-1}{2} \, (2-\nu) + 1 \right) N \hbar\omega \, . \tag{219}$$

In both cases $E = (|\ell| + N)\hbar\omega$, meaning that these are ground states for those two values of the angular momentum. For $\nu = 1$, in particular, these are fermion states constructed either from the one-particle states $z^j \psi_0$ or from the states $(z^*)^j \psi_0$, with $j = 0, 1, \ldots, N-1$. However, the energy

is higher than the fermion ground state energy for every $N > 2$, and the discrepancy increases with N, since the energy increases like N^2 for large N, whereas the fermionic ground state energy increases like $N\sqrt{N}$. The ground state energy is not exactly known for anyons close to fermions, even in the three-particle case.

We may act on these two states by creation operators, taking care to preserve the interchange symmetry and avoid generating singular states. In this way we get energy eigenstates that we may classify as type (I) or (II). Besides the fact that we do not get the fermion states of lowest energy, another way to see that not all energy eigenstates are of either type (I) or (II), is by considering the asymptotic behaviour of the wave functions when two particles approach each other. The states of class (I) will all have the asymptotic form $(z_j - z_k)^{\nu+2m}$ as $|z_j - z_k| \to 0$, whereas those of class (II) will have the asymptotic form $(z_j^* - z_k^*)^{2-\nu+2n}$, with m and n non-negative integers. But there must exist more general states that somehow mix these possibilities, an example is the three-fermion ground state,

$$\widetilde{\psi} = z_1 z_2^* + z_2 z_3^* + z_3 z_1^* - z_2 z_1^* - z_3 z_2^* - z_1 z_3^* \,. \tag{220}$$

Let us look more closely at the states of class (I). They are of the form

$$\widetilde{\psi} = f(z_1, \ldots, z_N, z_1^*, \ldots, z_N^*)\, \Delta^\nu \,, \tag{221}$$

where we have split off, as usual, the Gaussian factor ψ_0, and where f is a polynomial in the $2N$ independent variables $z_1, \ldots, z_N, z_1^*, \ldots, z_N^*$, symmetric under interchange of particle labels. The "reduced" Hamiltonian operator $\widetilde{H}$, acting on $\widetilde{\psi}$, is given by equation (213).

As a starting point for constructing an energy eigenstate, assume that f is a homogeneous polynomial, of degree J in $z_1, z_2, \ldots, z_N$ and degree K in $z_1^*, z_2^*, \ldots, z_N^*$. Then we have that

$$\begin{aligned} \frac{\widetilde{H}(f\Delta^\nu)}{\hbar\omega\Delta^\nu} = &\left(\frac{N(N-1)}{2}\,\nu + N + J + K\right) f \\ &- 2\sum_{j=1}^{N} \partial_j^* \partial_j f - 2\nu \sum_{k=2}^{N} \sum_{j=1}^{k-1} \frac{\partial_j^* f - \partial_k^* f}{z_j - z_k} \,. \end{aligned} \tag{222}$$

If the last two terms here do not vanish, they add up to a homogeneous function of degree $(J-1, K-1)$, which has to be compensated for by the addition to the polynomial f of a "counterterm", in fact the same homogeneous function multiplied by some constant coefficient. The counterterm may need a second counterterm, of degree $(J-2, K-2)$, and so on, until the variables $z_1^*, z_2^*, \ldots, z_N^*$ are eliminated and the process stops after K steps.

This construction of an energy eigenstate works smoothly and produces a non-singular wave function as long as the counterterms arising are all polynomials. We see from equation (222) that the necessary and sufficient condition is that the polynomial $\partial_j^* f - \partial_k^* f$ must always be divisible by $z_j - z_k$. This condition must hold when f is the homogeneous polynomial we start with, or any one of the counterterms that we construct successively. Since we are working with symmetric functions, it is enough to impose the single condition on f that $\partial_1^* f - \partial_2^* f$ must be divisible by $z_1 - z_2$. Equivalently, we must have $\partial_1^* f - \partial_2^* f = 0$ when $z_1 = z_2$. Remember that we treat z and z^* as independent variables, so that $z_1 = z_2$ does not imply $z_1^* = z_2^*$. Explicitly written out the condition is that

$$\begin{aligned}&[\partial_1^* f](z, z, z_3, \ldots, z_N, z_1^*, z_2^*, z_3^*, \ldots, z_N^*) = \\ &[\partial_2^* f](z, z, z_3, \ldots, z_N, z_1^*, z_2^*, z_3^*, \ldots, z_N^*).\end{aligned} \tag{223}$$

Let us call a symmetric polynomial f "good" if it satisfies this condition. Since the condition is linear, any linear combination of good polynomials is a good polynomial. Almost as easy is it to see that any product of good polynomials is again good.

Obviously, f is good if it does not depend on $z_1^*, z_2^*, \ldots, z_N^*$. Let S be the symmetrization operator,

$$S = \frac{1}{N!} \sum_{p \in S_n} p \,. \tag{224}$$

Then the polynomials

$$c_{j_1, j_2, \ldots, j_N} = \gamma_{j_1, j_2, \ldots, j_N} \, S({z_1}^{j_N} {z_2}^{j_{N-1}+j_N} \cdots {z_N}^{j_1+j_2+\cdots+j_N}) \,, \tag{225}$$

where $j_1, j_2, \ldots, j_N$ are non-negative integers and $\gamma_{j_1, j_2, \ldots, j_N}$ are (unspecified) normalization factors, form a basis for the vector space of symmetric polynomials in $z_1, z_2, \ldots, z_N$. These basis polynomials are homogeneous of degree $j_1 + 2j_2 + \cdots + Nj_N$. By definition, the elementary symmetric polynomial c_i in $z_1, z_2, \ldots, z_N$ is the basis polynomial for which $j_i = 1$ and $j_k = 0$ when $k \neq i$. Explicitly written out, we have for example,

$$\begin{aligned} c_1 &= z_1 + z_2 + \cdots + z_N \,, \\ c_2 &= z_1 z_2 + z_1 z_3 + \cdots + z_{N-1} z_N \,, \\ c_3 &= z_1 z_2 z_3 + z_1 z_2 z_4 + \cdots + z_{N-2} z_{N-1} z_N \,. \end{aligned} \tag{226}$$

The corresponding N-anyon energy levels of the harmonic oscillator are therefore

$$E_{j_1, j_2, \ldots, j_N} = \left(\frac{N(N-1)}{2} \nu + N + j_1 + 2j_2 + \cdots + Nj_N \right) \hbar\omega \,. \tag{227}$$

In the system of charged anyons in a magnetic field, discussed in the next section, the same wave functions are even more useful, since they describe the degenerate ground state, the lowest Landau level "LLL".

This is not yet the full story. In fact, any symmetric polynomial which is linear in $z_1^*, z_2^*, \ldots, z_N^*$, is also "good". It must be of the form

$$f = \sum_{j=1}^{N} f_j(z_1, z_2, \ldots, z_N)\, z_j^* \,, \tag{228}$$

with $f_j(z_1, z_2, \ldots, z_j, \ldots, z_N) = f_1(z_j, z_2, \ldots, z_1, \ldots, z_N)$, and with $f_1(z_1, z_2, \ldots, z_N)$ symmetric in the $N-1$ last arguments. The condition for f to be "good",

$$f_1(z, z, z_3, \ldots, z_N) = f_2(z, z, z_3, \ldots, z_N) \,, \tag{229}$$

is seen to hold automatically. It follows that all sums and products of symmetric polynomials linear in $z_1^*, z_2^*, \ldots, z_N^*$, are good polynomials. The tricky part of enumerating all such good polynomials is to avoid double counting.

The enumeration can be done as follows. Let

$$s_k = \sum_{j=1}^{N} {z_j}^k \,, \qquad t_k = \sum_{j=1}^{N} {z_j}^k z_j^* \,, \tag{230}$$

and let $\mathcal{G}$ ("$\mathcal{G}$" for "good") be the set of all polynomials in $s_1, s_2, \ldots, s_N$, $t_0, t_1, \ldots, t_{N-1}$. Then it can be shown that $\mathcal{G}$ contains every symmetric polynomial we can generate by taking sums and products of symmetric polynomials that are either independent of or linear in $z_1^*, z_2^*, \ldots, z_N^*$. Moreover, because the $2N$ variables $s_1, s_2, \ldots, s_N, t_0, t_1, \ldots, t_{N-1}$ are independent, the representation of a symmetric polynomial as a polynomial in these particular variables is unique.

In order to see that we actually get eigenfunctions of the Hamiltonian, we should start with the basis polynomial

$$f = {s_1}^{j_1} {s_2}^{j_2} \cdots {s_N}^{j_N} {t_0}^{k_1} {t_1}^{k_2} \cdots {t_{N-1}}^{k_N} \,, \tag{231}$$

in which $j_1, j_2, \ldots, j_N, k_1, k_2, \ldots, k_N$ are arbitrary non-negative integers, and carry out the construction as outlined above. The construction indeed works, and we get an energy eigenvalue which is

$$E^{(I)}_{j_1, \ldots, j_N, k_1, \ldots, k_N} = \Big(\frac{N(N-1)}{2}\, \nu + N + j_1 + 2j_2 + \cdots + N j_N + k_1 + 2k_2 + \cdots + N k_N \Big) \hbar\omega \,. \tag{232}$$

The energy eigenfunctions are

$$\psi^{(\mathrm{I})}_{j_1,\,\ldots\,,j_N,k_1,\,\ldots\,,k_N} = (A_1^\dagger)^{j_1}\cdots(A_N^\dagger)^{j_N}(B_0^\dagger)^{k_1}\cdots(B_{N-1}^\dagger)^{k_N}\,\Delta^\nu\,\psi_0\,. \quad (233)$$

when we write

$$A_k^\dagger = \sum_{j=1}^N (a_j^\dagger)^k\,, \qquad B_k^\dagger = \sum_{j=1}^N (a_j^\dagger)^k b_j^\dagger\,. \quad (234)$$

A parallel construction can be carried out for the states of class (II), leading to the energy eigenvalues

$$E^{(II)}_{j_1,\,\ldots\,,j_N,k_1,\,\ldots\,,k_N} = \left(\frac{N(N-1)}{2}(2-\nu) + N + j_1 + 2j_2 + \cdots + Nj_N + k_1 + 2k_2 + \cdots + Nk_N\right)\hbar\omega. \quad (235)$$

All of these energy eigenvalues depend linearly on the statistics parameter ν.

The partition function obtained by summing over the "linear" energy levels is then

$$\begin{aligned} Z_N^{\mathrm{L}}(\beta) &= \left(\mathrm{e}^{-\frac{N(N-1)}{2}\nu\xi} + \mathrm{e}^{-\frac{N(N-1)}{2}(2-\nu)\xi}\right)\mathrm{e}^{-N\xi}\prod_{n=1}^N \frac{1}{(1-\mathrm{e}^{-n\xi})^2} \\ &= \frac{1}{2^{2N-1}}\cosh\left(\frac{N(N-1)}{2}(1-\nu)\xi\right)\prod_{n=1}^N \frac{1}{\sinh^2\left(\frac{n\xi}{2}\right)}\,. \end{aligned} \quad (236)$$

The above arguments left a number of loose ends that we should try to tie up. Consider first the generating function for the elementary symmetric polynomials $c_1, \ldots, c_N$,

$$g(w) = 1 + \sum_{j=1}^N c_j w^j = \prod_{j=1}^N (1 + wz_j)\,. \quad (237)$$

On the one hand we have that

$$\begin{aligned} \ln g(w) &= \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}\left(\sum_{j=1}^N c_j w^j\right)^k = c_1 w + \left(c_2 - \frac{{c_1}^2}{2}\right)w^2 + \cdots \\ &= \sum_{j=1}^N \ln(1+wz_j) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}\,s_k w^k\,, \end{aligned} \quad (238)$$

and this gives every polynomial s_k as a polynomial in $c_1, \ldots, c_N$. On the other hand we have that

$$g(w) = \prod_{k=1}^{\infty} \exp\left(\frac{(-1)^{k-1}}{k} s_k w^k\right)$$
$$= 1 + s_1 w + \left(-s_2 + \frac{{s_1}^2}{2}\right) w^2 + \cdots , \tag{239}$$

and this gives $c_1, \ldots, c_N$ as polynomials in $s_1, \ldots, s_N$. The fact that the last product of exponentials is a polynomial of degree N gives relations that can be used to express s_k for every $k > N$ as a polynomial in $s_1, \ldots, s_N$.

The values of $s_1, s_2, \ldots, s_N$ determine the values of $c_1, c_2, \ldots, c_N$, which in turn determine the values of $z_1, z_2, \ldots, z_N$, in arbitrary order, as the roots of the equation

$$z^N + c_1 z^{N-1} + \cdots + c_N = 0 . \tag{240}$$

Next, given $z_1, \ldots, z_N$ in a definite order, the values of $t_1, \ldots, t_N$ determine uniquely $z_1^*, \ldots, z_N^*$. This shows that the $2N$ variables $s_1, \ldots, s_N, t_0, \ldots, t_{N-1}$ are independent, and that every symmetric function of $z_1, \ldots, z_N, z_1^*, \ldots, z_N^*$ must be functionally dependent on them. However, the functional dependence need not be polynomial, and there do indeed exist symmetric polynomials in $z_1, \ldots, z_N, z_1^*, \ldots, z_N^*$ that are non-polynomial functions of $s_1, \ldots, s_N, t_0, \ldots, t_{N-1}$.

As already noted, s_k for $k > N$ is always a polynomial in $s_1, \ldots, s_N$. Another way to see this is to observe that

$${z_j}^k + c_1 {z_j}^{k-1} + \cdots + c_N {z_j}^{k-N} = 0 \tag{241}$$

for every $k \geq N$ and $j = 1, 2, \ldots, N$, implying the following recursion relations,

$$\begin{aligned} s_k &= -c_1 s_{k-1} - \cdots - c_N s_{k-N} , \\ t_k &= -c_1 t_{k-1} - \cdots - c_N t_{k-N} . \end{aligned} \tag{242}$$

In general, we may write every symmetric polynomial in the $2N$ variables $z_1, \ldots, z_N, z_1^*, \ldots, z_N^*$ as a linear combination of the symmetric polynomials generated by the following generating function, where $w_{00} = 1$,

$$g(w_{10}, w_{01}, w_{20}, w_{11}, \ldots) = \prod_{j=1}^{N} \left(\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} w_{kl} {z_j}^k {z_j^*}^l\right)$$
$$= 1 + \sum_{j=1}^{N} c_j {w_{10}}^j + \ldots \tag{243}$$

By a similar reasoning as above we find that all such polynomials can in turn be expressed as polynomials in the special symmetric polynomials

$$s_{kl} = \sum_{j=1}^{N} {z_j}^k z_j^{*l} \,, \tag{244}$$

which include the polynomials s_k and t_k defined above.

Thus in particular, and as we already know, every symmetric polynomial in z, independent of z^*, is a polynomial in $s_1, s_2, \ldots s_N$. Furthermore, every symmetric polynomial in z and z^*, linear in z^*, is a polynomial in $s_1, \ldots, s_N, t_0, \ldots, t_{N-1}$, linear in $t_0, t_1, \ldots, t_{N-1}$.

One point we have not proved here is that our list of energy levels depending linearly on ν, is complete for states of class (I) when the polynomial part of the wave function is quadratic, cubic or of higher degree in z^*.

Another point worth noting is that there are three common operators in the two sets of creation operators producing the excited states of class (I) and (II), they are

$$A_1^\dagger = \sum_{j=1}^{N} a_j^\dagger \,, \qquad B_1^\dagger = \sum_{j=1}^{N} b_j^\dagger \,, \qquad B_2^\dagger = \sum_{j=1}^{N} a_j^\dagger b_j^\dagger \,. \tag{245}$$

Two are linear and one is quadratic in the $a^\dagger$ and $b^\dagger$ operators. The linear ones produce pure centre of mass excitations, whereas the quadratic operator

$$\widetilde{B}_2^\dagger = B_2^\dagger - \frac{1}{N} A_1^\dagger B_1^\dagger = \sum_{j=1}^{N} \left(a_j^\dagger - \frac{1}{N} A_1^\dagger \right) \left(b_j^\dagger - \frac{1}{N} B_1^\dagger \right) \tag{246}$$

produces excitations of the relative motion, which can be interpreted as radial excitations.

The distinguishing property of the two classes of so called "linear" wave functions is their asymptotic behaviour as two particles come together: $(z_j - z_k)^{\nu+2k}$ for class (I) and $(z_j^* - z_k^*)^{2-\nu+2k}$ for class (II), with $k = 0$, 1, 2, ... But all wave functions, even the "non-linear" ones, must show one or possibly both of these two asymptotic behaviours for one given pair of particles. Since both asymptotic behaviours are "good" for the operators $A_1^\dagger$, $B_1^\dagger$ and $B_2^\dagger$, it means that all wave functions belong to their domain of definition. Therefore all energy eigenstates, both "linear" and "non-linear", come in so called "towers", generated by the repeated action of $\widetilde{B}_2^\dagger$ on a "bottom state". Each tower has infinitely many states, with a constant energy spacing of $2\hbar\omega$.

7.4 The three-anyon problem

The three-anyon problem is the simplest example going beyond the solvable two-anyon case. It has received much attention, and yet nobody has succeeded in finding a set of coordinates making it completely solvable by separation of variables. The best choice seems to be the coordinates described below, for which the separation is almost complete. More precisely, an anticlockwise cyclic interchange of the three particles gives a phase factor of $\mathrm{e}^{2i\theta}$ in the wave function, and this condition is compatible with the separation of variables. The stumbling block is the condition due to the interchange of two particles, which can in general only be satisfied by a superposition of such separated wave functions.

We introduce the primitive cube root of unity

$$\eta = \exp\left(\frac{2i\pi}{3}\right) = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\,, \tag{247}$$

with the properties that $\eta^2 = -\eta - 1 = \eta^* = 1/\eta$, and define the dimensionless complex coordinates

$$\begin{aligned} Z &= \frac{1}{\sqrt{3}\lambda}\left(z_1 + z_2 + z_3\right), \\ u &= \frac{1}{\sqrt{3}\lambda}\left(z_1 + \eta z_2 + \eta^2 z_3\right), \\ v &= \frac{1}{\sqrt{3}\lambda}\left(z_1 + \eta^2 z_2 + \eta z_3\right). \end{aligned} \tag{248}$$

Here $\lambda = \sqrt{\hbar/m\omega}$, as before. Z is the centre of mass coordinate, with a slightly unusual normalization, while u and v are relative coordinates. The inverse transformation is

$$\begin{aligned} z_1 &= \frac{\lambda}{\sqrt{3}}\left(Z + u + v\right), \\ z_2 &= \frac{\lambda}{\sqrt{3}}\left(Z + \eta^2 u + \eta v\right). \\ z_3 &= \frac{\lambda}{\sqrt{3}}\left(Z + \eta u + \eta^2 v\right). \end{aligned} \tag{249}$$

This coordinate transformation is a discrete Fourier transformation, and it transforms the cyclic interchange of particle positions.

$$(z_1, z_2, z_3) \mapsto (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3) = (z_2, z_3, z_1)\,. \tag{250}$$

into the diagonal form

$$(Z, u, v) \mapsto (\widetilde{Z}, \widetilde{u}, \widetilde{v}) = (Z, \eta^2 u, \eta v)\,. \tag{251}$$

The interchange of particles 2 and 3 is just an interchange of u and v. These two permutations generate the whole symmetric group S_3. A similar treatment of permutation symmetry has been used for some time in nuclear physics [271, 272].

Three particles in the plane define a triangle. The ratio

$$s = \frac{z_3 - z_1}{z_2 - z_1} \tag{252}$$

is real when the triangle is degenerate so that the particles lie on a straight line. We define the orientation of a non-degenerate triangle as positive or negative depending on whether the imaginary part of s is positive or negative. Thus the orientation is positive when the loop $z_1 \to z_2 \to z_3 \to z_1$ is counterclockwise, and negative when the loop is clockwise. We have that

$$\frac{|u|}{|v|} = \frac{|s+\eta^2|}{|s+\eta|} \,. \tag{253}$$

Hence $|u| = |v|$ when the particles lie on a line, $|u| < |v|$ when the orientation of the triangle is positive, and $|u| > |v|$ when the orientation is negative.

The quantization of the centre of mass motion is trivial, and the interesting part of the problem is the simultaneous diagonalization of the relative Hamiltonian and angular momentum operators,

$$\begin{aligned} H_{\rm rel} &= \hbar\omega \left(-2\frac{\partial^2}{\partial u\, \partial u^*} - 2\frac{\partial^2}{\partial v\, \partial v^*} + \frac{|u|^2}{2} + \frac{|v|^2}{2} \right), \\ L_{\rm rel} &= \hbar \left(u\frac{\partial}{\partial u} - u^*\frac{\partial}{\partial u^*} + v\frac{\partial}{\partial v} - v^*\frac{\partial}{\partial v^*} \right). \end{aligned} \tag{254}$$

The three-particle configuration is completely described by a total scale factor $r > 0$, a relative scale factor $q \geq 0$, and two angles φ_1 and φ_2 such that

$$u = \frac{rq\,\mathrm{e}^{i\varphi_1}}{\sqrt{1+q^2}} \,, \qquad v = \frac{r\,\mathrm{e}^{i\varphi_2}}{\sqrt{1+q^2}} \,. \tag{255}$$

These are the hyperspherical coordinates of Kilpatrick and Larsen, except that they used the angles $\varphi_1 \pm \varphi_2$ instead of φ_1 and φ_2 [273]. We now have that

$$\begin{aligned} \frac{H_{\rm rel}}{\hbar\omega} &= -\frac{1}{2r^3}\frac{\partial}{\partial r}\, r^3 \frac{\partial}{\partial r} - \frac{1+q^2}{2r^2}\left(\frac{1+q^2}{q}\frac{\partial}{\partial q}\, q\frac{\partial}{\partial q} + \frac{1}{q^2}\frac{\partial^2}{\partial {\varphi_1}^2} + \frac{\partial^2}{\partial {\varphi_2}^2} \right) + \frac{r^2}{2}, \\ \frac{L_{\rm rel}}{\hbar} &= -i\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2} \right). \end{aligned} \tag{256}$$

Assume that the wave function of the relative motion is separable,

$$\psi = \psi(r, q, \varphi_1, \varphi_2) = f(r)\, g(q)\, \mathrm{e}^{\,i(j\varphi_1 + k\varphi_2)} \,. \tag{257}$$

Then the eigenvalue equation $H_{\rm rel}\,\psi = E\psi$ separates into an angular eigenvalue equation,

$$(1+q^2)\left(-\frac{1+q^2}{q}\frac{\mathrm{d}}{\mathrm{d}q}\,q\,\frac{\mathrm{d}}{\mathrm{d}q}+\frac{j^2}{q^2}+k^2\right)g=\Lambda g\,, \tag{258}$$

with Λ as eigenvalue, and a radial equation,

$$-\frac{1}{2}f''(r)-\frac{3}{2r}f'(r)+\left(\frac{\Lambda}{2r^2}+\frac{r^2}{2}\right)f(r)=\frac{E}{\hbar\omega}f(r)\,. \tag{259}$$

A general wave function can be written as a linear combination of such separated wave functions. As will be seen below, we need linear combinations, where Λ and $j+k$ are constant but $j-k$ varies, in order to satisfy the anyonic boundary conditions.

The radial wave function must have the form $f(r) = r^{\mu}\,\mathrm{e}^{-r^2/2}\,f_1(r^2)$, with f_1 a polynomial of degree $n_r = 0, 1, 2, \ldots$, and in the above radial equation we have to choose

$$\Lambda=\mu(\mu+2)\,,\qquad E=(2+\mu+2n_r)\hbar\omega\,. \tag{260}$$

We must take $\mu \geq 0$ if we require the wave function to be finite as $r \to 0$, or $\mu > -2$ if we only require it to be normalizable.

Equation (258) has two asymptotic solutions $q^{\pm j}$ in the limit $q \to 0$. We exclude the singular solution (for $j = 0$ the singularity is logarithmic). In fact there is no reason for any singularity at $q = 0$, where the configuration is an equilateral triangle. The solution

$$g(q)=q^{|j|}\,(1+q^2)^{\kappa}\,F(a,b;c;-q^2) \tag{261}$$

is unique up to normalization. The constant κ may be chosen in one of two ways,

$$\kappa=\frac{\mu}{2}+1\qquad\text{or}\qquad\kappa=-\frac{\mu}{2}\,, \tag{262}$$

giving two different representations of the same solution. The constants

$$a=\frac{|j|+|k|}{2}+\kappa\,,\qquad b=\frac{|j|-|k|}{2}+\kappa\,,\qquad c=1+|j|\,, \tag{263}$$

define the hypergeometric series

$$F(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(c)_m}\,\frac{x^m}{m!}\,, \tag{264}$$

where, *e.g.* $(a)_0 = 1$, $(a)_{n+1} = (a)_n\,(a+n)$. The convergence radius for this series is 1. A more convergent representation is, *e.g.*,

$$g(q)=q^{|j|}\,(1+q^2)^{\kappa-a}\,F\left(a,c-b;c;\frac{q^2}{1+q^2}\right)\,. \tag{265}$$

We now need the boundary conditions in order to determine the allowed values of the quantum numbers j, k and $\Lambda = \mu(\mu+2)$ in equation (258). For three identical particles there is a six-fold identification of points in the relative space. We will restrict the wave functions to the region $0 \leq q \leq 1$, which corresponds to all the positively oriented triangles, but is still a three-fold covering of the true configuration space. The boundary conditions defining the particles to be anyons are of two types, since there are two classes of non-trivial permutations. The first class contains the three-particle cyclic permutations, which leave q invariant. The second class contains the two-particle interchanges, which transform q into $1/q$, and so give boundary conditions at $q = 1$.

Consider first a continuous, counterclockwise and cyclic deformation of the configuration, as defined in equation (250), with no extra overall rotation of the triangle. It gives a phase factor $\mathrm{e}^{2i\theta}$ in the wave function, where $\theta = \nu\pi$ is the statistics parameter. We should keep $|u| < |v|$ all the time during the deformation, that is, all the deformed configurations should be positive triangles, since this is the region where we require the wave function to be defined, and since this will ensure that no pair of particles wind around each other separately. Then the phase of v increases continuously from φ_2 to $\varphi_2 + (2\pi/3)$, whereas the phase of u changes from φ_1 to $\varphi_1 - (2\pi/3) + 2m'\pi$, where m' is any integer. Note that $u = 0$ represents a positively oriented equilateral triangle, and by means of small deformations close to $u = 0$ we may change the phase of u by an arbitrary multiple of 2π. We can not change the phase of v similarly without rotating the whole triangle, or deforming it so that its orientation becomes negative. The condition on the wave function is, therefore,

$$\psi\left(r, q, \varphi_1 - \frac{2\pi}{3} + 2m'\pi\,,\, \varphi_2 + \frac{2\pi}{3}\right) = \mathrm{e}^{\,2i\theta}\,\psi(r, q, \varphi_1, \varphi_2)\,. \tag{266}$$

That is,

$$j\left(-\frac{2\pi}{3} + 2m'\pi\right) + k\,\frac{2\pi}{3} = 2(n' + \nu)\pi\,, \tag{267}$$

for some integer n'. Since m' is an arbitrary integer, j must be an integer. Then

$$k = j + 3(n + \nu)\,, \tag{268}$$

where $n = n' - jm'$ is an arbitrary integer, and the eigenvalue of the relative angular momentum L_{rel} is $\ell\hbar$ with

$$\ell = j + k = 2j + 3(n + \nu)\,. \tag{269}$$

These relations take care of the cyclic permutations of all three particles. What remains is only to take care of one of the three cases where two

particles are interchanged, for example $z_2 \leftrightarrow z_3$, or equivalently, $u \leftrightarrow v$. This is the same as $q \leftrightarrow 1/q$ and $\varphi_1 \leftrightarrow \varphi_2$, if we define angles so that $u = v$ corresponds to $\varphi_1 = \varphi_2$. To be more precise, we consider a continuous interchange, with $q = 1$ at the beginning and end, and $q < 1$ during the interchange. The interchange should be anticlockwise, which means that we start with $\varphi_1 > \varphi_2$, and end with $\varphi_1 < \varphi_2$. There is one further restriction, that $|\varphi_1 - \varphi_2| < (2\pi/3)$ when $q = 1$, meaning that the particle position z_1 must not be encircled.

Thus, the boundary condition on ψ at $q = 1$ is

$$\psi(r, 1, \varphi_2, \varphi_1) = \mathrm{e}^{i\theta}\, \psi(r, 1, \varphi_1, \varphi_2)\ , \tag{270}$$

for $0 < \varphi_1 - \varphi_2 < (2\pi/3)$. It is a special case of the general condition

$$\psi(r, 1/q, \varphi_2, \varphi_1) = \mathrm{e}^{i\theta}\, \psi(r, q, \varphi_1, \varphi_2)\ . \tag{271}$$

Since the Schrödinger equation is second order in the q derivative, we need boundary conditions at $q = 1$ both for the wave function ψ and its normal derivative $\psi_q = \partial\psi/\partial q$. The derivative condition is easily deduced,

$$\psi_q(r, 1, \varphi_2, \varphi_1) = -\mathrm{e}^{i\theta}\, \psi_q(r, 1, \varphi_1, \varphi_2)\ . \tag{272}$$

The boundary conditions for ψ and ψ_q can not in general be satisfied by a wave function which is separable in q, φ_1 and φ_2. But we may quantize the relative angular momentum ℓ, and according to equation (269) $\ell - 3\nu = 2j + 3n$ is an integer, either even or odd. Let $\nu' = \nu$ if $n = 2m$ and $\nu' = \nu + 1$ if $n = 2m + 1$, with m integer. Then

$$j = \frac{\ell}{2} - 3\left(m + \frac{\nu'}{2}\right), \qquad k = \frac{\ell}{2} + 3\left(m + \frac{\nu'}{2}\right). \tag{273}$$

Let $g_m(q)$ be the function $g(q)$ as given by equation (261). Introducing $\varphi = (\varphi_1 + \varphi_2)/2$ and $\xi = 3(\varphi_1 - \varphi_2)$, and summing over m, including an as yet undetermined coefficient γ_m for each m, we get the following angular wave function,

$$\begin{aligned}\Omega(q, \varphi_1, \varphi_2) &= \sum_{m=-\infty}^{\infty} \gamma_m\, g_m(q)\, \mathrm{e}^{\,i\,(j\varphi_1 + k\varphi_2)} \\ &= \mathrm{e}^{\,i\,\ell\varphi} \sum_{m=-\infty}^{\infty} \gamma_m\, g_m(q)\, \mathrm{e}^{\,-i(m + (\nu'/2))\xi}.\end{aligned} \tag{274}$$

It is natural to call Ω an *anyonic spherical harmonic function*, whenever it satisfies the anyonic boundary conditions. The two boundary conditions

that must hold for $0 < \xi < 2\pi$ are

$$\sum_{m=-\infty}^{\infty} \gamma_m \, g_m(1) \, \mathrm{e}^{\,im\xi} \;=\; \mathrm{e}^{\,i(\nu\pi-\nu'\xi)} \sum_{m=-\infty}^{\infty} \gamma_m \, g_m(1) \, \mathrm{e}^{\,-im\xi} \,,$$

$$\sum_{m=-\infty}^{\infty} \gamma_m \, g'_m(1) \, \mathrm{e}^{\,im\xi} \;=\; -\mathrm{e}^{\,i(\nu\pi-\nu'\xi)} \sum_{m=-\infty}^{\infty} \gamma_m \, g'_m(1) \, \mathrm{e}^{\,-im\xi} \,. \quad (275)$$

Recall that $g_m(1)$ and $g'_m(1)$ depend on the three parameters μ, ℓ and m. For each given ℓ, the parameter μ, which determines the energy E, has to be adjusted so that the above boundary conditions have non-trivial solutions for the coefficients γ_m. For each ℓ there will be many solutions, possibly more than one with the same μ, and this procedure should give the complete set of anyonic spherical harmonics.

The left hand sides in equation (275) are two functions of ξ with Fourier components $\gamma_m \, g_m(1)$ and $\gamma_m \, g'_m(1)$, respectively. They are periodic in ξ with period 2π, and may be regarded as functions on the interval $[0, 2\pi]$. There is a natural scalar product between any two functions $\phi = \phi(\xi)$ and $\chi = \chi(\xi)$, with Fourier components ϕ_m and χ_m,

$$(\phi, \chi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\xi \, (\phi(\xi))^* \, \chi(\xi) = \sum_{m=-\infty}^{\infty} \phi_m^* \, \chi_m \,. \quad (276)$$

Define the linear operator A by

$$[A\phi](\xi) = \mathrm{e}^{\,i(\nu\pi-\nu'\xi)} \, \phi(2\pi - \xi) \,, \quad (277)$$

for $0 < \xi < 2\pi$. Then A is Hermitean with respect to the natural scalar product, and $A^2 = I$, the identity operator. Note that A is a somewhat singular operator, unless ν is an integer, since the factor $\mathrm{e}^{\,i(\nu\pi-\nu'\xi)}$, extended by periodicity outside the interval $[0, 2\pi]$, is discontinuous at every integer multiple of 2π. This shows up in the slow asymptotic falloff of the matrix elements of A with respect to the basis functions $\mathrm{e}^{\,im\xi}$,

$$A_{mn} = A_{nm} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\xi \, \mathrm{e}^{\,-im\xi} \, \mathrm{e}^{\,i(\nu\pi-\nu'\xi)} \, \mathrm{e}^{\,in(2\pi-\xi)} = \frac{\sin(\nu\pi)}{\pi(m+n+\nu)} \,. \quad (278)$$

The bosonic limit $\nu \to 0$ is $A_{mn} = \delta_{m,-n}$, and the fermionic limit $\nu \to 1$ is $A_{mn} = -\delta_{m,-n-1}$.

Define operators G and G' that are diagonal in the Fourier representation, with matrix elements

$$G_{mn} = g_m(1)\delta_{mn} \,, \qquad G'_{mn} = g'_m(1)\delta_{mn} \,. \quad (279)$$

Then the above boundary conditions may be written as

$$(I - A)G\gamma = 0 \,, \qquad (I + A)G'\gamma = 0 \,. \quad (280)$$

Since A is a real symmetric matrix and $A^2 = I$, the two vectors $(I - A)G\gamma$ and $(I + A)G'\gamma$ are orthogonal, and we may for example add the two conditions to get one single, equivalent condition,

$$(G + G' - A(G - G'))\gamma = 0 \,. \tag{281}$$

There exist nontrivial solutions for γ whenever the operator $G+G'-A(G-G')$ is singular. If we truncate the equation, the condition is that the determinant must vanish. Since the determinant is real, and a small change in ν could move a zero of the determinant, but not remove it, or introduce a new zero (at least not without removing or adding a double zero at once) we conclude, by continuity in ν, that the anyonic solutions are in one to one correspondence with the bosonic and fermionic solutions.

To find numerical solutions we must truncate to a finite number M of coefficients. Remarkably enough, this method is capable of giving many energy levels with non-trivial accuracy even if we take M to be very small. This is so when the low Fourier components dominate. On the other hand, the convergence as $M \to \infty$ is sometimes very slow. This is clearly related to the fact that the wave functions for non-integer ν have non-integer power behaviour at $\xi = 0$, where two particles meet. Hence the approximation by means of a finite Fourier series converges slowly.

An empirical rule is that the leading correction term for finite M is of order $M^{-2\nu}$. Using two different M one may therefore extrapolate to $M = \infty$, and this improves the convergence considerably. Another point to note is that one may take advantage of the supersymmetry in order to get more accurate energy levels.

8 The anyon gas

We will discuss in this section the cluster and virial expansions for the anyon gas, which are by now fairly well understood, although not completely solved. The fact that these expansions exist for anyons, is in itself non-trivial.

Three complementary methods for computations, all with their own limitations, are perturbation theory, non-perturbative numerical computation of energy levels, and direct computation of partition functions by the Monte Carlo method. All three methods rely on regularization techniques to obtain the thermodynamic limit from finite systems. Perturbation theory has given important exact information, but will be mentioned only very briefly here. A fourth method is mean field theory, which should be covered in other lectures.

The present discussion may seem like an evasion of the most interesting topic, which is the low temperature behaviour of a system of anyons. However, the low temperature problem is a hard nut to crack, and the high

temperature side is one possible direction of approach. Attemts to understand more directly the low temperature behaviour are usually based on mean field theory.

8.1 The cluster and virial expansions

The *virial expansion* for the equation of state of a gas,

$$\beta P = \rho + \sum_{n=2}^{\infty} A_n \rho^n, \tag{282}$$

is called so because it is related to the virial theorem, see *e.g.* [274]. Here $\beta = 1/k_B T$, k_B is Boltzmann's constant, T the temperature, P the pressure, ρ the number density, and $A_n = A_n(T)$ is the n-th virial coefficient.

Another representation of the equation of state is the *cluster expansion*,

$$\beta P = \sum_{n=1}^{\infty} b_n z^n, \qquad \rho = z\,\frac{\partial(\beta P)}{\partial z} = \sum_{n=1}^{\infty} n b_n z^n. \tag{283}$$

Any power series with a non-zero radius of convergence defines an analytic function, which in general is well defined in a region in the complex plane at least as large as the circle of convergence of the series. In fact the convergence radius is the smallest distance from the origin to any singularity of the analytic function. In general there need not be any direct relation between the convergence radius of the virial expansion and the region where the equation of state it represents, is physically valid. We will use the cluster and virial expansions here without worrying too much about questions of convergence.

Eliminating z in equation (283) gives equation (282), with the following relations between the virial and cluster coefficients,

$$\begin{aligned} A_2 &= -\frac{b_2}{b_1^2}, \\ A_3 &= -\frac{2b_3}{b_1^3} + \frac{4b_2^2}{b_1^4}, \\ A_4 &= -\frac{3b_4}{b_1^4} + \frac{18 b_2 b_3}{b_1^5} - \frac{20 b_2^3}{b_1^6}, \end{aligned} \tag{284}$$

and so on. Or inversely,

$$\begin{aligned} b_2 &= -b_1^2 A_2, \\ b_3 &= -\frac{b_1^3}{2}\left(A_3 - 4A_2^2\right), \\ b_4 &= -\frac{b_1^4}{3}\left(A_4 - 9A_3 A_2 + 16 A_2^3\right), \end{aligned} \tag{285}$$

and so on. Note that the coefficient b_1 is not determined by the virial coefficients, because the normalization of z is not fixed by the equations (282) and (283). We may define $z = e^{\beta\mu}$, where μ is the chemical potential. z is usually called fugacity, although strictly speaking it is only proportional to the fugacity, which has the dimension of pressure.

If the particles interact by a two-particle potential, then a necessary condition for the existence of the virial expansion in the thermodynamic limit of infinite volume and constant density, is that the potential has sufficiently short range. For example, if it decreases as r^{-n} at large distance r, then the condition is that $n > d$, the configuration space dimension. However, this criterion does not apply to a vector potential, and the virial expansion exists for non-interacting anyons, even though anyons are two-dimensional and have a statistics interaction which may be represented by a vector potential proportional to $1/r$. One may argue that the interaction range is short in the sense that the statistics flux is localized exactly at the particle positions, and the statistics interaction is present only when the particles are close enough to interchange positions or encircle each other.

The first clear evidence was the finite and exact result for the second virial coefficient of a gas of free anyons [66, 128]. Perturbation theory gave finite results for the expansion of the higher virial coefficients to first and second order around the boson and fermion points [102,129,130,134–141]. The third virial coefficient was proved to be finite for all θ, and was calculated numerically [122,143–145].

A general proof can be based on the path integral representation for the N-particle partition function, which leads to a path integral representation for the cluster coefficients, valid quite generally for anyons in two dimensions, as well as for bosons and fermions in any dimension, interacting by general scalar and vector potentials. It follows from this representation that the cluster coefficients are finite when the interaction range is "short" in a well defined sense, although it does not follow at the same time that the cluster expansion converges.

In particular, the cluster and virial coefficients of the gas of free anyons are finite. What counts is the pointlike nature of the flux more than the $1/r$ dependence of the vector potential. The range is temperature dependent, however, because the statistics interaction is effective when the particle paths wind around each other, and each path in the path integral represents Brownian motion of a particle in the plane, covering an area inversely proportional to the temperature.

8.2 First and second order perturbative results

An important source of exact information about the cluster and virial expansion for anyons is perturbation theory, to first and second order in θ at the boson point $\theta = 0$ and at the fermion point $\theta = \pi$.

The straightforward way to do perturbation theory is to work, not in the parallel gauge we have used so far most of the time, but in the bosonic gauge, where all wave functions are symmetric, or in the fermionic gauge, where all wave functions are antisymmetric. Then the statistics gauge potential is treated as a perturbation of the bosonic or fermionic Hamiltonian. Since this gauge potential is singular at those points where two particles come together, it is not obvious that perturbation theory should work, especially in the bosonic gauge, where the unperturbed wave functions need not vanish at coincidence points. Furthermore, those wave functions that do not vanish at coincidence points, depend on $|\theta|$ rather than θ in their asymptotic behaviour at such points. Thus it might be necessary to treat θ and $|\theta|$ as independent expansion parameters.

Note that θ is transformed into $-\theta$ by either a parity inversion or a time reversal. Equivalently we might say that $\theta-\pi$ is transformed into $-(\theta-\pi)$, since $-\theta-\pi$ and $-\theta+\pi$ represent the same statistics. Thus, if either parity invariance or time reversal invariance hold, implying in particular that there is no external magnetic field to break these invariances, then the energy spectrum and the partition function (but not the energy eigenfunctions) will depend only on $|\theta|$, or equivalently, on $|\theta-\pi|$.

On the other hand, at the fermion point $\theta=\pi$ it must also be possible to treat wavefunctions, energy eigenvalues and the partition function as analytic functions of θ. In fact, the source of the non-analyticity at the boson point $\theta=0$ is the asymptotic behaviour of some energy eigenfunctions, that $|\boldsymbol{x}_j-\boldsymbol{x}_j|^\mu$ as $|\boldsymbol{x}_j-\boldsymbol{x}_j|\to 0$, with $\mu=|\theta|/\pi$. It is mathematically possible to choose $\mu=-|\theta|/\pi$ for small enough $|\theta|$, but that would make the wave functions diverge in the limit $|\boldsymbol{x}_j-\boldsymbol{x}_j|\to 0$, which is usually considered a physically unacceptable alternative.

From this argument we conclude that when we make perturbation expansions around the fermion point $\theta=\pi$, we have a free choice whether we want to use $\theta-\pi$ or $|\theta-\pi|$ as our expansion parameter. This means that the expansion of the partition function will contain only even powers of $\theta-\pi$.

To be specific, let us consider the transformation from the parallel gauge to the bosonic gauge. It is most easily discussed in terms of dimensionless complex variables $z_j=(x_j+iy_j)/\lambda$, where λ is some standard length. Like before, we define $\partial_j=\partial/\partial z_j$ and $\partial_j^*=\partial/\partial z_j^*$. Then the kinetic energy operator of particle j is, in the parallel gauge,

$$\frac{\boldsymbol{p}_j^{\,2}}{2m}=-\frac{2\hbar^2}{m\lambda^2}\,\partial_j^*\partial_j\ . \tag{286}$$

Let ψ be the symmetric wave function in the bosonic gauge, and let Φ be some fixed multivalued function, such that $\Phi\psi$ is the multivalued wave function in the parallel gauge. The gauge independent (covariant) derivatives

are ∂_j and ∂_j^* in the parallel gauge, whereas in the bosonic gauge they are $\partial_j - ia_j$ and $\partial_j^* - ia_j^*$, where by definition

$$\partial_j(\Phi\psi) = \Phi\left((\partial_j - ia_j)\psi\right), \qquad \partial_j^*(\Phi\psi) = \Phi\left((\partial_j^* - ia_j^*)\psi\right). \tag{287}$$

Thus the "statistics vector potentials" a_j and a_j^* are

$$a_j = i\partial_j(\ln\Phi), \qquad a_j^* = i\partial_j^*(\ln\Phi). \tag{288}$$

The question remaining to be settled is how to choose the gauge transformation factor Φ. One possible choice is the pure phase factor

$$\Phi_0 = \left(\frac{\Delta}{|\Delta|}\right)^\nu = \left(\frac{|\Delta|}{\Delta^*}\right)^\nu = \left(\frac{\Delta}{\Delta^*}\right)^{\frac{\nu}{2}}, \tag{289}$$

where $\Delta = \prod_{j<k}(z_j - z_k)$. One problem with this is that the unperturbed wave functions will have the wrong asymptotic behaviour as $|z_j - z_k| \to 0$, since the behaviour of the exact wave functions depends on ν, for example as $|z_j - z_k|^{|\nu|}$. Another problem is that the gauge potentials, which are

$$\begin{aligned} a_{0j} &= i\partial_j(\ln\Phi_0) = \; i\frac{\nu}{2}\sum_{k\neq j}\frac{1}{z_j - z_k}, \\ a_{0j}^* &= i\partial_j^*(\ln\Phi_0) = -i\frac{\nu}{2}\sum_{k\neq j}\frac{1}{z_j^* - z_k^*}, \end{aligned} \tag{290}$$

give rise to three-body terms in the Hamiltonian, of the type

$$a_{0j}^* a_{0j} = \frac{\nu^2}{4}\sum_{k\neq j}\sum_{l\neq j}\frac{1}{(z_j - z_k)(z_j^* - z_l^*)}. \tag{291}$$

A better method is therefore to split off explicitly not only the phase factor Φ_0, but also the factor $|\Delta|^{|\nu|}$. Thus we define

$$\Phi = |\Delta|^{|\nu|}\Phi_0 = \Delta^{\frac{|\nu|+\nu}{2}}(\Delta^*)^{\frac{|\nu|-\nu}{2}}. \tag{292}$$

The corresponding vector potentials are

$$\begin{aligned} a_j &= i\partial_j(\ln\Phi) = i\frac{|\nu|+\nu}{2}\sum_{k\neq j}\frac{1}{z_j - z_k}, \\ a_j^* &= i\partial_j^*(\ln\Phi) = i\frac{|\nu|-\nu}{2}\sum_{k\neq j}\frac{1}{z_j^* - z_k^*}. \end{aligned} \tag{293}$$

Note that $a_j = 0$ if $\nu \leq 0$, and $a_j^* = 0$ if $\nu \geq 0$. This gives that

$$\begin{aligned} \sum_{j=1}^N(\partial_j^* - ia_j^*)(\partial_j - ia_j) = \sum_{j=1}^N \partial_j^*\partial_j + \frac{|\nu|+\nu}{2}\sum_{j<k}\frac{\partial_j^* - \partial_k^*}{z_j - z_k} \\ + \frac{|\nu|-\nu}{2}\sum_{j<k}\frac{\partial_j - \partial_k}{z_j^* - z_k^*}, \end{aligned} \tag{294}$$

and there are no longer any three-body terms present. If we take $\nu \geq 0$, we have $\nu = |\nu|$, and

$$\sum_{j=1}^{N}(\partial_j^* - ia_j^*)(\partial_j - ia_j) = \sum_{j=1}^{N}\partial_j^*\partial_j + |\nu|\sum_{j<k}\frac{\partial_j^* - \partial_k^*}{z_j - z_k}\,. \tag{295}$$

If instead $\nu \leq 0$, we have $\nu = -|\nu|$, and the complex conjugate operator,

$$\sum_{j=1}^{N}(\partial_j^* - ia_j^*)(\partial_j - ia_j) = \sum_{j=1}^{N}\partial_j^*\partial_j + |\nu|\sum_{j<k}\frac{\partial_j - \partial_k}{z_j^* - z_k^*}\,. \tag{296}$$

Multipyling with $-2\hbar^2/(m\lambda^2)$, we have here suitable kinetic energy operators for doing perturbation theory. If parity invariance and/or time reversal invariance hold, then the wave functions and energies can be expanded as power series in $|\nu|$, except that the wave function for ν is the complex conjugate of the corresponding wave function for $-\nu$.

The gauge transformation $\psi \mapsto \Phi_0\psi$ from the bosonic to the parallel gauge is unitary, since Φ_0 as defined in equation (289) is a pure phase factor (we do not mind that it is multivalued and hence singular if ν is not an integer). The gauge factor Φ defined in equation (292), on the other hand, is more than a pure phase factor, and hence defines a *non-unitary* "gauge transformation". One result of the non-unitarity is that the kinetic energy operator of either equation (295) or equation (296) is not Hermitian in the standard scalar product of the bosonic Hilbert space. This is one of the subtleties involved in the perturbation theoretic treatment of the statistics vector potential [275]. To second order in $|\nu|$, Dasnières de Veigy and Ouvry have computed the following result, which does not look particularly encouraging for anybody wanting to attempt third or fourth order computations [138,276],

$$\begin{aligned}\Lambda^2\beta P = {} & \pm\sum_{n=1}^{\infty}\frac{(\pm z)^n}{n^2} - |\nu|\,\frac{1\pm 1}{2}\,(\ln(1\mp z))^2 \\ & + \nu^2\,\frac{1\pm z + 2(1-(1/24))z}{2(1\mp z)}\,(\ln(1\mp z))^2 \\ & + \nu^2\sum_{s,t\geq 1}\sum_{u,v\geq 0}\frac{(\pm z)^{s+t+u+v}}{s+t+u+v}\left(\frac{1}{(s+v)^2}\right. \\ & \left. + \frac{1}{(s+u)(t+v)}\right)C(s,t,u,v).\end{aligned} \tag{297}$$

The upper signs hold if $\nu = 0$ is defined as Bose statistics, whereas the lower signs hold if one instead defines $\nu = 0$ to mean Fermi statistics.

The coefficient C is defined as

$$C(s,t,u,v) = \frac{(1-\eta_+)(\eta_+ - s)(\eta_+ + v)}{\eta_+ - \eta_-} \ln\left|\frac{1-\eta_+}{\eta_+}\right| + (\eta_+ \leftrightarrow \eta_-)\,, \quad (298)$$

with

$$\eta_\pm = \frac{st - uv \pm \sqrt{(s+u)(s+v)(t+u)(t+v)}}{s+t+u+v}. \quad (299)$$

The first order term in $|\nu|$ vanishes at the fermion point, as it should. The cluster expansion to second order in ν has the convergence radius $|z| = 1$.

8.3 Regularization by periodic boundary conditions

We want to consider free particles in two dimensions, but in order to keep the partition functions finite we need some kind of regularization, by confinement of the particles inside a finite region. We may use a confining potential, such as a harmonic oscillator potential, with soft walls, or a square box, with hard walls, but the fastest convergence to the limit of infinite system size is obtained by using periodic boundary conditions so that there are no edge effects due to the reflecting walls. The periodicity is then used only for normalization, and when we speak about anyons in the path integral formalism, the only restriction is that the starting points of trajectories should be inside the given area. Otherwise the particles propagate freely in the plane and not on the torus.

The one-particle partition function is, with periodic boundary conditions in a square box of area A,

$$\begin{aligned} Z_1(\beta) &= \left[\sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2 \Lambda^2}{A}\right)\right]^2 \\ &= \frac{A}{\Lambda^2}\left[1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{\pi n^2 A}{\Lambda^2}\right)\right]^2 . \end{aligned} \quad (300)$$

The last formula is a Poisson resummation, by Fourier expansion of

$$f(x) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi(n+x)^2\Lambda^2}{A}\right), \quad (301)$$

a periodic function of x [277]. Λ is the thermal de Broglie wave length,

$$\Lambda = \hbar\sqrt{\frac{2\pi\beta}{m}}\,, \quad (302)$$

with m the particle mass. Below we will take

$$Z_1(\beta) = \frac{A}{\Lambda^2} \,, \tag{303}$$

using only the leading term in the limit $A \to \infty$. The correction terms for finite A are exponential in A. For comparison, the hard wall boundary conditions would give

$$\begin{aligned} Z_1(\beta) &= \left[\sum_{n=1}^{\infty} \exp\left(-\frac{\pi n^2 \Lambda^2}{A}\right)\right]^2 \\ &= \frac{A}{\Lambda^2}\left[\frac{1}{2} - \frac{\Lambda}{2\sqrt{A}} + \sum_{n=1}^{\infty} \exp\left(-\frac{\pi n^2 A}{\Lambda^2}\right)\right]^2 , \end{aligned} \tag{304}$$

with relative correction terms of order $1/\sqrt{A}$, which is the ratio between circumference and area of the square box.

The formula (303) implies the following scaling relation, valid for one free particle in two dimensions,

$$Z_1(L\beta) = \frac{Z_1(\beta)}{L} \cdot \tag{305}$$

The general relation in dimension d is, with V the d-dimensional volume,

$$Z_1(\beta) = (\sqrt{L})^d Z_1(L\beta) = \frac{V}{\Lambda^d} \cdot \tag{306}$$

It is convenient to introduce here the following notation. We defined a partition of N as a sequence of non-negative integers, $\mathcal{P} = (\nu_1, \nu_2, \ldots)$, with $\sum_{L=1}^{\infty} L\nu_L = N$. Let $\mathcal{C}_N$ denote the set of all partitions of N, and let $\mathcal{C} = \bigcup_{N=0}^{\infty} \mathcal{C}_N$ and $\mathcal{C}' = \bigcup_{N=1}^{\infty} \mathcal{C}_N$. In this notation we have that

$$\sum_{\mathcal{P}\in\mathcal{C}} = \sum_{N=0}^{\infty} \sum_{\mathcal{P}\in\mathcal{C}_N} = \sum_{\nu_1=0}^{\infty} \sum_{\nu_2=0}^{\infty} \cdots \sum_{\nu_L=0}^{\infty} \cdots \tag{307}$$

We will frequently use also another notation, writing for example 311 for the partition $5 = 3 + 1 + 1$, which we would otherwise refer to as $\mathcal{P} = (2, 0, 1, 0, \ldots)$.

The grand canonical partition function is a function of the inverse temperature β and the chemical potential μ,

$$\Xi(\beta, \mu) = 1 + \sum_{N=1}^{\infty} z^N Z_N(\beta) \,, \tag{308}$$

and the relation between quantum mechanics and statistical mechanics is the equation

$$\ln \Xi = A\beta P = A \sum_{n=1}^{\infty} b_n z^n \ . \tag{309}$$

The area A enters in the two-dimensional case, but the formula is valid in arbitrary dimension d when A is replaced by the d-dimensional volume. An immediate consequence is that

$$\Xi = \prod_{n=1}^{\infty} \exp(Ab_n z^n) = \sum_{\mathcal{P}\in\mathcal{C}} \prod_{L=1}^{\infty} \frac{\left(Ab_L z^L\right)^{\nu_L}}{\nu_L!} \ , \tag{310}$$

and hence,

$$Z_N = \sum_{\mathcal{P}\in\mathcal{C}_N} \prod_{L=1}^{\infty} \frac{\left(Ab_L\right)^{\nu_L}}{\nu_L!} \cdot \tag{311}$$

We are more interested in the inverse relation, which follows from the expansion

$$\begin{aligned} \ln \Xi &= \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \left(\sum_{L=1}^{\infty} z^L \, Z_L \right)^{\nu} \\ &= \sum_{\mathcal{P}\in\mathcal{C}'} (-1)^{\nu-1} (\nu-1)! \prod_{L=1}^{\infty} \frac{\left(z^L \, Z_L\right)^{\nu_L}}{\nu_L!} \cdot \end{aligned} \tag{312}$$

Here $\nu = \sum_{L=1}^{\infty} \nu_L$ is the total number of cycles in the partition $\mathcal{P}$. This gives the cluster coefficients in terms of the N-particle partition functions,

$$Ab_N = \sum_{\mathcal{P}\in\mathcal{C}_N} (-1)^{\nu-1} (\nu-1)! \prod_{L=1}^{\infty} \frac{{Z_L}^{\nu_L}}{\nu_L!} \cdot \tag{313}$$

In particular,

$$\begin{aligned} Ab_1 &= Z_1 \ , \\ Ab_2 &= Z_2 - \frac{{Z_1}^2}{2} \ , \\ Ab_3 &= Z_3 - Z_2 Z_1 + \frac{{Z_1}^3}{3} \ , \\ Ab_4 &= Z_4 - Z_3 Z_1 - \frac{{Z_2}^2}{2} + Z_2 {Z_1}^2 - \frac{{Z_1}^4}{4} \cdot \end{aligned} \tag{314}$$

Using these, we obtain the virial coefficients from equation (284),

$$\begin{aligned} A_2 &= \frac{A}{2}\left(1-2\,\frac{Z_2}{Z_1^2}\right), \\ A_3 &= \frac{A^2}{3}\left(1-6\,\frac{Z_2}{Z_1^2}+12\,\frac{Z_2^2}{Z_1^4}-6\,\frac{Z_3}{Z_1^3}\right), \\ A_4 &= \frac{A^3}{4}\left(1-12\,\frac{Z_2}{Z_1^2}+54\,\frac{Z_2^2}{Z_1^4}-80\,\frac{Z_2^3}{Z_1^6}-24\,\frac{Z_3}{Z_1^3}+72\,\frac{Z_3Z_2}{Z_1^5}-12\,\frac{Z_4}{Z_1^4}\right). \end{aligned} \tag{315}$$

Clearly many cancellations of leading order terms are needed to produce finite values for the cluster and virial coefficients in the free particle limit $A \to \infty$.

Next we use our general expansion of the N-particle partition function as a sum over partitions of N, equation (160), reading explicitly as follows,

$$\begin{aligned} Z_2(\beta) &= \frac{1}{2}\,F_{11}\,Z_1(\beta)^2+\frac{1}{2}\,F_2\,Z_1(2\beta)\,, \\ Z_3(\beta) &= \frac{1}{6}\,F_{111}\,Z_1(\beta)^3+\frac{1}{2}\,F_{21}\,Z_1(2\beta)Z_1(\beta)+\frac{1}{3}\,F_3\,Z_1(3\beta)\,, \\ Z_4(\beta) &= \frac{1}{24}\,F_{1111}\,Z_1(\beta)^4+\frac{1}{4}\,F_{211}\,Z_1(2\beta)Z_1(\beta)^2+\frac{1}{8}\,F_{22}\,Z_1(2\beta)^2 \\ &\quad +\frac{1}{3}\,F_{31}\,Z_1(3\beta)Z_1(\beta)+\frac{1}{4}\,F_4\,Z_1(4\beta)\,. \end{aligned} \tag{316}$$

We get then that

$$\begin{aligned} Ab_N &= Z_N+\cdots=\sum_{\mathcal{P}\in\mathcal{C}_N}\left(F_{\mathcal{P}}\prod_L\frac{1}{\nu_L!}\left(\frac{Z_1(L\beta)}{L}\right)^{\nu_L}+\cdots\right) \\ &= Z_1(\beta)\sum_{\mathcal{P}\in\mathcal{C}_N}G_{\mathcal{P}}\prod_L\frac{1}{\nu_L!}\left(\frac{Z_1(L\beta)}{LZ_1(\beta)}\right)^{\nu_L}\,, \end{aligned} \tag{317}$$

in terms of a new set of coefficients,

$$G_{\mathcal{P}}=(F_{\mathcal{P}}+\cdots){Z_1}^{\nu-1}\,. \tag{318}$$

The "$\cdots$" in the last formula represents a sum of terms that are products of "F" coefficients. The "G" coefficients are useful especially because they tend to finite limits in the thermodynamic limit $A \to \infty$, when all interactions have sufficiently short range, as we will prove below. $G_{\mathcal{P}}$ is the "connected part" of $F_{\mathcal{P}}$ for any partition $\mathcal{P}$. The concept of connectedness will also be made more precise below.

We have explicitly that $G_1 = F_1 = 1$, $G_N = F_N$ for $N = 2, 3, 4, \ldots$,

and

$$\begin{aligned} G_{11} &= (F_{11} - 1)Z_1 \,, \\ G_{111} &= (F_{111} - 3F_{11} + 2){Z_1}^2 \,, \\ G_{21} &= (F_{21} - F_2)Z_1 \,, \\ G_{1111} &= (F_{1111} - 4F_{111} - 3{F_{11}}^2 + 12F_{11} - 6){Z_1}^3 \,, \\ G_{211} &= (F_{211} - 2F_{21} - F_2F_{11} + 2F_2){Z_1}^2 \,, \\ G_{22} &= (F_{22} - {F_2}^2)Z_1 \,, \\ G_{31} &= (F_{31} - F_3)Z_1 \,. \end{aligned} \tag{319}$$

So far our formulae are dimension independent. From the specifically two-dimensional equations (303) and (305) we now get the equation

$$\Lambda^2 b_N = \sum_{\mathcal{P}\in\mathcal{C}_N} G_{\mathcal{P}} \prod_L \frac{1}{\nu_L!\, L^{2\nu_L}} \,, \tag{320}$$

in which all quantities are finite in the $A \to \infty$ limit. In particular,

$$\begin{aligned} \Lambda^2 b_2 &= \frac{G_{11}}{2} + \frac{G_2}{4} \,, \\ \Lambda^2 b_3 &= \frac{G_{111}}{6} + \frac{G_{21}}{4} + \frac{G_3}{9} \,, \\ \Lambda^2 b_4 &= \frac{G_{1111}}{24} + \frac{G_{211}}{8} + \frac{G_{22}}{32} + \frac{G_{31}}{9} + \frac{G_4}{16} \,. \end{aligned} \tag{321}$$

8.4 Regularization by a harmonic oscillator potential

Another useful regularization scheme is by means of an external potential $V_1 = V_1(\boldsymbol{x})$. Assuming that V_1 varies slowly enough, we can simply take the pressure, density and fugacity in equation (283) to be functions of the position $\boldsymbol{x}$. In particular, the local fugacity is

$$z(\boldsymbol{x}) = \mathrm{e}^{\beta(\mu - V_1(\boldsymbol{x}))} = z_0\, \mathrm{e}^{-\beta V_1(\boldsymbol{x})} \,, \tag{322}$$

with $z_0 = \mathrm{e}^{\beta\mu}$ independent of position. In this case we should replace equation (309) by the following equation [131],

$$\ln \Xi = \int \mathrm{d}^d\boldsymbol{x}\, \beta P(\boldsymbol{x}) = \ln\left(1 + \sum_{N=1}^{\infty} z_0^N Z_N\right) . \tag{323}$$

To evaluate the integral in equation (323) explicitly, we use the local form of equation (283), with $z = z(\boldsymbol{x})$ as given by equation (322), and with an external potential of the harmonic oscillator form,

$$V_1(\boldsymbol{x}) = \frac{1}{2}\, m\omega^2 |\boldsymbol{x}|^2 . \tag{324}$$

Here m is the particle mass, and ω the angular frequency of the oscillator. We take $\omega > 0$ in order to get a discrete energy spectrum, but in the end we want to take the free particle limit $\omega \to 0$. It is convenient to introduce the dimensionless parameter $\xi = \hbar\omega\beta$. Then we have that

$$\int \mathrm{d}^d\boldsymbol{x}\, \beta P(\boldsymbol{x}) = \sum_{n=1}^{\infty} b_n z_0^n \int \mathrm{d}^d\boldsymbol{x}\, \mathrm{e}^{-n\beta V_1(\boldsymbol{x})} = \sum_{n=1}^{\infty} b_n z_0^n \left(\frac{\Lambda}{\xi\sqrt{n}}\right)^d . \quad (325)$$

Comparison of equation (325) with equation (323) gives the following formulae for the cluster coefficients in terms of the partition functions, in dimension $d = 2$,

$$\begin{aligned}
\frac{\Lambda^2 b_1}{\xi^2} &= Z_1, \\
\frac{\Lambda^2 b_2}{2\xi^2} &= Z_2 - \frac{Z_1^2}{2}, \\
\frac{\Lambda^2 b_3}{3\xi^2} &= Z_3 - Z_2 Z_1 + \frac{Z_1^3}{3}, \\
\frac{\Lambda^2 b_4}{4\xi^2} &= Z_4 - Z_3 Z_1 - \frac{Z_2^2}{2} + Z_2 Z_1^2 - \frac{Z_1^4}{4} .
\end{aligned} \quad (326)$$

Comparing with equation (314), we see that we may adopt the mechanical rule of substitution $Ab_n \to \Lambda^2 b_n/(n\xi^2)$.

From these equations and equation (284) follow the virial coefficients,

$$\begin{aligned}
A_2 &= \left(\frac{\Lambda}{\xi}\right)^2 \left(1 - 2\,\frac{Z_2}{Z_1^2}\right), \\
A_3 &= \left(\frac{\Lambda}{\xi}\right)^4 \left(2 - 10\,\frac{Z_2}{Z_1^2} + 16\,\frac{Z_2^2}{Z_1^4} - 6\,\frac{Z_3}{Z_1^3}\right), \\
A_4 &= \left(\frac{\Lambda}{\xi}\right)^6 \left(5 - 42\,\frac{Z_2}{Z_1^2} + 138\,\frac{Z_2^2}{Z_1^4} - 160\,\frac{Z_2^3}{Z_1^6}\right. \\
&\qquad \left. -42\,\frac{Z_3}{Z_1^3} + 108\,\frac{Z_3 Z_2}{Z_1^5} - 12\,\frac{Z_4}{Z_1^4}\right).
\end{aligned} \quad (327)$$

Note that Planck's constant cancels in the ratio Λ/ξ, but reappears in the partition functions.

To obtain the virial coefficients for free particles we must take the limit $\omega \to 0$. The existence of the limit is again a far from trivial issue, since the cancellation of the singular factor $(\Lambda/\xi)^{2(n-1)} \propto \omega^{-2(n-1)}$ in A_n depends on the vanishing of the factor containing the partition functions to all orders below $\omega^{2(n-1)}$. That A_3 is finite as $\omega \to 0$, for the gas of free anyons, was proved in reference [145], by means of the path integral expression for Z_3.

The computation of the free particle $G_{\mathcal{P}}$ coefficients by means of the harmonic oscillator regularization is slightly tricky, partly because the substitution rule $Ab_n \to \Lambda^2 b_n/(n\xi^2)$ involves a factor n, and partly because the scaling relation for the harmonic oscillator partition function, equation (179), in the limit $\omega \to 0$,

$$Z_1(L\beta) = \frac{Z_1(\beta)}{L^2} , \tag{328}$$

differs from the relation $Z_1(L\beta) = Z_1(\beta)/L$ we assumed when writing equation (320). The correct limit for obtaining the free particle $G_{\mathcal{P}}$ is

$$G_{\mathcal{P}} = \frac{n}{\prod_L L^{\nu_L}} \lim_{\omega\to 0} (F_{\mathcal{P}} + \cdots) {Z_1}^{\nu-1} . \tag{329}$$

8.5 Bosons and fermions

For bosons and fermions with no mutual interaction the probability generating functions can be factorized as

$$F_{\mathcal{P}} = \prod_L {F_L}^{\nu_L} , \tag{330}$$

where $F_L = 1$ for bosons and $F_L = (-1)^{L-1}$ for fermions. The factorization implies, by the equations (160) and (308), that

$$\Xi = \sum_{\mathcal{P}\in\mathcal{C}} \prod_L \frac{1}{\nu_L!} \left(\frac{z^L F_L Z_1(L\beta)}{L} \right)^{\nu_L} = \prod_L \exp\left(\frac{z^L F_L Z_1(L\beta)}{L} \right) . \tag{331}$$

This is nothing but the standard expression for the logarithm of the grand canonical partition function as a sum over the one-particle energy levels E_{1k}, for an ideal gas bosons or fermions,

$$\ln \Xi = \mp \sum_{k=1}^{\infty} \ln\left(1 \mp z\mathrm{e}^{-\beta E_{1k}}\right) = \pm \sum_{L=1}^{\infty} \frac{(\pm z)^L}{L} Z_1(L\beta) , \tag{332}$$

and it gives directly the cluster coefficients,

$$b_n = \frac{F_n Z_1(n\beta)}{nA} . \tag{333}$$

Thus we have $G_N = F_N = (\pm 1)^{N-1}$, and $G_{\mathcal{P}} = 0$ for every partition $\mathcal{P}$ containing two or more cycles.

In two dimensions we get, using equation (303), and in particular the scaling relation (305),

$$b_n = \frac{(\pm 1)^{n-1}}{n^2\Lambda^2} . \tag{334}$$

Similarly we get in dimension d, using equation (306),

$$b_n = \frac{(\pm 1)^{n+1}}{n(\Lambda\sqrt{n})^d} \,. \tag{335}$$

Note that the ideal gas cluster coefficients in any dimension are the same for bosons and fermions, except that the even numbered coefficients b_2, b_4, etc. have the opposite sign. This means that we can formally transform bosons into fermions and *vice versa* by substituting simultaneously $z \to -z$, $P \to -P$ and $\rho \to -\rho$. It follows that the virial coefficients possess the same symmetry as the cluster coefficients,

$$A_n^{\mathrm{F}} = (-1)^{n+1} A_n^{\mathrm{B}}, \tag{336}$$

with superscripts "B" for boson and "F" for fermion.

In particular, the density of the ideal gas of bosons or fermions in dimension $d = 2$ is

$$\rho = \sum_{n=1}^{\infty} n b_n z^n = \mp \frac{1}{\Lambda^2} \ln(1 \mp z) \,. \tag{337}$$

Hence the equation of state in two dimensions takes the form

$$\beta P = \sum_{n=1}^{\infty} b_n z^n = \pm \frac{1}{\Lambda^2} \sum_{n=1}^{\infty} \frac{\left(1 - \mathrm{e}^{\mp \Lambda^2 \rho}\right)^n}{n^2} = \pm \frac{1}{\Lambda^2} \operatorname{Li}_2\left(\mathrm{e}^{\mp \Lambda^2 \rho}\right) \,, \tag{338}$$

where Li_2 is the dilogarithm function [240].

Bose–Einstein condensation occurs in a three-dimensional ideal gas of bosons: above a certain critical density the extra particles condense in the lowest energy level and do not contribute to the pressure. The two-dimensional ideal boson gas almost, but not quite, shows the same phenomenon. In fact it has a finite pressure at infinite density,

$$P_\infty = \frac{\operatorname{Li}_2(0)}{\beta \Lambda^2} = \frac{\pi^2}{6 \beta \Lambda^2} = \frac{\pi m}{12 \hbar^2 \beta^2} \,. \tag{339}$$

From equation (338) we get that

$$\frac{\partial(\beta P)}{\partial \rho} = \frac{\pm \Lambda^2 \rho}{\mathrm{e}^{\pm \Lambda^2 \rho} - 1} = \sum_{N=0}^{\infty} \frac{B_n}{n!} \left(\pm \Lambda^2 \rho\right)^n \,, \tag{340}$$

where B_n is the n-th Bernoulli number [240]. Thus,

$$A_n = \frac{B_{n-1} \left(\pm \Lambda^2\right)^{n-1}}{n!} \,. \tag{341}$$

The function $\rho/(e^{\pm\Lambda^2\rho} - 1)$ is well behaved for all real values of ρ, but has poles in the complex ρ plane at $\Lambda^2\rho = 2n\pi i$, for any non-zero integer n. This means that equation (340) can be integrated to give P as a well defined function of ρ anywhere on the real axis, while the virial expansion converges for $\Lambda^2|\rho| < 2\pi$ but diverges for $\Lambda^2|\rho| > 2\pi$. Note that the sum in equation (338) has different convergence properties, it converges for bosons at any non-negative density ρ but for fermions only when $\Lambda^2\rho < \ln 2$.

The relation between the boson or fermion ideal gas virial coefficients in two dimensions, and the Bernoulli numbers, was derived by Sen and by Viefers [134, 147, 278]. In particular, the even numbered virial coefficients A_4, A_6, etc., all except A_2, vanish both for bosons and fermions. Since the odd numbered coefficients are the same for bosons and fermions, the only difference is the sign of A_2. The lowest coefficients are

$$A_2 = \mp\frac{\Lambda^2}{4}, \quad A_3 = \frac{\Lambda^4}{36}, \quad A_5 = -\frac{\Lambda^8}{3600}, \quad A_7 = \frac{\Lambda^{12}}{211\,680},$$
$$A_4 = A_6 = A_8 = \ldots = 0. \tag{342}$$

It may be instructive to rederive the results for bosons and fermions using the harmonic oscillator regularization. The one-particle partition function for the harmonic oscillator in d dimensions is

$$Z_1(\beta) = \left(\sum_{n=0}^{\infty} e^{-\xi\left(n+\frac{1}{2}\right)}\right)^d = \frac{1}{\left(2\sinh\left(\frac{\xi}{2}\right)\right)^d}, \tag{343}$$

with $\xi = \hbar\omega\beta$, as before. We may use equation (333), and translate from one regularization scheme to the other by the rule of substitution $Ab_n \to (\Lambda/(\sqrt{n}\xi))^d b_n$. This gives the same formula as before for the cluster coefficients of free bosons or fermions,

$$b_n = \left(\frac{\xi\sqrt{n}}{\Lambda}\right)^d \frac{(\pm 1)^{n+1} Z_1(n\beta)}{n} \underset{\omega\to 0}{\to} \frac{(\pm 1)^{n+1}}{n(\Lambda\sqrt{n})^d}. \tag{344}$$

8.6 Two anyons

The case of two anyons can be treated most simply by means of the harmonic oscillator regularization, since the energy spectrum is discrete and exactly known.

Let us define the periodic sawtooth function

$$\alpha(\theta + 2n\pi) = \frac{|\theta|}{\pi} \quad \text{for} \quad |\theta| \le \pi, \quad n = 0, \pm 1, \pm 2, \ldots \tag{345}$$

In terms of it the two-anyon partition function in a harmonic oscillator

potential is

$$Z_2(\beta) = \frac{\cosh((1-\alpha)\xi)}{\left(2\sinh\left(\frac{\xi}{2}\right)\right)^2 2\sinh^2\xi}\,, \tag{346}$$

where the factor $Z_1(\beta) = 1/(2\sinh(\xi/2))^2$ is due to the motion of the centre of mass. Note that, in spite of the fact that the function $\alpha = \alpha(\theta)$ is non-differentiable both at the boson point $\theta = 0$ and the fermion point $\theta = \pi$, the partition function Z_2, as a function of θ, is non-analytic only at the boson point.

From equation (346) and equation (327) we get the well-known result for the second virial coefficient of the ideal gas of anyons, in the limit $\xi \to 0$,

$$A_2 = \Lambda^2 \left(\frac{1}{4} - \frac{(1-\alpha)^2}{2}\right). \tag{347}$$

There are two partitions of two, one even, $1+1=2$, and one odd, $2=2$. According to equation (160) we may write Z_2 as a sum over these partitions,

$$Z_2(\beta) = \frac{1}{2} F_{11}(\theta)\,(Z_1(\beta))^2 + \frac{1}{2} F_2(\theta)\, Z_1(2\beta)\,. \tag{348}$$

By equation (157), $F_{11}(\theta+\pi) = F_{11}(\theta)$ and $F_2(\theta+\pi) = -F_2(\theta)$. The substitution $\theta \to \theta + \pi$ is essentially the supersymmetry transformation introduced by Sen [132, 133], and it changes α into $1-\alpha$. Hence, splitting the partition function in equation (346) in two parts, one even and one odd under the substitution $\alpha \to 1-\alpha$, we deduce that

$$F_{11}(\theta) = \frac{\cosh\left(\left(\alpha - \frac{1}{2}\right)\xi\right)}{\cosh\left(\frac{\xi}{2}\right)}\,, \qquad F_2(\theta) = -\frac{\sinh\left(\left(\alpha - \frac{1}{2}\right)\xi\right)}{\sinh\left(\frac{\xi}{2}\right)}\,. \tag{349}$$

From the probability generating functions F_{11} and F_2 we get the probability distributions of the winding number Q,

$$\begin{aligned} P_{11}(Q) &= \frac{2\xi\tanh\left(\frac{\xi}{2}\right)}{\xi^2 + (\pi Q)^2} \quad (Q \text{ even}), \\ P_2(Q) &= \frac{2\xi\coth\left(\frac{\xi}{2}\right)}{\xi^2 + (\pi Q)^2} \quad (Q \text{ odd}). \end{aligned} \tag{350}$$

Note that we have defined the winding number Q as *twice* the number of windings around the origin, so that the probability distribution of even winding numbers, $P_{11}(Q)$, is the same distribution as given by Wiegel and by Ouvry [130, 266].

If we do not distinguish between even and odd winding numbers, then we may write

$$Z_2(\beta) = F(\theta)\,\frac{\cosh\xi}{\left(2\sinh\left(\frac{\xi}{2}\right)\right)^2\,2\sinh^2\xi}\,, \tag{351}$$

where $F(\theta)$ is the total probability generating function for even and odd winding numbers,

$$F(\theta) = \sum_{Q=-\infty}^{\infty} P(Q)\,\mathrm{e}^{-i\theta Q} = \frac{\cosh((1-\alpha)\xi)}{\cosh\xi}\,, \tag{352}$$

and $P(Q)$ is the probability of the winding number Q,

$$P(Q) = \frac{\xi\,\tanh\xi}{\xi^2 + (\pi Q)^2}\,. \tag{353}$$

From the above expressions we also get the G coefficients of free anyons, by equation (329),

$$G_{11} = 2\lim_{\omega\to 0}(F_{11}-1)Z_1 = \alpha(\alpha-1)\,, \qquad G_2 = \lim_{\omega\to 0} F_2 = 1-2\alpha\,. \tag{354}$$

8.7 Three anyons

The harmonic oscillator regularization is useful also for three anyons, even though the "non-linear" part of the energy spectrum has to be computed numerically. The third virial coefficient A_3 has been calculated with high precision by the direct method of computing energy levels numerically and summing to get the partition function [122, 143, 144].

In order to compute the virial coefficient from the spectrum, we should incorporate all the exact knowledge we have. Therefore we subtract the known bosonic partition function Z_3^{B} and write the anyonic partition function as $Z_3 = Z_3^{\mathrm{B}} + \Delta Z_3$. Then we split the remainder further into contributions of energy levels that are linear and non-linear functions of the statistics angle θ, writing $\Delta Z_3 = \Delta Z_3^{\mathrm{L}} + \Delta Z_3^{\mathrm{NL}}$. The "linear" part is exactly known,

$$Z_3^{\mathrm{L}}(\beta) = \frac{\cosh(3(1-\alpha)\xi)}{32\sinh^2\left(\frac{\xi}{2}\right)\sinh^2\xi\sinh^2\left(\frac{3\xi}{2}\right)}\,, \tag{355}$$

and from Z_3^{L} we get ΔZ_3^{L} by subtracting the value for $\alpha = 0$. For the "non-linear" part the best we can do is to split off the Z_1 contribution due to the centre of mass, and the factor due to the tower structure of the spectrum, writing

$$\Delta Z_3^{\mathrm{NL}} = \frac{Z_1}{1-\mathrm{e}^{-2\xi}}\,\Delta\mathcal{Z}\,. \tag{356}$$

All in all we then get

$$A_3 = \lim_{\xi\to 0}\left(\frac{\Lambda}{\xi}\right)^4\left(2 - 10\,\frac{Z_2}{Z_1^2} + 16\,\frac{Z_2^2}{Z_1^4} - 6\,\frac{Z_3}{Z_1^3}\right)$$
$$= \Lambda^4\left(\frac{1}{36} - 3\lim_{\xi\to 0}\frac{\Delta\mathcal{Z}}{\xi}\right). \tag{357}$$

This is an important simplification, since $\Delta\mathcal{Z}$ has contributions only from "non-linear" states that are "bottom" states of infinite towers. All the θ dependence of the third virial coefficient is seen to come from these states.

One may calculate $\Delta\mathcal{Z}$ numerically by calculating the lowest energy levels, then summing over them and extrapolating to the infinite sum as best one can. The most accurate calculation, including all bottom of tower energy levels up to around $40\hbar\omega$, gave the result [122]

$$A_3 = \Lambda^2\left(\frac{1}{36} + \frac{\sin^2\theta}{12\pi^2} + a\sin^4\theta\right), \tag{358}$$

with a coefficient

$$a = -(1.652 \pm 0.012)\times 10^{-5} = -\frac{1}{(621\pm 5)\pi^4}\,. \tag{359}$$

The first and second order perturbative calculations give the exact coefficient of the $\sin^2\theta$ term, and it is a non-trivial consistency check that the numerical calculation reproduces the exact result with high precision. The coefficient of $\sin^4\theta$ could in principle be calculated exactly by fourth-order perturbation theory, although this possibility seems rather remote.

At this point some general observations might be in order. All observable properties of anyons must be periodic functions of θ with period 2π. Energy eigenvalues and eigenfunctions are analytic functions of θ, except that some are non-analytic at $\theta = 0$, varying like $|\theta|$ rather than θ. Hence the partition functions and all thermodynamic quantities derivable from them will be analytic functions of θ, even at the fermion point $\theta = \pi$, but generally not at the boson point $\theta = 0$.

In the absence of an external magnetic field, the theory is both time reversal and parity invariant if each of these transformations is defined so as to include a change in sign of θ. It follows that energy eigenvalues and thermodynamic quantities, as functions of θ, must be symmetric about $\theta = 0$, hence they are functions of the quantity $\alpha(\theta)$ defined in equation (345), which is non-analytic in θ at the boson and fermion points. Any even polynomial (or convergent power series) in α is analytic at the boson point, and any even polynomial in $1 - \alpha$ is analytic at the fermion point. An example is the exact second virial coefficient, which is an even polynomial in $1 - \alpha$.

The third virial coefficient is analytic at the boson as well as the fermion point, because it is "supersymmetric", *i.e.* symmetric under the substitution $\alpha \to 1-\alpha$ [132,133]. Being analytic everywhere, and periodic in θ with period π, it can not be polynomial in θ, because periodic polynomials do not exist, except constants. But it should be expandable as a rapidly converging power series in $\sin^2\theta$, as the above formula indicates. In fact, this is one possible way to represent a Fourier series with the desired properties.

The three-anyon partition function may be written as a sum over three partitions,

$$Z_3(\beta) = \frac{1}{6} F_{111}(\theta) \left(Z_1(\beta)\right)^3 + \frac{1}{3} F_3(\theta) Z_1(3\beta) + \frac{1}{2} F_{21}(\theta) Z_1(2\beta) Z_1(\beta) . \tag{360}$$

The first two terms are even and the last term is odd under the supersymmetry transformation of Sen, $\theta \to \theta + \pi$ [132,133]. As observed by Sen, the odd part of Z_3^{L}, which is

$$Z_3^{\mathrm{L}}(\beta,\alpha) - Z_3^{\mathrm{L}}(\beta,1-\alpha) = \frac{\cosh(3(1-\alpha)\xi) - \cosh(3\alpha\xi)}{32\sinh^2\left(\frac{\xi}{2}\right)\sinh^2\xi\sinh^2\left(\frac{3\xi}{2}\right)} , \tag{361}$$

when taken at $\alpha = 0$, completely accounts for the difference between the bosonic and the fermionic partition functions, which is simply $Z_1(2\beta)Z_1(\beta)$. Since the number of states does not vary with θ, it is therefore possible, and indeed true according to Sen, that the odd part of Z_3 is identical to the odd part of Z_3^{L} also for intermediate values of θ. We compute from this that

$$F_{21}(\theta) = -\frac{\sinh\left(\left(\alpha-\frac{1}{2}\right)3\xi\right)}{\sinh\left(\frac{3\xi}{2}\right)} . \tag{362}$$

Note the similarity between the two-particle and three-particle functions $F_2(\theta)$ and $F_{21}(\theta)$.

From F_{21} we get the exact G_{21} coefficient of free anyons, by equation (329),

$$G_{21} = \frac{3}{2} \lim_{\omega\to 0}(F_{21} - F_2)Z_1 = 2(1-2\alpha)\alpha(\alpha-1) = 2F_2\, G_{11} . \tag{363}$$

8.8 The Monte Carlo method

Although A_3 has been calculated with very high precision from numerical energy levels, it is useful to discuss here also how it can be calculated by the Monte Carlo method. This method is less precise for three anyons [145], but is so far the only one available for four anyons [146]. The Monte Carlo

method as such is important because it is a very natural approach to path integrals, and it leads to some general understanding of how they behave.

For a Monte Carlo calculation we need a more explicit expression for the propagator, as defined in equation (126). What we need here is only the propagator for the harmonic oscillator in two dimensions,

$$G(\boldsymbol{y},\boldsymbol{x};\tau) = \frac{m\omega}{2\pi\hbar\sinh(\omega\tau)} \exp\Big(-\frac{m\omega}{4\hbar}\Big(\tanh\Big(\frac{\omega\tau}{2}\Big)\,|\boldsymbol{y}+\boldsymbol{x}|^2 + \coth\Big(\frac{\omega\tau}{2}\Big)\,|\boldsymbol{y}-\boldsymbol{x}|^2\Big)\Big), \tag{364}$$

since the many-particle paths we want, may be reinterpreted as consisting of independent cyclic one-particle paths.

Consider one cyclic one-particle path from $\tau = 0$ to $\tau = L\hbar\beta$. In the path integral formula for the partition function $Z_1(L\hbar\beta)$ every point $\boldsymbol{x}(\tau)$ on the path is integrated out. If we integrate out every point except one, say the point $\boldsymbol{x}(0) = \boldsymbol{x}(L\hbar\beta) = \boldsymbol{x}$, then we get instead the propagator $G(\boldsymbol{x},\boldsymbol{x};L\hbar\beta)$. It follows that the probability density for the single point $\boldsymbol{x}$ on the path is

$$\rho(\boldsymbol{x}) = \frac{G(\boldsymbol{x},\boldsymbol{x};L\hbar\beta)}{Z_1(L\hbar\beta)} = \frac{m\omega}{\pi\hbar}\tanh\Big(\frac{L\xi}{2}\Big)\exp\Big(-\frac{m\omega}{\hbar}\tanh\Big(\frac{L\xi}{2}\Big)\,|\boldsymbol{x}|^2\Big). \tag{365}$$

This is a two-dimensional normal probability distribution of mean zero and standard deviation

$$\sigma_0 = \sqrt{\frac{\hbar}{2m\omega}\coth\Big(\frac{L\xi}{2}\Big)} \underset{\omega\to 0}{\longrightarrow} \frac{1}{\omega\sqrt{mL\beta}}\,. \tag{366}$$

Thus, loosely speaking, the point $\boldsymbol{x} = \boldsymbol{x}(0) = \boldsymbol{x}(L\hbar\beta)$ is located inside an area proportional to $1/\omega^2$ in the limit $\omega \to 0$.

By a similar reasoning, if we specify three imaginary times $\tau_a < \tau < \tau_b$ and the two points $\boldsymbol{x}(\tau_a) = \boldsymbol{x}_a$ and $\boldsymbol{x}(\tau_b) = \boldsymbol{x}_b$ on the path, then the probability density of the position $\boldsymbol{x}(\tau) = \boldsymbol{x}$ on the path is proportional to the product $G(\boldsymbol{x}_b,\boldsymbol{x};\tau_b-\tau)G(\boldsymbol{x},\boldsymbol{x}_a;\tau-\tau_a)$ of two propagators. Again this defines a normal distribution, of mean

$$\boldsymbol{x}_\tau = \frac{\sinh(\omega(\tau_b-\tau))\,\boldsymbol{x}_a + \sinh(\omega(\tau-\tau_a))\,\boldsymbol{x}_b}{\sinh(\omega(\tau_b-\tau_a))} \tag{367}$$

and standard deviation

$$\sigma_\tau = \sqrt{\frac{\hbar\sinh(\omega(\tau_b-\tau))\sinh(\omega(\tau-\tau_a))}{m\omega\sinh(\omega(\tau_b-\tau_a))}} \underset{\omega\to 0}{\longrightarrow} \sqrt{\frac{\hbar(\tau_b-\tau)(\tau-\tau_a)}{m(\tau_b-\tau_a)}}\,. \tag{368}$$

The fact that σ_τ tends to a finite, non-zero limit when $\omega \to 0$, means that the area covered by a single cyclic path tends to a finite limit.

The asymptotic behaviour of σ_0 and σ_τ in the limit $\omega \to 0$ means that the probability that two cycles overlap, tends to zero as ω^2. Furthermore, the probability that three cycles overlap simultaneously, tends to zero as ω^4. Hence the winding number distribution for the three distinct cycles belonging to the partition $1+1+1=3$ is determined by the two-particle windings, up to correction terms of order ω^4. Which means that we may write

$$F_{111}(\theta) = (F_{11}(\theta))^3 \left(1 + F_{111}^{(4)}(\theta)\xi^4 + \mathcal{O}(\xi^5)\right). \tag{369}$$

It also follows that the limit $F_3^{(0)}(\theta) = \lim_{\omega\to 0} F_3(\theta)$ is finite, so that

$$F_3(\theta) = F_3^{(0)}(\theta) + \mathcal{O}(\xi). \tag{370}$$

These results for the probability generating functions F_{111} and F_3, together with the exact result for F_{21}, implies that the third virial coefficient is finite,

$$A_3 = \Lambda^4 \left(\frac{1}{64} + \frac{7}{8}\left(\alpha - \frac{1}{2}\right)^2 + \frac{1}{4}\left(\alpha - \frac{1}{2}\right)^4 - F_{111}^{(4)}(\theta) - \frac{2}{9} F_3^{(0)}(\theta) \right). \tag{371}$$

The supersymmetry of Sen is manifest in this formula for A_3.

The equations (366, 367) and (368) are all we need in order to make a Monte Carlo simulation of the three-particle paths. The numerical results [145] suggested the following simple formula for $F_3^{(0)}(\theta)$, which has since been proved to all orders in perturbation theory [270],

$$F_3^{(0)}(\theta) = -\frac{1}{8} + \frac{9}{2}\left(\alpha - \frac{1}{2}\right)^2 = (1-3\alpha)\left(1 - \frac{3}{2}\alpha\right). \tag{372}$$

This formula, together with an estimate of the fourth order term $F_{111}^{(4)}(\theta)$, then gave the $\sin^2\theta$ interpolation between the second order perturbative results for bosons and fermions.

8.9 The path integral representation of the coefficients $G_{\mathcal{P}}$

The coefficient $G_{\mathcal{P}}$ for a given partition $\mathcal{P}$ representing a conjugation class in the symmetric group S_N, can be represented as a path integral over all paths inducing one given permutation represented by $\mathcal{P}$,

$$G_{\mathcal{P}} Z_1 = \mathcal{N}_{\mathcal{P}} \int \mathcal{D}(\boldsymbol{x}_1(\tau), \ldots, \boldsymbol{x}_N(\tau)) \exp\left(-\frac{S}{\hbar}\right) g_{\mathcal{P}} \,. \tag{373}$$

Here $\boldsymbol{x}_j(\tau)$ is the path of particle j, as a function of the imaginary time τ, and S is the free particle action in imaginary time,

$$S = \sum_{j=1}^{N} \int_0^{\hbar\beta} \mathrm{d}\tau \, \frac{m}{2} \left| \frac{\mathrm{d}\boldsymbol{x}_j(\tau)}{\mathrm{d}\tau} \right|^2 . \tag{374}$$

This representation is useful for numerical computation by the Monte Carlo method, but can also be used in order to prove that the cluster coefficients are finite.

We include the Gaussian factor $\exp(-S/\hbar)$ as part of the integration measure, so that it is the integrand $g_{\mathcal{P}}$ alone that represents the interaction of the particles, and we include a normalization factor $\mathcal{N}_{\mathcal{P}}$ so that $G_{\mathcal{P}} = {Z_1}^{\nu-1}$ if $g_{\mathcal{P}} = 1$ identically. Note that $\mathcal{N}_{\mathcal{P}}$ is then finite (*i.e.* A independent), since the path integral is proportional to ${Z_1}^{\nu}$ when $g_{\mathcal{P}} = 1$. Note also that this path integral representation is actually very general, and can be applied to any N-particle system with (short range) interactions in any dimension, not just to the N-anyon system considered here.

To see what the integrand $g_{\mathcal{P}}$ looks like in our case, let us take the partition 2+1+1 of 4 as an example. A closed path in the four-particle configuration space interchanges the positions of two particles, say particles 1 and 2, and takes the remaining two particles back to their starting points. The total winding number Q is the sum of six pairwise winding numbers,

$$Q = Q_{12} + (Q_{13} + Q_{23}) + (Q_{14} + Q_{24}) + Q_{34} \,. \tag{375}$$

Note that Q_{12} is an odd integer and Q_{34} an even integer (remember that the winding numbers are defined such that a complete revolution corresponds to the winding number 2), whereas $Q_{13}, Q_{23}, Q_{14}, Q_{24}$ are in general non-integer, because particles 1 and 2 do not return to their starting positions. However, the sums $Q_{(12)3} = Q_{13} + Q_{23}$ and $Q_{(12)4} = Q_{14} + Q_{24}$ are even integers. Hence Q is an odd integer. Let I be any subscript, and introduce the notation

$$e_I = 1 + f_I = \exp(-i\theta Q_I) \,. \tag{376}$$

In order to compute the coefficient $G_{211}Z_1 = (F_{211} - 2F_{21} - F_2F_{11} + 2F_2){Z_1}^3$ we take the integrand to be

$$\begin{aligned} g_{211} &= e_{12}\, e_{(12)3}\, e_{(12)4}\, e_{34} - e_{12}\, e_{(12)3} - e_{12}\, e_{(12)4} - e_{12}\, e_{34} + 2e_{12} \\ &= e_{12} \big(f_{(12)3}\, f_{(12)4}\, f_{34} + f_{(12)3}\, f_{(12)4} \\ &\quad + f_{(12)3}\, f_{34} + f_{(12)4}\, f_{34} \big) \,. \end{aligned} \tag{377}$$

For example, we compute $F_{211}{Z_1}^3$ by integrating

$$\exp(-i\theta Q) = e_{12}\, e_{(12)3}\, e_{(12)4}\, e_{34} \,, \tag{378}$$

and we compute $2F_{21}{Z_1}^3$ by integrating

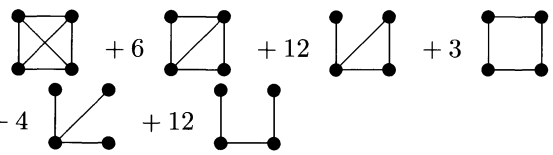
$$\begin{aligned} \exp(-i\theta(Q_{12} + Q_{(12)3})) + \exp(-i\theta(Q_{12} + Q_{(12)4})) \\ = e_{12}\, e_{(12)3} + e_{12}\, e_{(12)4} \,. \end{aligned} \tag{379}$$

Equation (377) may be represented diagrammatically as

$G_{211}Z_1 =$  (380)

The particles are represented as points (filled circles). The two-cycle is represented by e_{12} in the integrand and by a circle connecting two particles in the corresponding diagram. Each factor f_1 in the integrand is drawn as a single straight line in the diagram. Note that we should draw *labelled* graphs to represent the four terms in equation (377). But since the value of a graph is independent of the labelling, it is more natural to draw unlabelled graphs and include instead integer coefficients counting the number of ways each graph can be labelled. Hence the factor 2 in front of the last graph.

In a similar way we find the diagrammatic representation

$G_{1111}Z_1 =$  (381)

The coefficient in front of each diagram is again the number of inequivalent ways of labelling the nodes of the graph. We may also write

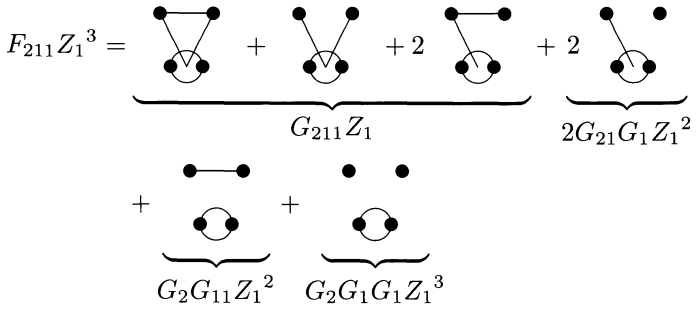
$G_{22}Z_1 =$  , $G_{31}Z_1 =$  (382)

We see that only connected diagrams contribute to the cluster coefficients. It follows that the latter are finite in the limit $A \to \infty$. Indeed, any path gives a non-zero contribution to the path integral represented by some diagram only if for every line in the diagram, the corresponding winding number is non-zero. The probability for this to happen for a connected diagram goes to zero as $(\Lambda^2/A)^{\nu-1}$ when $A \to \infty$, since every L-cycle path gives a Gaussian distribution of points which essentially covers only a finite area, proportional to Λ^2. Here ν is the number of cycles, and $\nu-1$ is the minimum number of links in a connected graph with ν nodes. The factor $A^{-\nu+1}$ cancels exactly the divergence of the factor ${Z_1}^{\nu-1}$ included in the definition of $G_{\mathcal{P}}$, equation (319).

The general meaning of the relations between the F and G coefficients should now be obvious. $F_{\mathcal{P}}$ is a sum of both connected and disconnected diagrams, whereas $G_{\mathcal{P}}$ is the part of the sum including only the connected diagrams. For example, the relation

$$F_{211}{Z_1}^3 = G_{211}Z_1 + 2G_{21}G_1{Z_1}^2 + G_2G_{11}{Z_1}^2 + G_2G_1G_1{Z_1}^3 \,, \qquad (383)$$

which follows from (319), is represented as



(384)

It is the last term that dominates in the thermodynamic limit, but it is G_{211} only that contributes to the cluster coefficient. Thus, as usual, the grand partition function is a sum of all diagrams but the thermodynamic potential is a sum of connected diagrams [279].

The Monte Carlo method consists in generating random paths according to the Gaussian distribution of paths valid for bosons. Each four-particle path is closed over the imaginary time interval $\hbar\beta$, in the sense that the final configuration is identical to the initial one, but with the particle positions interchanged by a permutation belonging to the class $\mathcal{P} \subset S_N$. Consider the partition $2 + 1 + 1 = 4$, as in the example above. Then particles 1 and 2 should interchange positions, while particles 3 and 4 should return to their starting points. We take, arbitrarily, the starting point for the path of particle 1 to be at the origin, this is then also the ending point for particle 2. Equivalently, it is the ending point for particle 1 over the imaginary time interval $2\hbar\beta$. The starting point for particle 2, equal to the position of particle 1 after half the imaginary time interval $2\hbar\beta$, can then be generated according to a Gaussian distribution around the origin. The starting and ending point for particle 3 is generated according to a flat distribution inside a square area A centered on the origin. Similarly for particle 4.

For each four-particle path generated we count the winding numbers Q_{12}, $Q_{(12)3}$, $Q_{(12)4}$, Q_{34} and increment a histogram $n(Q)$ in the following way. We compute the total winding number Q and add 1 to $n(Q)$, this takes care of the integrand $e_{12}\, e_{(12)3}\, e_{(12)4}\, e_{34}$. We subtract 1 from $n(Q_{12} + Q_{(12)3})$, in order to take care of the integrand $-e_{12}\, e_{(12)3}$. Similarly, we subtract 1 from $n(Q_{12} + Q_{(12)4})$ and from $n(Q_{12} + Q_{34})$, and we add 2 to $n(Q_{12})$. Finally, G_{211} is the Fourier transform of the histogram $n(Q)$, multiplied by the normalization factor ${Z_1}^2/n$, where n is the total number of four-particle paths generated. The net contribution to the histogram vanishes if more than one of the three winding numbers $Q_{(12)3}$, $Q_{(12)4}$ and Q_{34} is zero, and this is what ensures a finite limit as $A \to \infty$ for the computed G_{211}.

8.10 Exact and approximate polynomials

The first cluster coefficient, with our definition, is $b_1 = 1/\Lambda^2$. By the harmonic oscillator regularization we found that

$$G_{11} = \alpha(\alpha - 1) , \qquad G_2 = F_2 = 1 - 2\alpha . \tag{385}$$

G_{11} is even and G_2 is odd under the substitution $\alpha \to 1 - \alpha$, and together they give the second cluster coefficient of free anyons,

$$\Lambda^2 b_2 = \frac{G_{11}}{2} + \frac{G_2}{4} = \frac{1}{2}(1-\alpha)^2 - \frac{1}{4} \cdot \tag{386}$$

We also obtained the exact result

$$G_{21} = 2F_2 G_{11} . \tag{387}$$

The third virial coefficient,

$$A_3 = -2\frac{b_3}{{b_1}^3} + 4\frac{{b_2}^2}{{b_1}^4} , \tag{388}$$

is even under $\alpha \to 1-\alpha$, since the odd part of $-2b_3/{b_1}^3$, which is $-\Lambda^4 G_{21}/2$, cancels the odd part of $4{b_2}^2/{b_1}^4$, which is $\Lambda^4 G_{11} G_2$.

One further result [280], which is exact according to the perturbative calculation of reference [270], is

$$F_L = \prod_{k=1}^{L-1} \left(1 - \frac{L\alpha}{k}\right) \cdot \tag{389}$$

Various Monte Carlo simulations for single cycles of different lengths are consistent with this formula.

Note that only harmonic oscillator energy levels depending linearly on α contribute to the three quantities $G_2 = F_2$, G_{11} and G_{21}. The computation of $G_L = F_L$ for $L > 2$ by perturbation theory is much more non-trivial, because also states with non-linear α dependence contribute. This is essentially the only exact result known for the non-linear energy levels.

This is about as far as one can get with exact results. However, in the diagrammatic expansions shown above, one may argue quite generally that the tree graphs are expected to dominate, because every additional line in a diagram represents another factor of the type $f_I = \exp(-i\theta Q_I) - 1$ in the integrand, with Q_I an even integer. This factor vanishes when $Q_I = 0$, which will happen with a certain probability which is definitely non-zero, and even if it does not vanish it will often have an absolute value smaller than 1. Furthermore, one may argue that the path integral represented by a tree graph should approximately factorize in the same way as its integrand.

These two assumptions, of tree diagram dominance and factorization, lead in a not entirely trivial way to the following polynomial approximation for the general coefficient $G_{\mathcal{P}}$,

$$G_{\mathcal{P}} \simeq \widetilde{G}_{\mathcal{P}} = N^{\nu-2}\, {G_{11}}^{\nu-1} \prod_{L} (LF_L)^{\nu_L} \,. \tag{390}$$

There is a factor F_L for every cycle of length L, a factor $L_1 L_2 G_{11}$ for every single line connecting two different cycles of lengths L_1 and L_2 (each L-factor counts the number of ways the line can be connected to the cycle), and there is a sum over all $\nu^{\nu-2}$ possible ways to connect the cycles into a tree graph. It is perhaps not obvious how this leads to equation (390), but a simple way to understand the connection is by looking at low order examples: consider the case of 3 cycles of lengths L_1, L_2, and L_3. They can be connected to a tree graph in 3 possible ways. This gives a coefficient

$$F_{L_1} F_{L_2} F_{L_3} \,(L_1 L_2 G_{11}\, L_2 L_3 G_{11} + L_2 L_3 G_{11}\, L_3 L_1 G_{11} \tag{391}$$

$$+ L_3 L_1 G_{11}\, L_1 L_2 G_{11}) = (L_1 + L_2 + L_3)\, {G_{11}}^2 \prod_{i=1}^{3} L_i F_{L_i} \,, \tag{392}$$

which agrees with equation (390) since $L_1 + L_2 + L_3 = N$. We should point out that equation (390) was first derived empirically as an approximate representation of the Monte Carlo results.

Special cases where these polynomial formulae are exact, as already mentioned, are the cyclic coefficients $G_L = F_L$, as well as G_{11} and $G_{21} = 2F_2\, G_{11}$. In the three-particle case there is one approximate polynomial,

$$\widetilde{G}_{111} = 3{G_{11}}^2 \,. \tag{393}$$

The four-particle approximate polynomials are:

$$\begin{aligned} \widetilde{G}_{1111} &= 16{G_{11}}^3 \,, \quad \widetilde{G}_{211} = 8F_2{G_{11}}^2 \,, \\ \widetilde{G}_{22} &= 4{F_2}^2 G_{11} \,, \quad \widetilde{G}_{31} = 3F_3 G_{11} \,. \end{aligned} \tag{394}$$

The polynomial approximations for the G coefficients imply the following polynomial approximations for the cluster coefficients,

$$\Lambda^2 \widetilde{b}_N = \frac{(-1)^{N-1}}{N^2} \prod_{k=1}^{N-1} \left(1 - \frac{N(1-\alpha)^2}{k}\right) = \frac{1}{N^2} \prod_{k=1}^{N-1} \left(1 - \frac{Ng}{k}\right), \tag{395}$$

which imply that the virial coefficients are independent of the statistics, except for the second coefficient,

$$A_2 = \Lambda^2 \left(\frac{1}{4} - \frac{(1-\alpha)^2}{2}\right) = \Lambda^2 \left(-\frac{1}{4} + \frac{g}{2}\right). \tag{396}$$

The parameter g is defined here by

$$g = 1 - (1 - \alpha)^2 \,. \tag{397}$$

One nice property of these polynomials is that they are analytic functions of θ at the fermion point, as the exact cluster coefficients must be. However, they do not give the correct second derivatives at the boson and fermion points, known from perturbation theory, although they do give the correct first derivatives. An alternative way to introduce the same polynomials is to postulate that the second virial coefficient is given by equation (396), while all higher virial coefficients are independent of α. That is, these are just cluster coefficients for two-dimensional exclusion statistics [236], with the statistics parameter given by (397). The corresponding second order diagrams were identified in reference [138].

This correspondence with exclusion statistics is of course only approximate, and it is well known from perturbation theory that the higher virial coefficients of anyons all have a second order variation with θ at the boson and fermion points. Nevertheless it might be interesting to understand better the deeper reasons behind, if any.

8.11 The fourth virial coefficient of anyons

Since the third virial coefficient is analytic in θ everywhere, in contrast to the cluster coefficients, which are all non-analytic at the boson point, one may be bold enough to conjecture that all virial coefficients, with the exception of A_2, are analytic functions of θ. If A_4 is analytic, then it must have the form

$$A_4 = \Lambda^6 \left[\frac{\sin^2\theta}{16\pi^2} \left(\frac{1}{\sqrt{3}} \ln\left(\sqrt{3}+2\right) + \cos\theta \right) \right.$$
$$\left. + \sin^4\theta \left(c_4 + d_4 \cos\theta\right) + \cdots \right], \tag{398}$$

where the coefficients of the lowest order terms are fixed by perturbation theory at the boson and fermion points. A Monte Carlo calculation of A_4 gave indeed a result which was fitted to this form with no more than two parameters [146],

$$c_4 = -0.0053 \pm 0.0003 \,, \qquad d_4 = -0.0048 \pm 0.0009 \,. \tag{399}$$

This fit is shown in Figure 13.

It is rather remarkable how nearly constant the computed A_4 is, that is, how close to zero it is for all values of θ. In fact, it is closer to zero than the minimal Fourier series with $c_4 = d_4 = c_6 = d_6 = \ldots = 0$. This is one particular example of how well the anyon system realizes approximately Haldane's

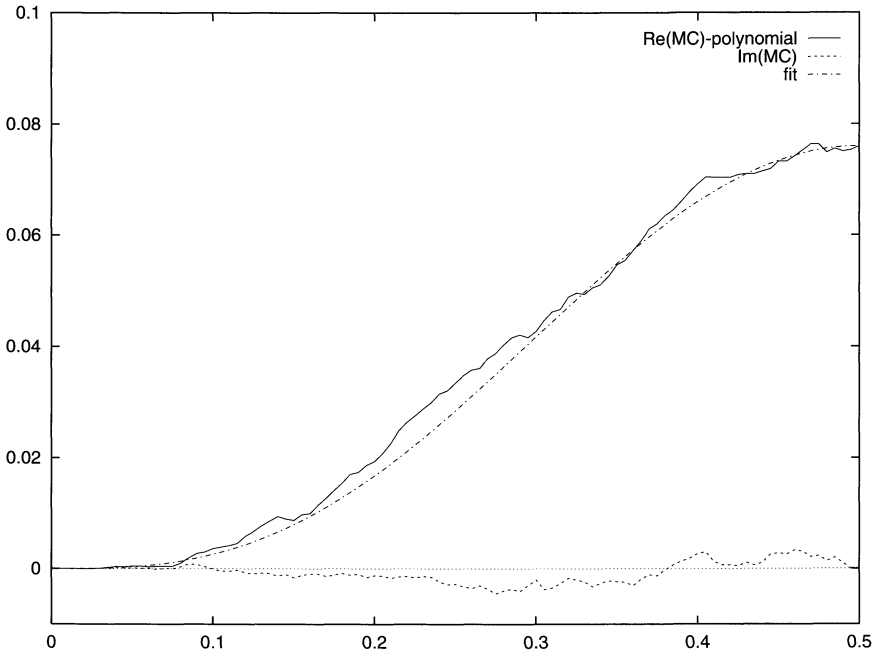


Fig. 7. $G_{1111} - 16(\alpha(\alpha-1))^3$ as a function of α. The imaginary part is plotted to indicate the statistical uncertainty of the real part of the Monte Carlo data. Only the interval $0 \leq \alpha \leq 1/2$ is plotted, because of the (anti)symmetry about $\alpha = 1/2$. The curve marked "fit" is given in equation (400). From [146].

so-called exclusion statistics [227], characterized by a continuously variable parameter g, for which only the second virial coefficient depends on g, in two dimensions [236].

The computed G coefficients are plotted in Figures 7 to 11, as functions of α.

In each case we subtract the polynomial approximation, which is the main contribution, and plot only the difference, marked by "Re(MC) − polynomial" in the figures. Because of the statistical errors, the Monte Carlo generated curve has also a non-zero imaginary part, marked "Im(MC)", which is useful because it indicates the statistical errors in the real part. Since the real part is even about $\alpha = 1/2$ and the imaginary part is odd, or vice versa, depending on whether the partition is even or odd, only the interval $0 \leq \alpha \leq 1/2$ is plotted in all figures.

Figure 7 shows the computed G_{1111} with the polynomial $16{G_{11}}^3 = 16(\alpha(\alpha-1))^3$ subtracted. The curve marked "fit" is mostly empirical, and

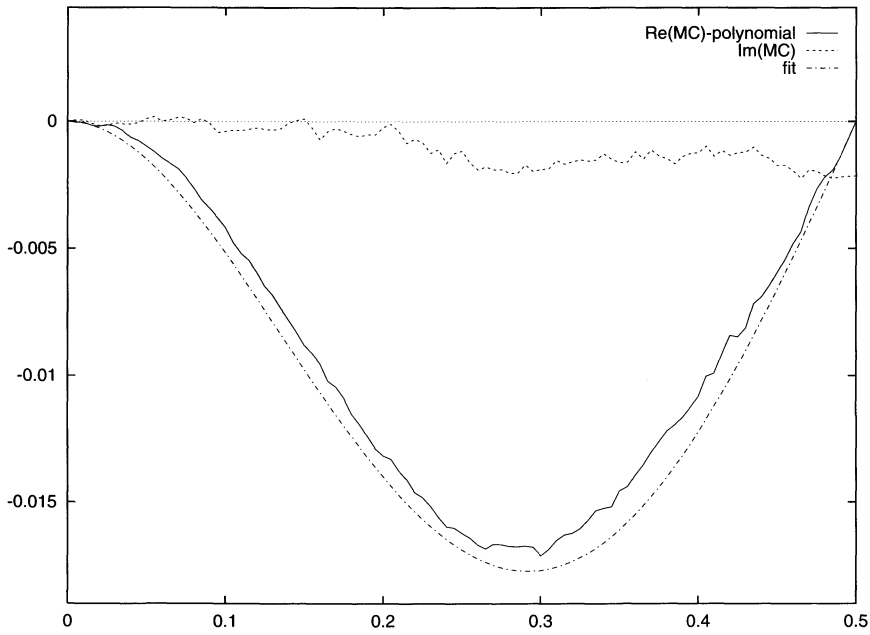


Fig. 8. $G_{211} - 8(1-2\alpha)(\alpha(\alpha-1))^2$ *versus* α. The curve marked "fit" is given in equation (401). From [146].

is given by

$$\text{fit} = -\frac{3}{\pi^2}\,\alpha(\alpha-1)\sin^2(\alpha\pi)\ . \tag{400}$$

The figure shows that this is a perfect fit to the Monte Carlo curve, within the statistical uncertainty as indicated by the imaginary part.

Figure 8 shows the computed G_{211} with the polynomial $8{F_2}{G_{11}}^2 = 8(1-2\alpha)(\alpha(\alpha-1))^2$ subtracted. The curve marked "fit" is partly empirical, but with a coefficient which is chosen so as to produce the correct second order derivative at $\alpha = 0$ [270]. The formula is:

$$\text{fit} = -\frac{2}{3\pi^2}\,(1-2\alpha)\sin^2(\alpha\pi)\ . \tag{401}$$

Figure 9 shows the computed G_{22} with $4{F_2}^2 G_{11} = 4(1-2\alpha)^2\alpha(\alpha-1)$ subtracted. The "fit" here is

$$\text{fit} = \frac{2}{\sqrt{3}\pi^2}\,\ln(\sqrt{3}+2)\sin^2(\alpha\pi)\cos^2(\alpha\pi)\ . \tag{402}$$

Figure 10 shows the computed G_{31} with $3F_3G_{11} = 3(1-3\alpha)(1-(3/2)\alpha)\alpha$

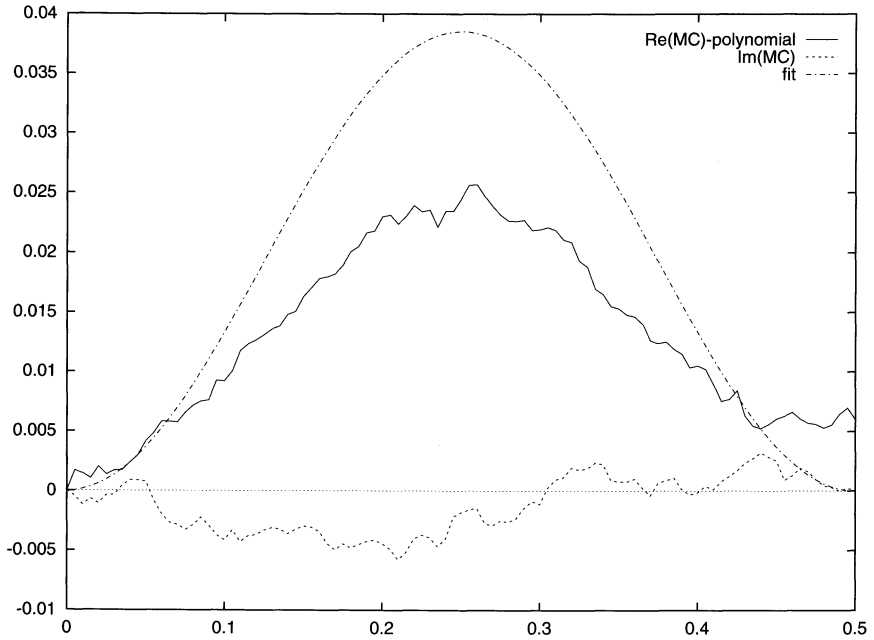


Fig. 9. $G_{22} - 4(1-2\alpha)^2\alpha(\alpha-1)$ *versus* α. The "fit" here is given in equation (402). From [146].

$(\alpha - 1)$ subtracted. Here we have chosen

$$\text{fit} = \frac{\sqrt{3}}{4\pi^2} \ln(\sqrt{3}+2) \sin^2(\alpha\pi) \cos^2(\alpha\pi) \ . \tag{403}$$

Figure 11 shows the computed $G_4 = F_4$ with the polynomial $(1 - 4\alpha)$ $(1 - 2\alpha)(1 - (4/3)\alpha)$ subtracted. The figure supports the claim that the polynomial is exact. Figure 12 shows the computed cluster coefficient, $\Lambda^2 b_4$ with the polynomial $\Lambda^2 b_4$ of equation (395) subtracted. The parabolas given by the second order perturbation theory at $\alpha = 0$ and $\alpha = 1$ are shown.

Figure 13 shows the computed virial coefficient, A_4/Λ^6. The parabolas given by the second order perturbation theory at $\alpha = 0$ and $\alpha = 1$ are shown. Also plotted are two Fourier series, as given in equation (398). The curve marked "Fourier 1" is a minimal Fourier series having only the two terms required by perturbation theory, *i.e.* $c_4 = d_4 = \ldots = 0$. The curve marked "Fourier 2" is a least squares fit with the coefficients $c_4 = -0.0053$ and $d_4 = -0.0048$. The minimal Fourier series is seen to be inconsistent with the Monte Carlo curve, unless there are important systematical errors that dominate over the statistical errors.

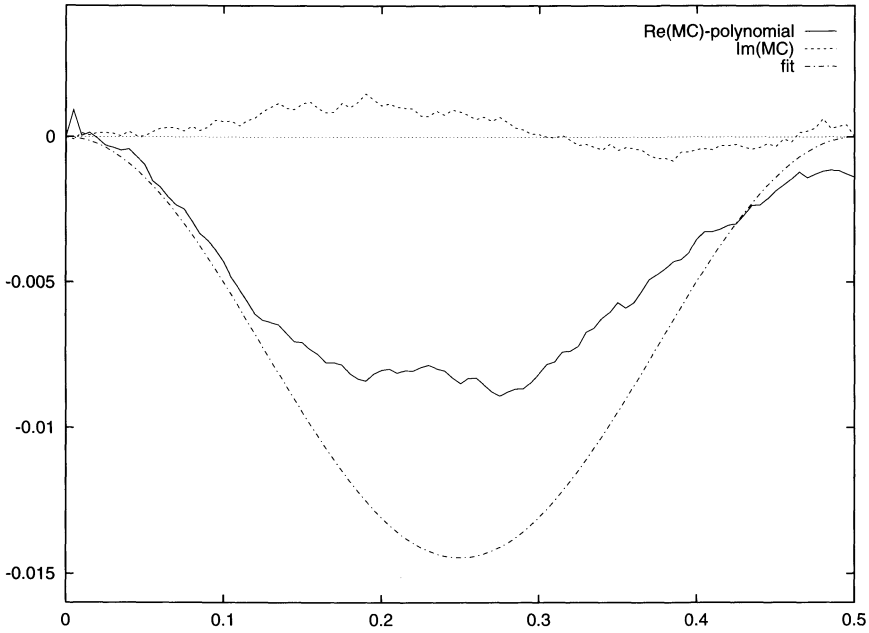


Fig. 10. $G_{31} - 3(1-3\alpha)(1-(3/2)\alpha)\alpha(\alpha-1)$ *versus* α. The "fit" is given in equation (403). From [146].

8.12 Two polynomial theorems

As a mathematical appendix to the present Section we will prove the result that the polynomial approximation (395) for the cluster coefficients is equivalent to a virial expansion which is the same as for the two-dimensional free non-relativistic Bose gas, except that the second virial coefficient is modified according to equation (396). We will also prove that the polynomial approximation (390) for $G_{\mathcal{P}}$ implies equation (395).

For simplicity we fix the temperature and choose units such that $\beta = \Lambda = 1$. Thus, *e.g.*, the fugacity is $z = \mathrm{e}^{\mu}$. We make use of the expansions

$$\begin{aligned} \rho &= \frac{\mathrm{d}P}{\mathrm{d}\mu} = \sum_{N=1}^{\infty} N b_N z^N \,, \\ \frac{\mathrm{d}\mu}{\mathrm{d}\rho} &= \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho} = \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho} = \sum_{N=1}^{\infty} N A_N \rho^{N-2} \,. \end{aligned} \tag{404}$$

We also define

$$\rho_g(\mu) = \sum_{N=1}^{\infty} \frac{z^N}{N} \prod_{k=1}^{N-1} \left(1 - \frac{Ng}{k}\right) = \sum_{N=1}^{\infty} z^N \frac{(-1)^{N-1}}{Ng} \binom{Ng}{N}, \tag{405}$$

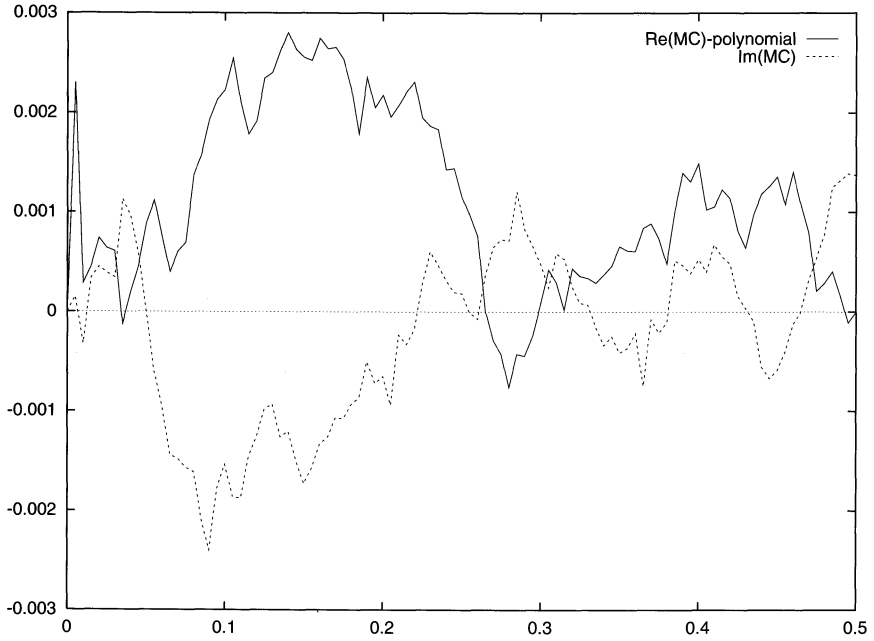


Fig. 11. $F_4 - (1-4\alpha)(1-2\alpha)(1-(4/3)\alpha)$ *versus* α. From [146].

which is the density corresponding to the cluster coefficients $\widetilde{b}_N$ of equation (395). For the Bose gas, with $g = 0$, we have $\rho = -\ln(1-z)$, or equivalently, $\mu = \ln z = \ln(1-\mathrm{e}^{-\rho})$. Shifting the second virial coefficient by an amount $\Delta A_2 = g/2$ then gives

$$\mu = \ln\left(1-\mathrm{e}^{-\rho}\right) + g\rho\,. \tag{406}$$

For every $g > 0$ and every μ, or for $g = 0$ and every $\mu < 0$, this equation clearly has a unique solution $\rho > 0$. We want to prove that the solution is $\rho = \rho_g(\mu)$.

For this purpose we rewrite equation (406) as

$$\rho = -\ln\left(1-z\mathrm{e}^{-g\rho}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n}\,\mathrm{e}^{-ng\rho}\,, \tag{407}$$

and apply the following theorem due to Lagrange (see Vol. 1, pp. 404–405 of [281], or [282]): The equation $\rho = f(\rho)$ has the solution

$$\rho = \sum_{M=1}^{\infty} \frac{1}{M!} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{M-1} f(r)^M \bigg|_{r=0} \tag{408}$$

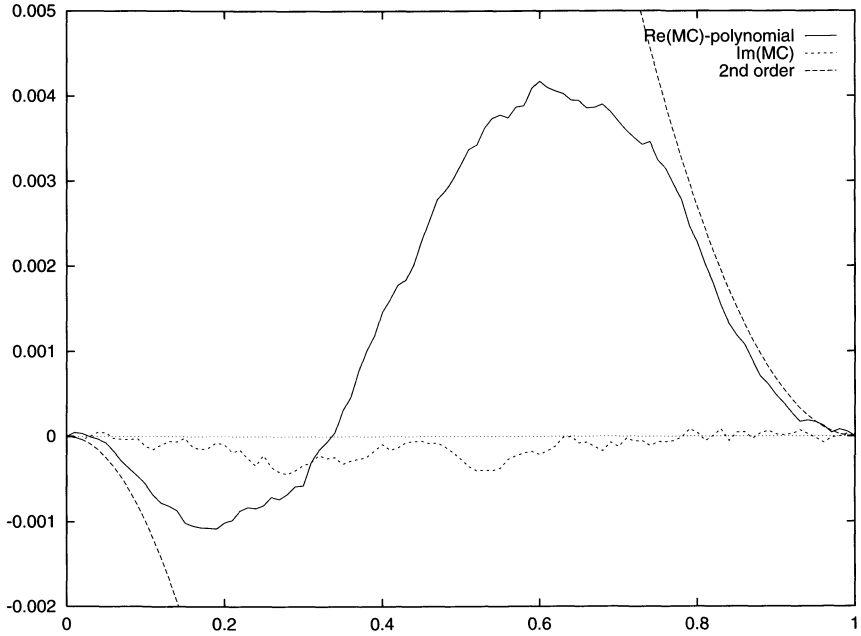


Fig. 12. The fourth cluster coefficient minus the polynomial of equation (395), $\Lambda^2(b_4 - \widetilde{b}_4)$, as a function of α. Also shown are the parabolas given by the second order perturbation theory at $\alpha = 0$ and $\alpha = 1$. From [146].

This gives

$$\rho = \sum_{M=1}^{\infty} \frac{1}{M!} \sum_{n_1=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} \frac{z^{n_1+\cdots+n_M}}{n_1 \cdots n_M} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{M-1} \mathrm{e}^{-(n_1+\cdots+n_M)gr}\Big|_{r=0}$$
$$= \sum_{N=1}^{\infty} z^N \sum_{M=1}^{N} (-Ng)^{M-1} C_{N,M} \,, \tag{409}$$

where

$$C_{N,M} = \frac{1}{M!} \sum_{n_1=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} \frac{\delta_{n_1+\cdots+n_M,N}}{n_1 \cdots n_M} \,. \tag{410}$$

What we need to show is that

$$\sum_{M=1}^{N} (-Ng)^{M-1} C_{N,M} = \frac{(-1)^{N-1}}{Ng} \binom{Ng}{N} \,. \tag{411}$$

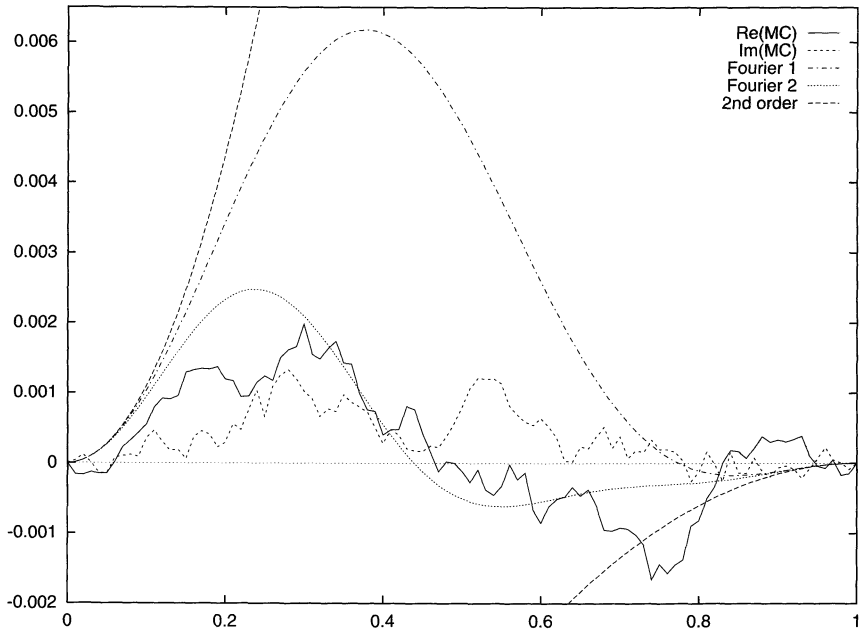


Fig. 13. The fourth virial coefficient, A_4/Λ^6, as a function of α. Also plotted are the parabolas given by the second order perturbation theory at $\alpha = 0$ and $\alpha = 1$, and two different Fourier series, as given in equation (398). The curve marked "Fourier 1" has $c_4 = d_4 = \ldots = 0$, whereas "Fourier 2" is the least squares fit with $c_4 = -0.0053$, $d_4 = -0.0048$. From [146], reprinted with permission.

It is straightforward to show that

$$\sum_{N=1}^{\infty} z^N \sum_{M=1}^{N} g^M C_{N,M} = \mathrm{e}^{-g\ln(1-z)} - 1 = \sum_{N=1}^{\infty} (-z)^N \binom{-g}{N}, \qquad (412)$$

and hence,

$$\sum_{M=1}^{N} g^{M-1} C_{N,M} = \frac{(-1)^N}{g} \binom{-g}{N}. \qquad (413)$$

Substituting $g \to -Ng$ we get equation (411), completing the proof.

We next turn to the cluster coefficients

$$b'_N = \sum_{\mathcal{P}\in\mathcal{C}_N} N^{\nu-2} {G_{11}}^{\nu-1} \prod_L \frac{{F_L}^{\nu_L}}{\nu_L!\, L^{\nu_L}}, \qquad (414)$$

given by the polynomial approximation in equation (390). We want to prove that $b'_N = \widetilde{b}_N$.

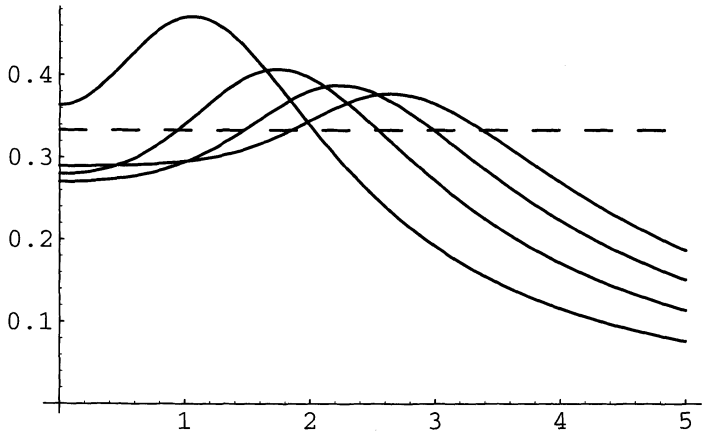


Fig. 14. $\mathrm{d}(\ln I_1)/\mathrm{d}\rho^2$ as a function of ρ. This quantity measures the charge ratio q_1/e, see equation (527). The curves are for $N = 2, 3, 4, 5$ electrons, the leftmost peak for $N = 2$ and the rightmost peak for $N = 5$. The dashed line is 1/3.

We may rewrite the above formula as

$$b'_N = \sum_{\nu=1}^{N} \frac{N^{\nu-2}\, {G_{11}}^{\nu-1}}{\nu!} \sum_{n_1=1}^{\infty} \cdots \sum_{n_\nu=1}^{\infty} \delta_{n_1+\cdots+n_\nu,N} \prod_{j=1}^{\nu} \frac{F_{n_j}}{n_j} \,. \tag{415}$$

To evaluate $\rho = \sum_{N=1}^{\infty} N b'_N z^N$ we insert equation (415), interchange the summation order of N and ν and use the relations $N^{\nu-1} z^N = (\mathrm{d}/\mathrm{d}\mu)^{\nu-1} z^N$ and $\sum_{n=1}^{\infty} z^n F_n/n = \rho_\alpha(\mu)$. We find

$$\begin{aligned} \rho &= \sum_{\nu=1}^{\infty} \frac{{G_{11}}^{\nu-1}}{\nu!} \left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^{\nu-1} (\rho_\alpha(\mu))^\nu \\ &= \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\nu-1} (\rho_\alpha(\mu + G_{11} r))^\nu \Big|_{r=0} \,. \end{aligned} \tag{416}$$

By the Lagrange theorem, equation (416) is the solution to the equation $\rho = \rho_\alpha(\mu + G_{11}\rho)$, which, as we saw above, is equivalent to

$$\mu + G_{11}\rho = \ln\left(1 - \mathrm{e}^{-\rho}\right) + \alpha\rho \,. \tag{417}$$

This is precisely equation (406) with $g = \alpha - G_{11} = 1 - (1-\alpha)^2$, which means that $b'_N = \widetilde{b}_N$ with $\widetilde{b}_N$ as given in equation (395).

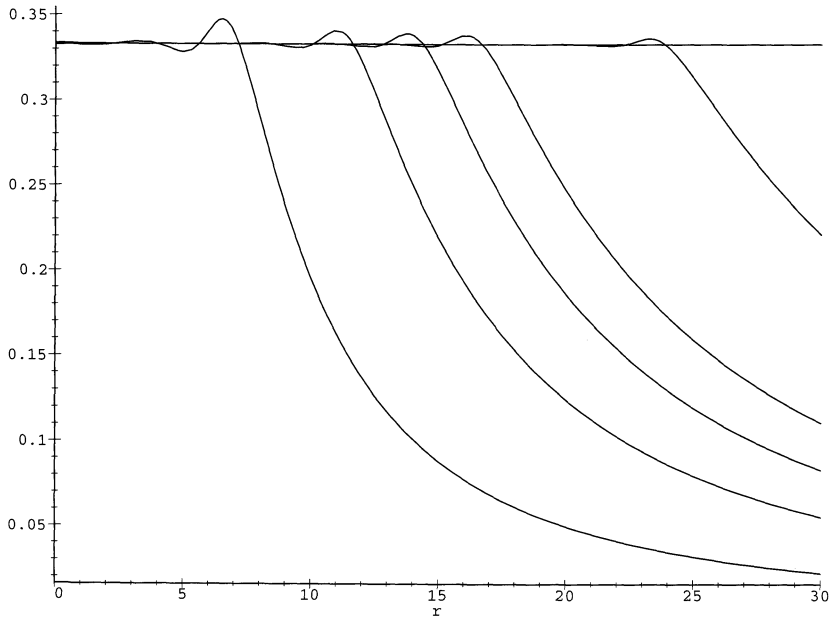


Fig. 15. The quasi-hole charge q_1/e, equation (527), as a function of ρ, the quasi-hole distance from the origin. The curves are, from left to right, for 20, 50, 75, 100 and 200 electrons. The horizontal line is 1/3. From [286], reprinted with permission.

9 Charged particles in a constant magnetic field

The quasi-particle excitations in the fractional quantum Hall system is so far the best (and maybe the only) experimental realization of anyons. The electrons of the two-dimensional electron gas, as well as the anyon-like quasi-particles, are electrically charged and therefore strongly influenced by the magnetic field. If the field is sufficiently strong, it effectively "freezes out" one degree of freedom, so that in a certain well defined sense the system becomes one-dimensional.

The quantization problem for charged particles in a constant magnetic field reduces to the simultaneous quantization of energy and total angular momentum in a harmonic oscillator potential, discussed at length in Section 7 above. In the present context we want to discuss also one more topic, the coherent states, which are interesting because they are maximally localized ground states in the one-particle system. In particular, coherent states of anyons are supposed to be models of localized quasi-particles in the fractional quantum Hall system.

9.1 One particle in a magnetic field

The Hamiltonian for one particle of mass m and charge q in a constant magnetic field, in two dimensions, is

$$H = \frac{\boldsymbol{\pi}^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} , \tag{418}$$

where $\boldsymbol{\pi} = \boldsymbol{p} - q\mathbf{A}$ is the *kinematical* and $\boldsymbol{p} = -i\hbar\nabla$ the *canonical* momentum. The vector potential $\mathbf{A} = \mathbf{A}(\boldsymbol{x})$ depends on the gauge. When the magnetic flux density B is constant, a convenient choice is the circular gauge, in which

$$\mathbf{A} = (A_x, A_y) = \frac{B}{2}(-y, x) . \tag{419}$$

In this gauge we have $\mathbf{A}^2 = B^2\boldsymbol{x}^2/4$, and $\boldsymbol{p}\cdot\mathbf{A} = \mathbf{A}\cdot\boldsymbol{p} = BL/2$, where $L = xp_y - yp_x$ is the canonical angular momentum. Hence,

$$H = \frac{1}{2m}(\boldsymbol{p} - q\mathbf{A})^2 = \frac{1}{2m}\boldsymbol{p}^2 + \frac{1}{8}m\omega^2\boldsymbol{x}^2 \mp \frac{1}{2}\omega L . \tag{420}$$

The last sign is $-$ or $+$ depending on whether the product qB is positive or negative, since we define the cyclotron frequency ω to be positive,

$$\omega = \frac{|qB|}{m}. \tag{421}$$

The commutator

$$[\pi_x, \pi_y] = i\hbar qB \tag{422}$$

is gauge independent, and implies that H is formally just the Hamiltonian of a *one-dimensional* harmonic oscillator. We may define

$$\begin{aligned} \pi_x^{\pm} &= p_x \pm \frac{|qB|}{2}y = -i\hbar\frac{\partial}{\partial x} \pm \frac{|qB|}{2}y , \\ \pi_y^{\pm} &= p_y \mp \frac{|qB|}{2}x = -i\hbar\frac{\partial}{\partial y} \mp \frac{|qB|}{2}x , \end{aligned} \tag{423}$$

so that $\boldsymbol{\pi} = \boldsymbol{\pi}^+$ if $qB > 0$ and $\boldsymbol{\pi} = \boldsymbol{\pi}^-$ if $qB < 0$. The four operators $\pi_x^{\pm}$ and $\pi_x^{\pm}$ are a complete set of observables in the four-dimensional phase space, and they commute, except that

$$[\pi_x^+, \pi_y^+] = -[\pi_x^-, \pi_y^-] = i\hbar|qB| . \tag{424}$$

Since only one degree of freedom contributes to the energy, the second degree of freedom contributes only to the degeneracy of the energy levels, which

are called Landau levels. Every level is infinitely degenerate, assuming of course that the system is infinite in extent.

Another point of view is that H in the circular gauge is the Hamiltonian of a two-dimensional harmonic oscillator of angular frequency $\omega/2$, plus an angular momentum term, and the large degeneracy is due to cancellations between the two contributions. One usually defines the magnetic length as

$$\lambda = \sqrt{\frac{\hbar}{|qB|}} = \sqrt{\frac{\hbar}{m\omega}} \,. \tag{425}$$

In terms of the dimensionless complex coordinate

$$z = \frac{x+iy}{\sqrt{2}\,\lambda} \,, \tag{426}$$

and the corresponding differential operators $\partial = \partial/\partial z$ and $\partial^* = \partial/\partial z^*$, we have

$$\begin{aligned} L &= \hbar(z\partial - z^*\partial^*) \,, \\ H &= \frac{\hbar\omega}{2}\left(-2\partial\partial^* + \frac{|z|^2}{2} \mp (z\partial - z^*\partial^*)\right). \end{aligned} \tag{427}$$

We define annihilation and creation operators, in the same way as before,

$$\begin{aligned} a &= \partial + \frac{z^*}{2} = \frac{\lambda}{\sqrt{2}\,\hbar}\left(\pi_y^- + i\pi_x^-\right) \,, \\ a^\dagger &= -\partial^* + \frac{z}{2} = \frac{\lambda}{\sqrt{2}\,\hbar}\left(\pi_y^- - i\pi_x^-\right) \,, \\ b &= \partial^* + \frac{z}{2} = \frac{\lambda}{\sqrt{2}\,\hbar}\left(-\pi_y^+ + i\pi_x^+\right) \,, \\ b^\dagger &= -\partial + \frac{z^*}{2} = \frac{\lambda}{\sqrt{2}\,\hbar}\left(-\pi_y^+ - i\pi_x^+\right) \,, \end{aligned} \tag{428}$$

such that $[a, a^\dagger] = [b, b^\dagger] = 1$ and $[a, b] = \ldots = 0$. Then the canonical angular momentum is

$$L = \hbar\left(a^\dagger a - b^\dagger b\right), \tag{429}$$

and the Hamiltonian is

$$\begin{aligned} H &= \frac{\hbar\omega}{2}\left(a^\dagger a + b^\dagger b + 1 \mp \left(a^\dagger a - b^\dagger b\right)\right) \\ &= \begin{cases} \hbar\omega\left(b^\dagger b + \frac{1}{2}\right) & \text{if} \quad qB > 0 \,, \\ \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) & \text{if} \quad qB < 0 \,. \end{cases} \end{aligned} \tag{430}$$

For simplicity, we will mostly assume from now on that $qB > 0$, the case $qB < 0$ is entirely analoguous.

The normalized wave function

$$\chi_0 = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|z|^2}{2}\right) \tag{431}$$

is annihilated by both operators a and b, and is one of the infinitely many ground states of the one-particle system. When $qB > 0$, a complete orthonormal set of wave functions in the lowest Landau level are

$$\chi_n = \frac{1}{\sqrt{n!}} (a^\dagger)^n \chi_0 = \frac{1}{\sqrt{\pi n!}} z^n \exp\left(-\frac{|z|^2}{2}\right), \qquad n = 0, 1, 2, \ldots \tag{432}$$

The wave function χ_0 is distinguished by being maximally localized near the origin. However, the system is completely translation invariant, in fact the operators π_x^- and π_y^- are generators of translation that commute with the Hamiltonian, when $qB > 0$. Therefore we may obtain a wave function of the lowest Landau level which is maximally localized near any arbitrary point $z = \zeta$, simply by translating χ_0. The translated wave function is

$$\begin{aligned} \chi_\zeta &= \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|z|^2 + |\zeta|^2}{2} + \zeta^* z\right) \\ &= \frac{1}{\sqrt{\pi}} \exp\left(\frac{\zeta^* z - \zeta z^*}{2}\right) \exp\left(-\frac{|z-\zeta|^2}{2}\right). \end{aligned} \tag{433}$$

In the last expression the first exponential gives the phase and the last exponential the magnitude of the wave function. Thus χ_ζ is complex, except when $\zeta = 0$. It is a coherent state in the sense that it is an eigenstate of the annihilation operator a [64,283],

$$a\chi_\zeta = \zeta^* \chi_\zeta \,. \tag{434}$$

For later use, let us define the non-normalized wave function

$$\psi_\zeta = \exp\left(-\frac{|z|^2}{2} + \zeta^* z\right), \tag{435}$$

and compute the overlap integral for two such wave functions,

$$\int \mathrm{d}^2 z \, (\psi_{\zeta_a}(z))^* \psi_{\zeta_b}(z) = \int \mathrm{d}^2 z \, \mathrm{e}^{-|z|^2 + \zeta_a z^* + \zeta_b^* z} = \pi \, \mathrm{e}^{\zeta_a \zeta_b^*} \,. \tag{436}$$

Perhaps the most direct way to obtain this answer is to integrate separately over the real and imaginary parts of z, writing

$$\begin{aligned} -|z|^2 + \zeta_a z^* + \zeta_b^* z = &- \left(\mathrm{Re}\, z - \frac{\zeta_a + \zeta_b^*}{2}\right)^2 \\ &- \left(\mathrm{Im}\, z - i\,\frac{-\zeta_a + \zeta_b^*}{2}\right)^2 + \zeta_a \zeta_b^* \,. \end{aligned} \tag{437}$$

The one-particle Hamiltonian in a constant magnetic field and in an external oscillator potential of angular frequency ω_0 is

$$H = \frac{1}{2m}(\boldsymbol{p} - q\mathbf{A})^2 + \frac{1}{2}m{\omega_0}^2\boldsymbol{x}^2 = \frac{1}{2m}\boldsymbol{p}^2 + \frac{1}{8}m(\omega^2 + 4{\omega_0}^2)\boldsymbol{x}^2 \mp \frac{1}{2}\omega L\,. \tag{438}$$

We introduce the external potential here for the purpose of regularization, thus we want to take the limit $\omega_0 \to 0$ at the end of our calculations. The oscillator frequency is changed from $\omega/2$ to $\gamma\omega/2$, with

$$\gamma = \sqrt{1 + \frac{4{\omega_0}^2}{\omega^2}}\,. \tag{439}$$

As a consequence, if we do not modify the definition of z, we should substitute everywhere $\sqrt{\gamma}z$ for z.

By quantizing L and H simultaneously, we find the energy eigenvalues

$$E_{j,k} = \left(j + \frac{1}{2}\right)\hbar\omega_1 + \left(k + \frac{1}{2}\right)\hbar\omega_2\,, \tag{440}$$

with $j, k = 0, 1, 2, \ldots$, and with

$$\begin{aligned}\omega_1 &= \frac{1}{2}\left(\sqrt{\omega^2 + 4{\omega_0}^2} + \omega\right),\\ \omega_2 &= \frac{1}{2}\left(\sqrt{\omega^2 + 4{\omega_0}^2} - \omega\right).\end{aligned} \tag{441}$$

We see that $\omega_1 \to \omega$ and $\omega_2 \to 0$ as $\omega_0 \to 0$, so that $\hbar\omega_1$ is the modified energy difference between Landau levels, whereas $\hbar\omega_2$ is the energy splitting within one Landau level due to the external potential. We see also that both $\omega_1 \to \omega_0$ and $\omega_2 \to \omega_0$ in the limit of zero magnetic field, $\omega \to 0$.

These one-particle energy levels give the partition function

$$Z_1 = \sum_{j,k} \mathrm{e}^{-\beta E_{j,k}} = \frac{\mathrm{e}^{-\frac{1}{2}\beta\hbar(\omega_1+\omega_2)}}{(1 - \mathrm{e}^{-\beta\hbar\omega_1})(1 - \mathrm{e}^{-\beta\hbar\omega_2})}\,. \tag{442}$$

9.2 Two anyons in a magnetic field

The problem of many particles in a magnetic field falls into the class of problems with quadratic Hamiltonian which are exactly solvable for bosons or fermions, but not for anyons, except in a few special cases. Among those special cases is the problem of two anyons in a magnetic field, and this is still solvable if we add a harmonic oscillator interaction potential or an external harmonic oscillator potential, or both.

The two-particle Hamiltonian, including an external potential, is

$$H = \frac{1}{2m}\left((\boldsymbol{p}_1 - q\mathbf{A}_1)^2 + (\boldsymbol{p}_2 - q\mathbf{A}_2)^2\right) + \frac{1}{2}\, m\omega_0{}^2\left(\boldsymbol{x}_1{}^2 + \boldsymbol{x}_2{}^2\right) . \qquad (443)$$

We introduce the anyon statistics by requiring an arbitrary wave function ψ to be multivalued, with

$$\psi(\boldsymbol{x}_2, \boldsymbol{x}_1) = \mathrm{e}^{i\theta}\, \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) \qquad (444)$$

for an anticlockwise interchange of particle positions. Thus there is no contribution to the vector potentials from the statistics interaction. We assume that $\theta = \nu\pi$ with $0 \leq \nu < 2$.

The motion of the two particles can again be decomposed into independent motions of the centre of mass position $\mathbf{X} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and the relative position $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, with canonically conjugate momenta $\mathbf{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ and $\boldsymbol{p} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2$. Let us introduce a similar notation for the vector potentials, writing

$$\mathbf{A} = \frac{\mathbf{A}_1 + \mathbf{A}_2}{2} , \qquad \boldsymbol{a} = \mathbf{A}_1 - \mathbf{A}_2 . \qquad (445)$$

This gives the following expression for the Hamiltonian,

$$H = \frac{1}{4m}\,(\mathbf{P} - 2q\mathbf{A})^2 + \frac{1}{m}\left(\boldsymbol{p} - \frac{q}{2}\,\boldsymbol{a}\right)^2 + m\omega_0{}^2\,\mathbf{X}^2 + \frac{1}{4}\, m\omega_0{}^2\boldsymbol{x}^2 . \qquad (446)$$

The centre of mass is a "particle" of mass $2m$ and charge $2q$, whereas the relative coordinate describes a "particle" having a "reduced mass" of $m/2$ and a "reduced charge" of $q/2$. The ratio of charge to mass is the same for both, so that they have the same cyclotron frequency $\omega = |qB|/m$.

We introduce the complex coordinates z_1 and z_2 by the same definition as before, equation (426), with the same magnetic length λ. Then we define, quite naturally, $Z = (z_1 + z_2)/2$ and $z = z_1 + z_2$. However, we modify slightly the definitions of annihilation and creation operators, including the scaling factor $\gamma = \sqrt{1 + (4\omega_0{}^2/\omega^2)}$,

$$\begin{aligned}
a &= \frac{1}{\sqrt{2\gamma}}\,\frac{\partial}{\partial Z} + \sqrt{\frac{\gamma}{2}}\, Z^* , & a^\dagger &= -\frac{1}{\sqrt{2\gamma}}\,\frac{\partial}{\partial Z^*} + \sqrt{\frac{\gamma}{2}}\, Z , \\
b &= \frac{1}{\sqrt{2\gamma}}\,\frac{\partial}{\partial Z^*} + \sqrt{\frac{\gamma}{2}}\, Z , & b^\dagger &= -\frac{1}{\sqrt{2\gamma}}\,\frac{\partial}{\partial Z} + \sqrt{\frac{\gamma}{2}}\, Z^* , \\
c &= \sqrt{\frac{2}{\gamma}}\,\frac{\partial}{\partial z} + \sqrt{\frac{\gamma}{2}}\,\frac{z^*}{2} , & c^\dagger &= -\sqrt{\frac{2}{\gamma}}\,\frac{\partial}{\partial z^*} + \sqrt{\frac{\gamma}{2}}\,\frac{z}{2} , \\
d &= \sqrt{\frac{2}{\gamma}}\,\frac{\partial}{\partial z^*} + \sqrt{\frac{\gamma}{2}}\,\frac{z}{2} , & d^\dagger &= -\sqrt{\frac{2}{\gamma}}\,\frac{\partial}{\partial z} + \sqrt{\frac{\gamma}{2}}\,\frac{z^*}{2} .
\end{aligned} \qquad (447)$$

The non-vanishing commutators among these operators are

$$[a, a^\dagger] = [b, b^\dagger] = [c, c^\dagger] = [d, d^\dagger] = 1 \,. \tag{448}$$

When $qB > 0$ we obtain the following form of the Hamiltonian,

$$H = \hbar\omega_1 \left(b^\dagger b + d^\dagger d + 1\right) + \hbar\omega_2 \left(a^\dagger a + c^\dagger c + 1\right), \tag{449}$$

with the angular frequencies ω_1 and ω_2 given by equation (441). Two energy eigenstates having the correct symmetry under particle interchange are

$$\begin{aligned} \psi_0^{(\mathrm{I})} &= z^\nu \exp\left(-\gamma\left(|Z|^2 + \frac{|z|^2}{2}\right)\right), \\ \psi_0^{(\mathrm{II})} &= (z^*)^{2-\nu} \exp\left(-\gamma\left(|Z|^2 + \frac{|z|^2}{2}\right)\right). \end{aligned} \tag{450}$$

They have energies $E_0^{(\mathrm{I})} = \hbar\omega_1 + (1+\nu)\hbar\omega_2$ and $E_0^{(\mathrm{II})} = (3-\nu)\hbar\omega_1 + \hbar\omega_2$.

A complete set of energy eigenstates are, with j, k, l, m independent non-negativ integers,

$$\begin{aligned} \psi_{j,k,l,m}^{(\mathrm{I})} &= (a^\dagger)^j\,(b^\dagger)^k\,(c^\dagger d^\dagger)^l\,(c^\dagger)^{2m}\,\psi_0^{(\mathrm{I})}\,, \\ \psi_{j,k,l,m}^{(\mathrm{II})} &= (a^\dagger)^j\,(b^\dagger)^k\,(c^\dagger d^\dagger)^l\,(d^\dagger)^{2m}\,\psi_0^{(\mathrm{II})}\,, \end{aligned} \tag{451}$$

and the corresponding energy levels are

$$\begin{aligned} E_{j,k,l,m}^{(\mathrm{I})} &= (j+l+1)\,\hbar\omega_1 + (k+l+2m+1+\nu)\,\hbar\omega_2\,, \\ E_{j,k,l,m}^{(\mathrm{II})} &= (j+l+2m+3-\nu)\,\hbar\omega_1 + (k+l+1)\,\hbar\omega_2\,. \end{aligned} \tag{452}$$

This gives the two-particle partition function

$$\begin{aligned} Z_2 = {} & \frac{\mathrm{e}^{-\beta\hbar(\omega_1+\omega_2)}}{(1-\mathrm{e}^{-\beta\hbar\omega_1})(1-\mathrm{e}^{-\beta\hbar\omega_2})(1-\mathrm{e}^{-\beta\hbar(\omega_1+\omega_2)})} \\ & \left(\frac{\mathrm{e}^{-\nu\beta\hbar\omega_2}}{1-\mathrm{e}^{-2\beta\hbar\omega_2}} + \frac{\mathrm{e}^{-(2-\nu)\beta\hbar\omega_1}}{1-\mathrm{e}^{-2\beta\hbar\omega_1}}\right). \end{aligned} \tag{453}$$

The above calculation was done under the assumption that $qB > 0$. Changing the sign of qB is the same as interchanging ω_1 and ω_2. The same effect is obtained by substituting $2 - \nu$ for ν, or even simpler by the naive trick of switching the sign of ω.

By definition, the lowest Landau level for $qB > 0$ consists of the energy levels

$$E_{0,k,0,m}^{(\mathrm{I})} = \hbar\omega_1 + (k+2m+1+\nu)\,\hbar\omega_2\,. \tag{454}$$

They lie lower than the lowest energy level of type (II), $E_0^{(\mathrm{II})} = (3-\nu)\hbar\omega_1 + \hbar\omega_2$, as long as k and m are small, and ω_0 is small so that $\omega_2 << \omega_1$. At least this is true in the bosonic limit $\nu \to 0^+$. In the other bosonic limit, $\nu \to 2^-$, we see that the Landau levels are no longer clearly separated. For $qB < 0$, on the other hand, it is in the limit $\nu \to 2^-$, and not as $\nu \to 0^+$, that it has meaning to speak of separated Landau levels.

9.3 *The anyon gas in a magnetic field*

We are now in a position to compute the second virial coefficient of a gas of charged anyons in a magnetic field, neglecting the Coulomb interaction. Using equation (327) we get [102, 284]

$$\begin{aligned} A_2 &= \lim_{\omega_0 \to 0} \left(\frac{\Lambda}{\beta\hbar\omega_0} \right)^2 \left(1 - 2\frac{Z_2}{Z_1^2} \right) \\ &= \frac{\Lambda^2}{\beta\hbar\omega} \left(\nu - 1 - \frac{\mathrm{e}^{(\nu-1)\beta\hbar\omega}}{\sinh(\beta\hbar\omega)} + \frac{1}{2} \coth\left(\frac{\beta\hbar\omega}{2} \right) \right) . \end{aligned} \tag{455}$$

In the limit of zero magnetic field we recover the well known result, equation (347),

$$\lim_{\omega \to 0} A_2 = \Lambda^2 \left(\frac{1}{4} - \frac{(\nu-1)^2}{2} \right) . \tag{456}$$

More interesting are the next terms in the expansion around $B = 0$,

$$\left. \frac{\partial A_2}{\partial B} \right|_{B=0} = \frac{q}{m} \left. \frac{\partial A_2}{\partial \omega} \right|_{\omega=0} = -\Lambda^4 \frac{q}{h} \frac{\nu(\nu-1)(\nu-2)}{6} , \tag{457}$$

and

$$\left. \frac{\partial^2 A_2}{\partial B^2} \right|_{B=0} = \Lambda^6 \left(\frac{q}{h} \right)^2 \frac{1 - 2\nu^2(\nu-2)^2}{24} . \tag{458}$$

The first derivative at $B = 0$ vanishes for bosons, $\nu = 0$ and $\nu = 2$, and for fermions, $\nu = 1$, but for no other types of anyons. This means that a gas of charged anyons that are not bosons or fermions, should develop a spontaneous magnetization even in the absence of an external magnetic field.

The N-anyon problem in a magnetic field can only be solved numerically when $N > 2$, but it is still possible to treat exactly the limiting case when the field is strong enough, or the temperature low enough, that all energy levels not belonging to the lowest Landau level can be neglected. As already noted, in order to speak meaningfully of separated Landau levels, we have to exclude the bosonic limits $\nu \to 2^-$ when $qB > 0$, and $\nu \to 0^+$ when $qB < 0$.

The N-particle Hamiltonian is

$$H = \sum_{j=1}^{N} \left(\frac{1}{2m} \boldsymbol{p}_j{}^2 + \frac{1}{8} m(\omega^2 + 4\omega_0{}^2) \boldsymbol{x}_j{}^2 \mp \frac{1}{2} \omega L_j \right). \tag{459}$$

Using our results for the harmonic oscillator we can immediately write down a complete non-orthogonal set of wave functions in the lowest Landau level,

$$\psi^{\mathrm{LLL}}_{j_1, j_2, \ldots, j_N} = s_1{}^{j_1} s_2{}^{j_2} \cdots s_N{}^{j_N} \Delta^{\nu} \exp\left(-\frac{\gamma}{2} \sum_{j=1}^{N} |z_j|^2 \right), \tag{460}$$

with $j_1, j_2, \ldots, j_N$ non-negative integers, $s_k = \sum_{j=1}^{N} z_j{}^k$ and $\Delta = \prod_{j<k} (z_j - z_k)$, and the energies,

$$E^{\mathrm{LLL}}_{j_1, j_2, \ldots, j_N} = \frac{N}{2} \hbar\omega_1 + \left(j_1 + 2j_2 + \cdots + N j_N + \frac{N}{2} + \frac{N(N-1)}{2} \nu \right) \hbar\omega_2 . \tag{461}$$

Hence the N-anyon partition function of the lowest Landau level is

$$Z_N^{\mathrm{LLL}} = \frac{\mathrm{e}^{-\frac{N}{2}\beta\hbar\omega_1} \mathrm{e}^{-\frac{N}{2}(1+(N-1)\nu)\beta\hbar\omega_2}}{(1 - \mathrm{e}^{-\beta\hbar\omega_2})(1 - \mathrm{e}^{-2\beta\hbar\omega_2}) \cdots (1 - \mathrm{e}^{-N\beta\hbar\omega_2})} . \tag{462}$$

Special cases are bosons, $\nu = 0$, and fermions, $\nu = 1$, which can be derived from the grand canonical partition functions

$$\begin{aligned} \Xi^{\mathrm{B}} &= \prod_{j=0}^{\infty} \frac{1}{1 - z\mathrm{e}^{-\frac{1}{2}\beta\hbar(\omega_1+\omega_2)} \mathrm{e}^{-j\beta\hbar\omega_2}} , \\ \Xi^{\mathrm{F}} &= \prod_{j=0}^{\infty} \left(1 + z\mathrm{e}^{-\frac{1}{2}\beta\hbar(\omega_1+\omega_2)} \mathrm{e}^{-j\beta\hbar\omega_2} \right). \end{aligned} \tag{463}$$

In order to compute the cluster coefficients we use equation (326), substituting $\xi = \beta\hbar\omega_0$. In the free particle limit $\omega_0 \to 0$ we then get

$$b_n = \rho_L \frac{\mathrm{e}^{-\frac{n}{2}\beta\hbar\omega}}{n} \prod_{k=1}^{n-1} \left(1 - \frac{n\nu}{k} \right), \tag{464}$$

where ρ_L is the surface density of quantum states in the lowest Landau level,

$$\rho_L = \frac{\beta\hbar\omega}{\Lambda^2} = \frac{|qB|}{h} = \frac{1}{2\pi\lambda^2} . \tag{465}$$

Note the close resemblance of equation (464) with equation (395), the only significant difference is the factor $1/n$ in one formula *versus* $1/n^2$ in the other.

Similarly, we may compute the virial coefficients using equation (327). The general result is

$$A_n = \frac{\nu^n - (\nu - 1)^n}{n{\rho_L}^{n-1}} \cdot \tag{466}$$

In particular, the second virial coefficient

$$A_2 = \frac{1}{\rho_L}\left(\nu - \frac{1}{2}\right) \tag{467}$$

is the same as we get from the exact equation (455) in the strong field limit, or more precisely, when $\beta\hbar\omega$ is large. The difference between (467) and (455) should be a useful measure of the error we make by neglecting the higher Landau levels.

An explicit proof that the virial coefficients (466) follow from equation (464), can be given by means of the results from Subsection 8.12. It follows from those results that the equation

$$\beta P = \sum_{n=1}^{\infty} b_n z^n = \rho_L \sum_{n=1}^{\infty} \frac{(z')^n}{n} \prod_{k=1}^{n-1}\left(1 - \frac{n\nu}{k}\right), \tag{468}$$

with $z' = z\mathrm{e}^{-\frac{1}{2}\beta\hbar\omega}$, is equivalent to the equation

$$z' = \exp\left(\nu\frac{\beta P}{\rho_L}\right) - \exp\left((\nu - 1)\frac{\beta P}{\rho_L}\right) \cdot \tag{469}$$

Operating on this equation with $z'(\partial/\partial z')$, and using that $z'(\partial(\beta P)/\partial z') = \rho$, we get that

$$z' = \left(\nu\exp\left(\nu\frac{\beta P}{\rho_L}\right) - (\nu - 1)\exp\left((\nu - 1)\frac{\beta P}{\rho_L}\right)\right)\frac{\rho}{\rho_L} \cdot \tag{470}$$

Equating these two formulae for z' we obtain the equation of state

$$\beta P = \rho_L \ln\left(\frac{\rho_L - (\nu - 1)\rho}{\rho_L - \nu\rho}\right), \tag{471}$$

which is equivalent to equation (466).

We see that the pressure diverges when the density ρ approaches the maximum value ρ_L/ν. This result for anyons is a direct generalization of the fact that the maximum density of fermions in one Landau level is ρ_L. The divergence is clearly unphysical, and would have been avoided if we

had been able to include the higher Landau levels. At some density before the first Landau level is full, the particles should start filling the next level.

It was assumed in the above that $qB > 0$. In the opposite case, $qB < 0$, the same equation of state holds with the substitution $\nu \to 2 - \nu$. That is, we have

$$\beta P = \rho_L \ln\left(\frac{\rho_L - (1-\nu)\rho}{\rho_L - (2-\nu)\rho}\right). \tag{472}$$

10 Interchange phases and geometric phases

In this final Section we will discuss the relation between quantum mechanical phases of somewhat different origins, the interchange phases in systems of identical particles, and the geometric phases, also called Berry phases, associated with cyclic evolution of quantum systems. This relation may be used for studying the statistics of particles or of particle-like excitations in a physical system.

10.1 Introduction to geometric phases

The two-level time dependent Hamiltonian

$$H(t) = \hbar\omega \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha)\,\mathrm{e}^{-2i\beta t} \\ \sin(2\alpha)\,\mathrm{e}^{2i\beta t} & -\cos(2\alpha) \end{pmatrix}, \tag{473}$$

with ω, α and β real constants, illustrates well the phenomenon of the geometric phase, or Berry phase [15,57–61]. The instantaneous eigenvectors of $H(t)$, with eigenvalues $\pm\hbar\omega$, are

$$\chi_+(t) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\,\mathrm{e}^{2i\beta t} \end{pmatrix}, \qquad \chi_-(t) = \begin{pmatrix} \sin\alpha\,\mathrm{e}^{-2i\beta t} \\ -\cos\alpha \end{pmatrix}. \tag{474}$$

The exact solution of the time dependent Schrödinger equation

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = H\psi \tag{475}$$

is of the form $\psi(t) = U(t)\,\psi(0)$, with

$$U(t) = \begin{pmatrix} \mathrm{e}^{-i\beta t}\,(\cos(\omega_1 t) - i\gamma\sin(\omega_1 t)) & -i\delta\,\mathrm{e}^{-i\beta t}\sin(\omega_1 t) \\ -i\delta\,\mathrm{e}^{i\beta t}\sin(\omega_1 t) & \mathrm{e}^{i\beta t}\,(\cos(\omega_1 t) + i\gamma\sin(\omega_1 t)) \end{pmatrix}, \tag{476}$$

and with

$$\omega_1 = \sqrt{\omega^2 + \beta^2 - 2\omega\beta\cos(2\alpha)}\,, \qquad \gamma = \frac{\omega\cos(2\alpha) - \beta}{\omega_1},$$
$$\delta = \frac{\omega\sin(2\alpha)}{\omega_1}\,. \tag{477}$$

In the adiabatic limit when β is very small, but the product βt is not necessarily small, we may put

$$\omega_1 t \approx \omega t - \beta t \cos(2\alpha) \,, \qquad \gamma \approx \cos(2\alpha) \,, \qquad \delta \approx \sin(2\alpha) \,. \tag{478}$$

With these approximations the time evolution of the eigenvectors $\chi_\pm(0)$ of $H(0)$ is that

$$U(t)\,\chi_\pm(0) \approx \mathrm{e}^{\mp i(\omega_1+\beta)t}\,\chi_\pm(t) \approx \mathrm{e}^{\mp i\omega t}\,\mathrm{e}^{\mp i\beta t(1-\cos(2\alpha))}\,\chi_\pm(t) \,. \tag{479}$$

In other words, an eigenvector of $H(0)$ evolves approximately into an eigenvector of $H(t)$. The phase $\mp\omega t$ in equation (479) is readily understood as due to the time evolution of states with energies $\pm\hbar\omega$. But there is an additional phase,

$$\vartheta(t) = \vartheta_\pm(t) = \mp\beta t\,(1-\cos(2\alpha)) \,, \tag{480}$$

which can be interpreted as an effect of the geometry of the Hilbert space of spinors to which the eigenvectors $\chi_\pm(t)$ belong.

In fact, given $\chi(t) = \chi_+(t)$ or $\chi_-(t)$, normalized such that $|\chi|^2 = \chi^\dagger\chi = 1$, let us ask for the time dependent real phase $\vartheta(t)$ such that the curve $\psi(t) = \mathrm{e}^{i\vartheta(t)}\,\chi(t)$ in the Hilbert space has minimal length. The length is, with the time derivative $\mathrm{d}/\mathrm{d}t$ denoted by a dot,

$$\int \mathrm{d}t\,|\dot\psi| = \int \mathrm{d}t\,|i\dot\vartheta\chi + \dot\chi| = \int \mathrm{d}t\,\sqrt{(\dot\vartheta)^2 + |\dot\chi|^2 - i\dot\vartheta\,(\chi^\dagger\dot\chi - \dot\chi^\dagger\chi)} \,. \tag{481}$$

To minimize the integral we must minimize the integrand, *i.e.* choose $\vartheta(t)$ such that

$$\dot\vartheta = \frac{i}{2}\,(\chi^\dagger\dot\chi - \dot\chi^\dagger\chi) = -\,\mathrm{Im}\,(\chi^\dagger\dot\chi) = i\chi^\dagger\dot\chi \,. \tag{482}$$

The last equality follows because χ is normalized, so that

$$\mathrm{Re}\,(\chi^\dagger\dot\chi) = \frac{1}{2}\,(\chi^\dagger\dot\chi + \dot\chi^\dagger\chi) = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\chi^\dagger\chi) = 0 \,. \tag{483}$$

If $\chi = C\chi_0$, where χ_0 is unnormalized and C is a positive normalization factor such that χ is normalized, then equation (482) takes the form

$$\dot\vartheta = -\,\mathrm{Im}\,(\chi^\dagger\dot\chi) = -\,\mathrm{Im}\,\Big(C\chi_0^\dagger\,\big(C\dot\chi_0 + \dot C\chi_0\big)\Big) = -C^2\,\mathrm{Im}\,\big(\chi_0^\dagger\dot\chi_0\big) \,. \tag{484}$$

For the phases $\vartheta(t) = \vartheta_\pm(t)$ corresponding to $\chi(t) = \chi_\pm(t)$ we get the two equations

$$\begin{aligned} \dot\vartheta_+ &= i\,(\chi_+)^\dagger\,\dot\chi_+ = -2\beta\sin^2\alpha = -\beta\,(1-\cos(2\alpha)) \,, \\ \dot\vartheta_- &= i\,(\chi_-)^\dagger\,\dot\chi_- = \beta\,(1-\cos(2\alpha)) \,, \end{aligned} \tag{485}$$

which are the same as equation (480) if we take $\vartheta_\pm(0) = 0$.

The time dependence of $H(t)$ is periodic with period $T = \pi/\beta$, and the eigenvectors $\chi_\pm(0)$ of $H(0)$ evolve over one whole period T into

$$U(T)\,\chi_\pm(0) \approx \mathrm{e}^{\mp i\omega T}\,\mathrm{e}^{\mp i\pi(1-\cos(2\alpha))}\,\chi_\pm(0)\ . \tag{486}$$

The additional phase over one complete cycle of the variation of $H(t)$ is the Berry phase

$$\vartheta_\pm^O = \mp\pi\,(1-\cos(2\alpha))\ . \tag{487}$$

The superscript "O" indicates that it has to do with a closed loop. It is independent of the period T, it depends only on the sequence of eigenstates $\chi_\pm(t)$ gone through and not on the specific parametrization of the curve. Hence it makes sense to speak of a geometric phase associated with any closed loop in the space of state vectors, irrespective of whether the loop is a physical time evolution due to the adiabatic deformation of a Hamiltonian.

In fact, the Berry phase ϑ^O is unchanged if the eigenvector $\chi(t)$ is multiplied by an arbitrary t-dependent phase factor, as long as $\chi(T) = \chi(0)$. Thus it depends only on the sequence of *physical states* gone through. Remember that a physical state is represented in quantum mechanics not by one unique vector in the Hilbert space, but rather by a one-dimensional subspace. That is, two unit vectors in the Hilbert space represent the same physical state if they differ only by a phase factor.

10.2 One particle in a magnetic field

As an example of the geometric phase, or generalized Berry phase, let us calculate the phase induced when a charged particle is moved around a loop in a magnetic field. Both the one- and two-particle cases have been discussed by Leinaas [153]. See also [64]. The original derivation of the Berry phase applied to a non-degenerate energy level of a Hamiltonian which was time dependent, although varying slowly. The present example is of a diametrically opposite kind, since the Hamiltonian is time independent and all energy levels are infinitely degenerate.

Assume that the localized quantum state χ_ζ, equation (433), is moved once around the circle $|\zeta| = \rho$, in the anticlockwise direction. That is, we parametrize $\zeta = \rho\mathrm{e}^{i\alpha}$ and let the angle α increase from 0 to 2π. Note that, by equation (426), the dimensionless radius ρ corresponds to a dimensioned radius $r = \sqrt{x^2+y^2} = \sqrt{2}\,\lambda\rho$. In principle, the circular motion could be induced by a weak central electric field, since a charged particle in crossed electric and magnetic fields drifts perpendicularly to both fields. By direct generalization from equation (482) we define a geometric phase ϑ such that

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} = i\int \mathrm{d}^2z\ \chi_\zeta^*\,\frac{\partial\chi_\zeta}{\partial\alpha} = \zeta^*\int \mathrm{d}^2z\ z|\chi_\zeta|^2 = |\zeta|^2 = \rho^2\ , \tag{488}$$

where we have used that

$$\frac{\partial \chi_\zeta}{\partial \alpha} = \frac{\partial \zeta}{\partial \alpha}\frac{\partial \chi_\zeta}{\partial \zeta} + \frac{\partial \zeta^*}{\partial \alpha}\frac{\partial \chi_\zeta}{\partial \zeta^*} = i\zeta\frac{\partial \chi_\zeta}{\partial \zeta} - i\zeta^*\frac{\partial \chi_\zeta}{\partial \zeta^*} = -i\zeta^* z\chi_\zeta \,. \tag{489}$$

Integrated over α from 0 to 2π this gives the generalized Berry phase, which is independent of the phase convention for the localized states χ_ζ,

$$\vartheta^O = 2\pi\rho^2 = \frac{|qB|}{\hbar}\,\pi r^2 \,. \tag{490}$$

Note that we get $\vartheta^O > 0$ because we assumed that $qB > 0$. The case $qB < 0$ corresponds to the complex conjugate wave function, which will give $\vartheta^O < 0$. Thus we may drop the absolute value sign in equation (490) and write

$$\vartheta^O = \frac{qB}{\hbar}\,\pi r^2 = 2\pi\,\frac{\Phi}{\Phi_0} \,, \tag{491}$$

where $\Phi = B\pi r^2$ is the magnetic flux encircled, $\Phi_0 = h/q$ is the flux quantum, and both Φ and Φ_0 may have either sign.

There is an alternative way to compute the same Berry phase, using the non-normalized coherent state wave function ψ_ζ defined in equation (435). By generalization from equation (484), we have that

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} = -\frac{1}{I}\,\mathrm{Im}\left(\int \mathrm{d}^2z\;\psi_\zeta^*\,\frac{\partial \psi_\zeta}{\partial \alpha}\right), \tag{492}$$

where I is the one-particle normalization integral, which by rotation invariance is independent of α,

$$I = I(\rho) = \int \mathrm{d}^2z\;|\psi_\zeta|^2 = \pi\,\mathrm{e}^{\rho^2}. \tag{493}$$

The point now is that ψ_ζ depends on α only through $\zeta = \rho\mathrm{e}^{i\alpha}$, and is an analytic function of ζ^*, so that

$$\begin{aligned}\frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} &= \frac{1}{I}\,\mathrm{Im}\left(i\zeta^*\,\frac{\partial I}{\partial \zeta^*}\right) = \frac{1}{2I}\left(\zeta\,\frac{\partial I}{\partial \zeta} + \zeta^*\,\frac{\partial I}{\partial \zeta^*}\right)\\ &= \rho^2\,\frac{\mathrm{d}}{\mathrm{d}\rho^2}\,\ln I(\rho) = \rho^2 \,.\end{aligned} \tag{494}$$

Here we have used that

$$2\rho^2\,\frac{\partial}{\partial \rho^2} = \rho\,\frac{\partial}{\partial \rho} = \rho\,\frac{\partial \zeta}{\partial \rho}\frac{\partial}{\partial \zeta} + \rho\,\frac{\partial \zeta^*}{\partial \rho}\frac{\partial}{\partial \zeta^*} = \zeta\,\frac{\partial}{\partial \zeta} + \zeta^*\,\frac{\partial}{\partial \zeta^*} \,. \tag{495}$$

We have assumed here that $qB > 0$. As already mentioned, if $qB < 0$ instead, we have to use the complex conjugate wave function. Because

it depends analytically on ζ instead of ζ^*, there is a change of sign in equation (494), so that we get

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} = -\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I(\rho) = -\rho^2 \,. \tag{496}$$

10.3 Two particles in a magnetic field

In the case of two bosons or fermions in a magnetic field, in the lowest Landau level, the one-particle coherent states may be used to construct two-particle states where both particles are maximally localized. Simply take the product of the one-particle wave functions localized at ζ_a and ζ_b, and symmetrize if the particles are bosons, or antisymmetrize if they are fermions. A continuous variation of the parameters ζ_a and ζ_b induces a deformation of the two-particle wave function, and if ζ_a is changed continuously into ζ_b, and *vice versa*, this deformation is a closed loop starting and ending with the same physical state. It is in effect an interchange of the particles, and not unexpectedly, the corresponding Berry phase turns out to be related to the symmetry or antisymmetry of the wave function.

For two bosons we define the non-normalized wave function

$$\psi^{\mathrm{B}}_{\zeta_a,\zeta_b}(z_1,z_2) = \psi_{\zeta_a}(z_1)\,\psi_{\zeta_b}(z_2) + \psi_{\zeta_a}(z_2)\,\psi_{\zeta_b}(z_1) \,, \tag{497}$$

with ψ_ζ as defined in equation (435). For two fermions we define

$$\psi^{\mathrm{F}}_{\zeta_a,\zeta_b}(z_1,z_2) = \frac{\psi_{\zeta_a}(z_1)\,\psi_{\zeta_b}(z_2) - \psi_{\zeta_a}(z_2)\,\psi_{\zeta_b}(z_1)}{\zeta_a^* - \zeta_b^*} \,, \tag{498}$$

dividing by an extra factor $\zeta_a^* - \zeta_b^*$ which serves two purposes. Since the wave functions $\psi^{\mathrm{F}}_{\zeta_a,\zeta_b}$ and $\psi^{\mathrm{F}}_{\zeta_b,\zeta_a}$ represent the same physical state, we want them to be completely identical and not just identical up to a sign. Also, it is nice to have $\psi^{\mathrm{F}}_{\zeta_a,\zeta_b}$ well defined in the limit $|\zeta_a - \zeta_b| \to 0$. The boson and fermion wave functions are both analytic functions of ζ_a^* and ζ_b^*.

To simplify, let us take the one-particle coherent states to be localized symmetrically about the origin, with $\zeta_a = -\zeta_b = \zeta = \rho \mathrm{e}^{i\alpha}$. Then, since the wave function is analytic in ζ^*, the Berry phase is related to the normalization integral in the same way as in the one-particle case above. The boson and fermion normalization integrals are, respectively,

$$\begin{aligned} I^{\mathrm{B}}(\rho) &= \int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, |\psi^{\mathrm{B}}_{\zeta,-\zeta}(z_1,z_2)|^2 = 2\pi \left(\mathrm{e}^{2\rho^2} + \mathrm{e}^{-2\rho^2}\right) , \\ I^{\mathrm{F}}(\rho) &= \int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, |\psi^{\mathrm{F}}_{\zeta,-\zeta}(z_1,z_2)|^2 = \frac{\pi \left(\mathrm{e}^{2\rho^2} - \mathrm{e}^{-2\rho^2}\right)}{2\rho^2} \,. \end{aligned} \tag{499}$$

Keeping the radius ρ fixed and increasing the angle α from 0 to π corresponds to an anticlockwise interchange. The Berry phase is, for two bosons,

$$\vartheta^{\mathrm{B}}(\pi) = \pi \frac{\mathrm{d}\vartheta^{\mathrm{B}}}{\mathrm{d}\alpha} = \pi\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I^{\mathrm{B}}(\rho) = 2\pi\rho^2 \tanh(2\rho^2) \,. \tag{500}$$

And, for two fermions,

$$\vartheta^{\mathrm{F}}(\pi) = \pi \frac{\mathrm{d}\vartheta^{\mathrm{F}}}{\mathrm{d}\alpha} = \pi\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I^{\mathrm{F}}(\rho) = \pi \left(2\rho^2 \coth(2\rho^2) - 1\right) . \tag{501}$$

The asymptotic limit as $\rho \to \infty$ is $2\pi\rho^2$ in both cases. This we recognize as the one-particle contribution, due to the displacement of each of the two one-particle coherent states around a half circle. Subtracting this one-particle contribution, we are left with the genuine two-particle Berry phases,

$$\begin{aligned} \vartheta_{\mathrm{s}}^{\mathrm{B}} &= \pi\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \left(\ln I_2^{\mathrm{B}}(\rho) - 2 \ln I_1(\rho)\right) = 2\pi\rho^2 (\tanh(2\rho^2) - 1) \,, \\ \vartheta_{\mathrm{s}}^{\mathrm{F}} &= \pi\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \left(\ln I_2^{\mathrm{F}}(\rho) - 2 \ln I_1(\rho)\right) \\ &= \pi \left(2\rho^2 (\coth(2\rho^2) - 1) - 1\right) . \end{aligned} \tag{502}$$

Here $I_1(\rho)$ is the one-particle and $I_2(\rho)$ the two-particle normalization integral. We will refer to ϑ_{s} as the *statistics Berry phase*. The asymptotic values 0 for bosons and $-\pi$ for fermions, when $\rho \to \infty$, justify the terminology. Since we define ϑ_{s} to depend on the distance ρ, it is not surprising that there is a deviation from the asymptotic values 0 and $-\pi$ when ρ is so small that the two one-particle coherent states overlap significantly.

Note that for $\zeta_a = -\zeta_b = \zeta$, the boson and fermion two-particle coherent states defined in equations (497) and (498) have a common form, apart from constant factors,

$$\psi^{(\nu)} = \sum_{k=0}^{\infty} \frac{(\zeta^*)^{2k}}{\Gamma(2k + \nu + 1)} z^{2k+\nu} \psi_0 \,, \tag{503}$$

with

$$\psi_0 = \exp\left(-\frac{|z_1|^2 + |z_2|^2}{2}\right) = \exp\left(-|Z|^2 - \frac{|z|^2}{4}\right) \,, \tag{504}$$

and with $Z = (z_1 + z_2)/2$, $z = z_1 - z_2$. The boson state has $\nu = 0$ and the fermion state $\nu = 1$. Note however that in equation (503) we could take for example $\nu = 0, 2, 4, \ldots$ and get infinitely many different bosonic two-particle coherent states, with different asymptotic behaviour as $z \to 0$.

The basis states $z^{2k+\nu}\psi_0$ for fixed ν are orthogonal, and they are normalizable whenever $2k+\nu > -1$. The normalization integral for $\psi^{(\nu)}$ is, again apart from constant factors,

$$I^{(\nu)} = \sum_{k=0}^{\infty} \frac{(2\rho^2)^{2k}}{\Gamma(2k+\nu+1)} \,. \tag{505}$$

We could generalize from bosons and fermions to various two-anyon states that localize each of the particles more or less well. The most obvious generalization is simply to allow ν in equation (503) to take any real value, with the only restriction that $\nu > -1$, for normalizability. This can be interpreted as the anyon coordinate eigenstate projected onto the lowest Landau level [285]. Another two-anyon state which has also been proposed as a natural generalization is the coherent state of a particular $su(1,1)$ algebra [64,285],

$$\psi_{\rm c}^{(\nu)} = \sum_{k=0}^{\infty} \frac{(\zeta^*)^{2k}}{2^k\sqrt{k!\,\Gamma(k+\nu+\frac{1}{2})\Gamma(2k+\nu+1)}}\, z^{2k+\nu}\,\psi_0 \,. \tag{506}$$

The normalization integral for this is given by a modified Bessel function $I_{\nu-(1/2)}$,

$$I_{\rm c}^{(\nu)} = \sum_{k=0}^{\infty} \frac{(\rho^2)^{2k}}{k!\,\Gamma(k+\nu+\frac{1}{2})} = \frac{I_{\nu-\frac{1}{2}}(2\rho^2)}{\rho^{2\nu-1}} \,. \tag{507}$$

The statistics Berry phase

$$\vartheta_{\rm s}^{(\nu)} = \pi\rho^2 \frac{\rm d}{{\rm d}\rho^2} \left(\ln I_2^{(\nu)}(\rho) - 2\ln I_1(\rho)\right) \tag{508}$$

has the asymptotic value of $-\nu\pi$ for both these two-anyon states, but there is a difference between them for small ρ. Note that $\vartheta_{\rm s}^{(\nu)} \to 0$ for $\rho \to 0$, independent of ν. This does not mean that the bosons, fermions or anyons are not pointlike particles, what it means is that they are not sharply localized. Sharp localization is impossible as long as we admit only states belonging to the lowest Landau level.

We may now turn the whole argument around and use the Berry phase to define a distance dependent "anyon parameter"

$$\nu_{\rm Berry} = -\rho^2 \frac{\rm d}{{\rm d}\rho^2} \left(\ln I_2^{(\nu)}(\rho) - 2\ln I_1(\rho)\right) \,, \tag{509}$$

which is then asymptotically equal to the actual statistics parameter ν at large distances.

In the discussion so far we have assumed that $qB > 0$. The only difference in the case $qB < 0$ is that we have to take the complex conjugates of the wave functions $\psi^{(\nu)}$ and $\psi_{\rm c}^{(\nu)}$, defined in equations (503) and (506), but when we change z into z^*, we have to change z^ν into $(z^*)^{-\nu}$, in order to preserve the meaning of the anyon parameter ν. Thus the anyon states of negative charge are

$$\psi^{(\nu)} = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{\Gamma(2k-\nu+1)} (z^*)^{2k-\nu} \psi_0 \,, \tag{510}$$

and

$$\psi_{\rm c}^{(\nu)} = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{2^k \sqrt{k!\,\Gamma(k-\nu+\frac{1}{2})\Gamma(2k-\nu+1)}} z^{2k-\nu} \psi_0 \,. \tag{511}$$

These states are well defined for $\nu < 1$, and are singular as $|z| \to 0$ if $0 < \nu < 1$.

The complex conjugate wave functions depend analytically on ζ instead of ζ^*, which implies an opposite sign in the relation between the Berry phase and the normalization integral. Thus, for $qB < 0$, equation (509) is replaced by

$$\nu_{\rm Berry} = \rho^2 \frac{\rm d}{{\rm d}\rho^2} \left(\ln I_2^{(\nu)}(\rho) - 2 \ln I_1(\rho) \right) . \tag{512}$$

10.4 Interchange of two anyons in potential wells

The results just derived for two particles in a magnetic field indicate a general relation between the geometric phase, or Berry phase, and the interchange phase in a system of identical particles. The existense of such a relation is not entirely trivial, since the two phases are conceptually rather different. One phase has to do with the geometry, or more precisely the metric, in the Hilbert space of quantum state vectors, the other has to do with the topology of the configuration space. One phase arises when the whole wave function is changed continuously, the other arises when the argument of one single wave function is changed.

As another example, we may imagine two identical particles in two dimensions trapped inside two separate deep potential wells, and interchange the particle positions by interchanging the wells [63]. If only one potential well is present at the origin, let ψ_0 denote its ground state wave function, of energy E_0. For simplicity we assume that the well is rotationally symmetric, so that ψ_0 has angular momentum zero. Let $\psi_{\boldsymbol{a}}$ be the wave function ψ_0 translated to the position $\boldsymbol{a}$, that is,

$$\psi_{\boldsymbol{a}}(\boldsymbol{x}) = \psi_0(\boldsymbol{x} - \boldsymbol{a}) \,. \tag{513}$$

Table 1. The approximate energies, with the exact and approximate wave functions for two bosons or two fermions in the double well. Normalization constants are ignored.

Energy	Boson wave function	Fermion wave function
$2E_0+\epsilon$	$\psi_-(\boldsymbol{x}_1)\psi_-(\boldsymbol{x}_2)$ $\approx \psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)+\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)$ $-\psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)-\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)$	
$2E_0$	$\psi_-(\boldsymbol{x}_1)\psi_+(\boldsymbol{x}_2)+\psi_+(\boldsymbol{x}_1)\psi_-(\boldsymbol{x}_2)$ $\approx \psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)-\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)$	$\psi_-(\boldsymbol{x}_1)\psi_+(\boldsymbol{x}_2)-\psi_+(\boldsymbol{x}_1)\psi_-(\boldsymbol{x}_2)$ $\approx \psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)-\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)$
$2E_0-\epsilon$	$\psi_+(\boldsymbol{x}_1)\psi_+(\boldsymbol{x}_2)$ $\approx \psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)+\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)$ $+\psi_{\boldsymbol{a}}(\boldsymbol{x}_1)\psi_{\boldsymbol{b}}(\boldsymbol{x}_2)+\psi_{\boldsymbol{b}}(\boldsymbol{x}_1)\psi_{\boldsymbol{a}}(\boldsymbol{x}_2)$	

Then if the two wells are located at $\boldsymbol{a}$ and at $\boldsymbol{b}$, and if the overlap between the two wave functions $\psi_{\boldsymbol{a}}$ and $\psi_{\boldsymbol{b}}$ is small, the one-particle ground state is nearly degenerate, since there are two energy eigenstates

$$\psi_\pm \approx \psi_{\boldsymbol{a}} \pm \psi_{\boldsymbol{b}} \,. \tag{514}$$

The energies are $E_0 \mp \epsilon/2$, and the energy splitting ϵ is small.

The lowest energies and the corresponding wave functions for two particles in the two wells are tabulated in Table 1, for the boson and fermion cases. Note that the single fermionic energy eigenstate has essentially only one particle in each well, but in all three of the bosonic energy eigenstates the probability of finding both particles in the same well is either 50% or 100%, approximately. It follows by interpolation to the anyon case that there is in general no anyonic energy eigenstate with the two particles in separate wells. On the other hand, if the energy splitting ϵ is very small, then there surely exists an approximate energy eigenstate with two anyons in separate wells, and the transition probability from this state to other states is small. It is this particular state we are interested in here. Call its normalized wave function χ_0.

It is convenient to introduce polar coordinates, (R,Φ) for the centre of mass position and (r,ϕ) for the relative position, and to work in what we have called the parallel gauge, so that the statistics vector potential vanishes and every wave function ψ satisfies the following periodicity condition,

$$\psi(R,\Phi,r,\phi+\pi) = \mathrm{e}^{i\theta}\psi(R,\Phi,r,\phi) \,. \tag{515}$$

Assume now that the positions of the two wells are $\boldsymbol{a}$ and $\boldsymbol{b}=-\boldsymbol{a}$, and that they are interchanged simply by a rotation an angle π about the origin. Define a set of wave functions χ_α, depending on the real parameter α, such that

$$\chi_\alpha(R,\Phi,r,\phi) = \mathrm{e}^{i\nu\alpha}\chi_0(R,\Phi-\alpha,r,\phi-\alpha) \,, \tag{516}$$

with $\nu = \theta/\pi$. The phase factor $\mathrm{e}^{i\nu\alpha}$ is introduced in order that $\chi_\pi = \chi_0$. Apart from that, the wave function χ_α is just χ_0 rotated anticlockwise by an angle α, in other words it is the approximate eigenstate for the Hamiltonian where the two wells have been rotated into new positions.

The geometric phase ϑ associated with a change in α is determined by the equation

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} = i \int (R\,\mathrm{d}R)\mathrm{d}\Phi\,(r\,\mathrm{d}r)\mathrm{d}\phi\;\chi_\alpha^* \frac{\partial \chi_\alpha}{\partial \alpha} = -\nu + \frac{\langle L \rangle}{\hbar}\,. \tag{517}$$

$\langle L \rangle$ is the expectation value in the state χ_α of the angular momentum operator

$$L = -i\hbar \left(\frac{\partial}{\partial \Phi} + \frac{\partial}{\partial \phi} \right). \tag{518}$$

Since we have assumed that the two wells are so far separated that there is negligible overlap of the two ground state wave functions $\psi_{\boldsymbol{a}}$ and $\psi_{-\boldsymbol{a}}$, it can not matter for the expectation value $\langle L \rangle$ whether the particles are bosons, fermions or anyons. Hence we conclude that $\langle L \rangle = 0$ always, as is the case for bosons. Consequently, the geometric phase associated with an interchange by rotation an angle π is the negative of the anyonic statistics angle θ,

$$\vartheta^O = \pi \frac{\mathrm{d}\vartheta}{\mathrm{d}\alpha} = -\nu\pi = -\theta\,. \tag{519}$$

10.5 Laughlin's theory of the fractional quantum Hall effect

Arovas *et al.* used the concept of the geometric phase in order to calculate the charge and statistics of the elementary excitations in Laughlin's theory of the fractional quantum Hall effect [62,175]. Although the original idea of Laughlin was very simple and elegant, it applied only to the simple fractions 1/3, 1/5, etc., and the hierarchical extensions of the theory needed for other fractions become rather complicated [171,172]. As our final example, we will discuss the calculation of Arovas, Schrieffer and Wilczek for sufficiently small number of electrons that it can be done either exactly or numerically [285,286].

Since we want to discuss both particles of positive and of negative charge, we will assume throughout that the magnetic field is positive, $B > 0$. Then since electrons have negative charge $q = -e$, we will have $qB < 0$ for the electrons. The canonical unit of length is the magnetic length for electrons,

$$\lambda_\mathrm{e} = \sqrt{\frac{\hbar}{eB}}\,, \tag{520}$$

and the one-particle basis states are the complex conjugates of the wave functions χ_n in equation (432). In the state χ_n the particle is located approximately at a distance $|z| = \sqrt{n}$ from the origin, since that is where the probability density $|\chi_n|^2$ is maximal. Hence, if we put an upper limit on n, say $n < M$, this means that we have an approximate description of a system of finite radius $r = \lambda_e|z| < \lambda_e\sqrt{M}$.

If we now distribute N electrons among the M first one-particle states in the lowest Landau level, the filling fraction is $\nu_f = N/M$. The particles may be electrons, and we may make the (somewhat dubious) assumption that the electron spin is completely polarized in the magnetic field, so that the N-electron system is described by a wave function which is a totally antisymmetric function of the particle positions $z_1, z_2, \ldots, z_N$. For a filling fraction $\nu_f = 1/\mu$, where μ is an odd integer, the non-normalized wave function proposed by Laughlin is

$$\psi_0^\mu = \prod_{j<k} (z_j^* - z_k^*)^\mu \exp\left(-\frac{1}{2}\sum_{j=1}^N |z_j|^2\right). \tag{521}$$

It is antisymmetric when μ is odd, and is obviously a ground state of a system of non-interacting electrons, built from the one-particle states $\chi_0^*, \chi_1^*, \ldots, \chi_{M-1}^*$, where

$$M = \mu(N-1) + 1 \approx \mu N\,. \tag{522}$$

Due to the factor $(z_j^* - z_k^*)^\mu$, the wave function ψ_0^μ minimizes very well, if not perfectly, the probability of finding two particles j and k close together, so that it is still an approximate ground state of the system when we take into account the Coulomb repulsion between the electrons.

The simplest excitation of the system is a vortex at an arbitrary position $z = \zeta$, described by the wave function

$$\psi_1 = \prod_{j=1}^N (z_j^* - \zeta^*)\ \psi_0^\mu\,. \tag{523}$$

The vortex represents a "quasihole", since it repels the electrons and thereby creates an excess of positive background charge at ζ (we assume that the total electron charge is neutralized by a uniform background density of positive charge). Note that it takes μ vortices at the same position to create a positive excess charge equal in magnitude to the electron charge, thus we expect naively that the charge of the vortex is $1/\mu$ (or rather $-1/\mu$) of the electron charge.

We now ask for the geometric phase ϑ_1 arising when the vortex is moved around the circle $|\zeta| = \rho$. That is, we parametrize $\zeta = \rho e^{i\alpha}$ and let α increase

from 0 to 2π. The total phase is

$$\vartheta_1^O = \int_0^{2\pi} \mathrm{d}\alpha \, \frac{\mathrm{d}\vartheta_1}{\mathrm{d}\alpha} = 2\pi \frac{\mathrm{d}\vartheta_1}{\mathrm{d}\alpha} \cdot \tag{524}$$

Since ψ_1 depends on α only through $\zeta = \rho \mathrm{e}^{i\alpha}$, and is an analytic function of ζ^*, we have, by a similar reasoning as before, that

$$\frac{\mathrm{d}\vartheta_1}{\mathrm{d}\alpha} = -\frac{1}{I_1} \operatorname{Im} \left(\int \prod_{j=1}^{N} \mathrm{d}^2 z_j \; \psi_1^* \frac{\partial \psi_1}{\partial \alpha} \right) = \rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I_1(\rho) \; , \tag{525}$$

where $I_1(\rho)$ is the normalization integral,

$$\begin{aligned} I_1(\rho) &= \int \prod_{j=1}^{N} \mathrm{d}^2 z_j \; |\psi_1|^2 \\ &= \sum_{k=0}^{N} \rho^{2k} \int \prod_{j=1}^{N} \mathrm{d}^2 z_j \; |c_{N-k}(z_1^*, \; \ldots \, , z_N^*)|^2 |\psi_0^\mu|^2 \; . \end{aligned} \tag{526}$$

Here c_k are the elementary symmetric polynomials encountered earlier, see equations (226) and (237).

We may try to interpret the phase ϑ_1^O as due to the motion of a charge q_1 in the magnetic field, and then the vortex charge q_1 is related in the following way to the absolute value of the electron charge, $|q| = e$. According to equation (491), $\mathrm{d}\vartheta_1/\mathrm{d}\alpha$ is proportional to the charge q_1, and according to equation (488) we have $\mathrm{d}\vartheta/\mathrm{d}\alpha = \rho^2$ for a positive charge e. Hence,

$$\frac{q_1}{e} = \frac{1}{\rho^2} \frac{\mathrm{d}\vartheta_1}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I_1(\rho) \; . \tag{527}$$

In particular, we conclude immediately that q_1 is positive.

The normalization integral $I_1(\rho)$ is tabulated in Table 2 for $\mu = 3$ and up to 5 electrons. Based on this table, the quantity $\mathrm{d}(\ln I_1(\rho))/\mathrm{d}\rho^2$ is plotted in Figure 14 as a function of ρ. Although $N = 5$ is a very small number, the plot already suggests that $q_1/e = 1/3$ for large N. This is confirmed by the results of Monte Carlo integrations with $N = 20$, 50, 75, 100 and 200, as shown in Figure 15 [286]. Note that the wave function ψ_1 describes essentially a system of finite radius $\rho_N \approx \sqrt{\mu(N-1)+1}$ (which is 3.6 for $\mu = 3$ and $N = 5$, and 24.5 for $N = 200$), and that $\mathrm{d}(\ln I_1(\rho))/\mathrm{d}\rho^2$ approaches N/ρ^2 for $\rho > \rho_N$. One should therefore consider only the region $\rho < \rho_N$.

The wave function for a state with two vortices, at the positions ζ and $-\zeta$, is

$$\psi_2 = \prod_{j=1}^{N} (z_j^* - \zeta^*) \prod_{j=1}^{N} (z_j^* + \zeta^*) \; \psi_0^\mu = \prod_{j=1}^{N} \left((z_j^*)^2 - (\zeta^*)^2 \right) \; \psi_0^\mu . \tag{528}$$

Table 2. The one-vortex normalization integral I_1 for $\mu = 3$ and for $N = 2, 3, 4, 5$ particles.

N	I_1
2	$8 \cdot 3 \quad (11 + 4\rho^2 + 2\rho^4)$
3	$64 \cdot 81 \cdot 5 \quad (2 \cdot 761 + 2 \cdot 3 \cdot 71\,\rho^2 + 3 \cdot 31\,\rho^4 + 31\,\rho^6)$
4	$2^{15} \cdot 3^8 \cdot 5^2 \cdot 7 \quad (8 \cdot 625 \cdot 41 + 4 \cdot 11 \cdot 1259\,\rho^2 + 4 \cdot 3 \cdot 5 \cdot 157\,\rho^4 + 4 \cdot 353\,\rho^6 + 353\,\rho^8)$
5	$2^{25} \cdot 3^{13} \cdot 5^5 \cdot 7^2 \quad (4 \cdot 122\,297\,213 + 4 \cdot 35\,404\,417\,\rho^2 + 2 \cdot 5 \cdot 787 \cdot 2711\,\rho^4$ $+2 \cdot 27 \cdot 47\,317\,\rho^6 + 3 \cdot 5 \cdot 109 \cdot 179\,\rho^8 + 3 \cdot 109 \cdot 179\,\rho^{10})$

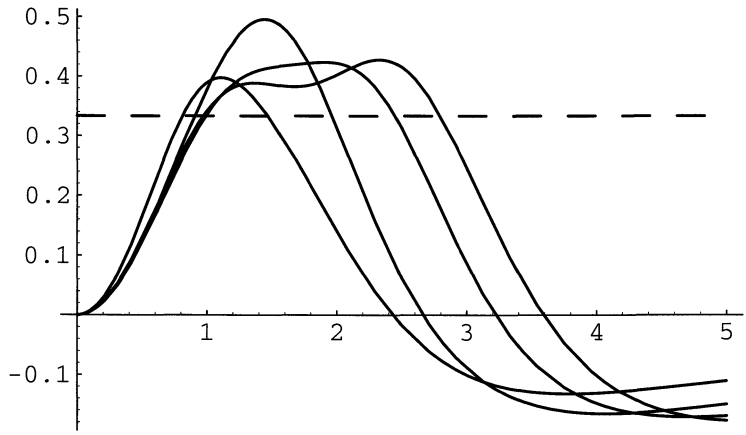


Fig. 16. The distance dependent statistics parameter $-\rho^2(\mathrm{d}/\mathrm{d}\rho^2)(\ln I_2 - 2\ln I_1)$ as a function of ρ. The curves are for $N = 2, 3, 4, 5$ electrons, the leftmost peak for $N = 2$ and the rightmost peak for $N = 5$. The dashed line is $1/3$.

If the two vortices are interchanged by an anticlockwise circular motion an angle π, this again gives rise to a geometric phase

$$\vartheta_2^O = \int_0^\pi \mathrm{d}\alpha \, \frac{\mathrm{d}\vartheta_2}{\mathrm{d}\alpha} = \pi \frac{\mathrm{d}\vartheta_2}{\mathrm{d}\alpha} \cdot \tag{529}$$

By the same derivation as above, we have that

$$\frac{\mathrm{d}\vartheta_2}{\mathrm{d}\alpha} = \rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I_2(\rho) \, , \tag{530}$$

where $I_2(\rho)$ is the normalization integral for ψ_2,

$$I_2(\rho) = \int \prod_{j=1}^{N} \mathrm{d}^2 z_j \; |\psi_2|^2$$
$$= \sum_{k=0}^{N} \rho^{4k} \int \prod_{j=1}^{N} \mathrm{d}^2 z_j \; |c_{N-k}((z_1^*)^2, \ldots, (z_N^*)^2)|^2 |\psi_0^\mu|^2 \; . \qquad (531)$$

It is tabulated in Table 3 for $\mu = 3$ and up to 5 particles.

Subtracting the one-vortex contribution $2\vartheta_1$ from the two-vortex phase ϑ_2, we would like to interpret the remaining phase as due to the quantum statistics of the vortices. More precisely, we use the definition (509) of the distance dependent anyon parameter,

$$\nu_{\text{Berry}} = -\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \left(\ln I_2(\rho) - 2 \ln I_1(\rho) \right) \; . \qquad (532)$$

This is plotted as a function of ρ in Figure 16. It clearly depends on the distance between the vortices, at least for small distances. We should in fact expect the vortex statistics to depend on distance, for small distances, since the vortices are not point particles, but have a finite size. More remarkably, the plot indicates that when the vortices are well separated, then there is an approximately constant part of the Berry phase, giving

$$\nu_{\text{Berry}} \approx \frac{1}{3} \; . \qquad (533)$$

This is again confirmed by Monte Carlo integrations, with N from 20 and up to 200, as shown in Figure 17.

In order to compare with the statistics phases of the anyon states given in equations (503) and (506), we have to remember that our present length scale is the electron magnetic length λ_{e}, corresponding to the elementary charge e, whereas the vortices that we want to describe as anyons have charge e/μ, with $\mu = 3$ in our numerical example, so that the "vortex magnetic length" is $\lambda = \sqrt{\mu}\lambda_{\text{e}}$. This stretching of the length scale must be compensated for by dividing the dimensionless length in the anyon system by $\sqrt{\mu}$ in every formula we use. The comparison is shown in Figure 18, it indicates that the anyon model with $\nu = 1/3$ is a reasonably good description of the Laughlin quasi-hole states for the filling fraction $\nu_{\text{f}} = 1/3$.

Laughlin also proposed wave functions representing quasi-electron excitations, in which the electron density is increased locally. These quasi-electron states were examined in reference [284], and within the approximations used, the results imply that the charge and statistics parameter should have the values $-e/\mu$ and $+1/\mu$, respectively, for a filling fraction of $1/\mu$ with μ odd.

Table 3. The two-vortex normalization integral I_2 for $\mu = 3$ and for $N = 2, 3, 4, 5$ particles.

N	I_2
2	$16 \cdot 3 \quad (64 + 2 \cdot 11\,\rho^4 + \rho^8)$
3	$64 \cdot 81 \cdot 5 \quad (16 \cdot 47 \cdot 223 + 8 \cdot 3 \cdot 1097\,\rho^4 + 2 \cdot 3 \cdot 11 \cdot 23\,\rho^8 + 31\,\rho^{12})$
4	$2^{15} \cdot 3^8 \cdot 5^2 \cdot 7 \quad (128 \cdot 5 \cdot 17 \cdot 26\,561 + 64 \cdot 709 \cdot 839\,\rho^4 + 64 \cdot 3 \cdot 47 \cdot 181\,\rho^8$ $+32 \cdot 7 \cdot 11 \cdot 13\,\rho^{12} + 353\,\rho^{16})$
5	$2^{25} \cdot 3^{13} \cdot 5^5 \cdot 7^2 \quad (256 \cdot 25 \cdot 13 \cdot 23 \cdot 5\,670\,473 + 64 \cdot 7 \cdot 467 \cdot 6\,931\,193\,\rho^4$ $+8 \cdot 5 \cdot 1\,030\,094\,323\,\rho^8 + 4 \cdot 3 \cdot 72\,197\,057\,\rho^{12}$ $+2 \cdot 9 \cdot 13 \cdot 61 \cdot 593\,\rho^{16} + 3 \cdot 109 \cdot 179\,\rho^{20})$

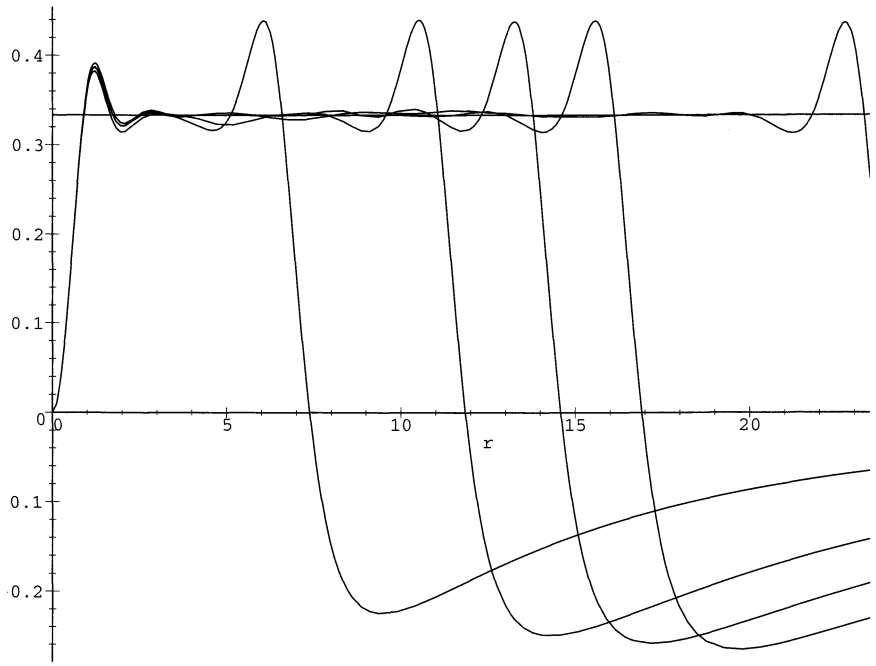


Fig. 17. The quasi-hole statistics parameter ν_{Berry}, equation (532), as a function of ρ, half the distance between the two quasi-holes. The curves are, from left to right, for 20, 50, 75, 100 and 200 electrons, and the horizontal line is 1/3. From [286].

The proposed wave function for one quasi-electron located at the position ζ is the following polynomial in ζ,

$$\psi_1' = \psi_0 \left(\prod_{i=1}^{N} (\partial_i^* - \zeta) \right) (\Delta^*)^\mu, \tag{534}$$

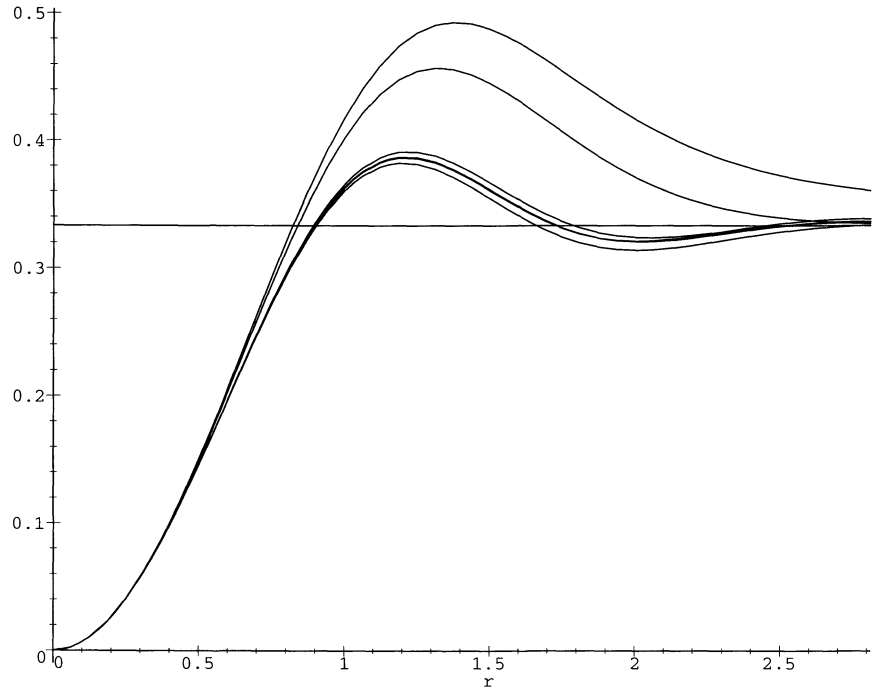


Fig. 18. Comparison of the distance dependent statistics parameter for two Laughlin quasi-holes, and for localized states in the system of two anyons. The lowest lying curve is for 75 electrons, then follows a common curve for the three cases of 20, 50 and 100 electrons, and the third curve is for 200 electrons. Somewhat higher lies the Berry phase curve calculated from the anyon position eigenstate projected onto the lowest Landau level, and even higher the one calculated from the coherent state of the $SU(1,1)$ algebra. From [286].

with $\Delta = \prod_{j<k}(z_j - z_k)$ and $\psi_0 = \exp\left(-\frac{1}{2}\sum_j |z_j|^2\right)$.

The normalization integral may be rewritten by partial integration as

$$I_1' = \int \mathrm{d}^{2N}z\, {\psi_0}^2\, |\Delta|^{2\mu} \prod_{k=1}^{N} (|z_k - \zeta|^2 - 1)\,. \tag{535}$$

It is again a polynomial in $\rho = |\zeta|$, due to rotational invariance. The difference from the quasi-hole normalization integral is the extra -1 in each factor $|z_k - \zeta|^2 - 1$. Without this -1, the quasi-electron and quasi-hole

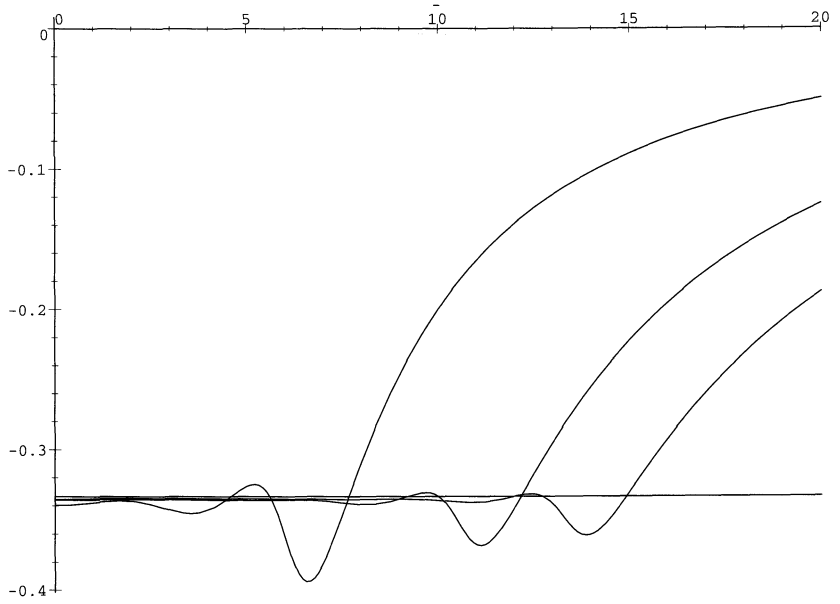


Fig. 19. The quasi-electron charge q_1'/e, equation (536), as a function of ρ, the quasi-electron distance from the origin. The curves are, from left to right, for 20, 50 and 75 electrons. The horizontal line is $-1/3$. From [286].

integrals would be identical, and the charge of the quasi-electron,

$$\frac{q_1'}{e} = -\frac{\mathrm{d}}{\mathrm{d}\rho^2} \ln I_1(\rho) \,, \tag{536}$$

would be the same as that of the quasi-hole, just with an opposite sign, because the quasi-electron wave function depends on ζ and the quasi-hole wave function on ζ^*. This approximation seems hard to justify, nevertheless it may be valid asyptotically for large N, as the results of Monte Carlo integrations shown in Figure 19 seem to indicate.

Two quasi-electrons at ζ and $-\zeta$ are described, according to Laughlin, by the wave function

$$\psi_2' = \psi_0 \left(\prod_{i=1}^{N} (\partial_i^* - \zeta)(\partial_i^* + \zeta) \right) (\Delta^*)^\mu \,, \tag{537}$$

which yields, by partial integration, the normalization integral

$$I_2' = \int \mathrm{d}^{2N} z \, {\psi_0}^2 \, |\Delta|^{2\mu} \prod_{k=1}^{N} \left(|{z_k}^2 - \zeta^2|^2 - 4|z_k|^2 + 2 \right) . \tag{538}$$

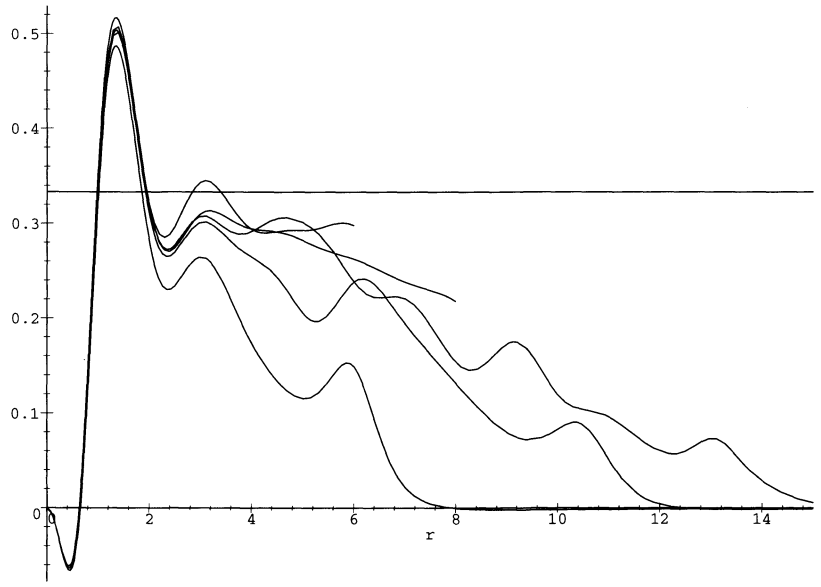


Fig. 20. The distance dependent statistics parameter for quasi-electrons, equation (539), *versus* ρ, half the distance between the quasi-electrons. The lowest lying curve is for 20 electrons. Next are curves for 50 and 75 electrons. The 100 electron curve is cut at $\rho = 8$ and the 200 electron curve at $\rho = 6$, to avoid numerical problems. The horizontal line is 1/3. From [286].

Comparing with the quasi-hole integral in equation (531), we see that for two quasi-electrons there are correction terms, like in the case of a single quasi-electron.

The distance dependent statistics parameter is now

$$\nu'_{\text{Berry}} = \rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \left(\ln I'_2(\rho) - 2 \ln I'_1(\rho) \right) . \tag{539}$$

This quantity, as computed by Monte Carlo integration, is plotted in Figure 20. It is found to be positive, although one can hardly justify the conclusion that the asymptotic value is 1/3, as one might want it to be.

In Figure 21 the Monte Carlo data are compared with the curves for anyons of negative charge $-e/3$ and with the anyon parameter $\nu = 1/3$. Note that these two-anyon states are actually singular where the anyon coordinates coincide. It is seen that the small distance behaviour is well represented by the anyon model, but the behaviour at larger distances is not at all well represented. It may be that the calculations were done with

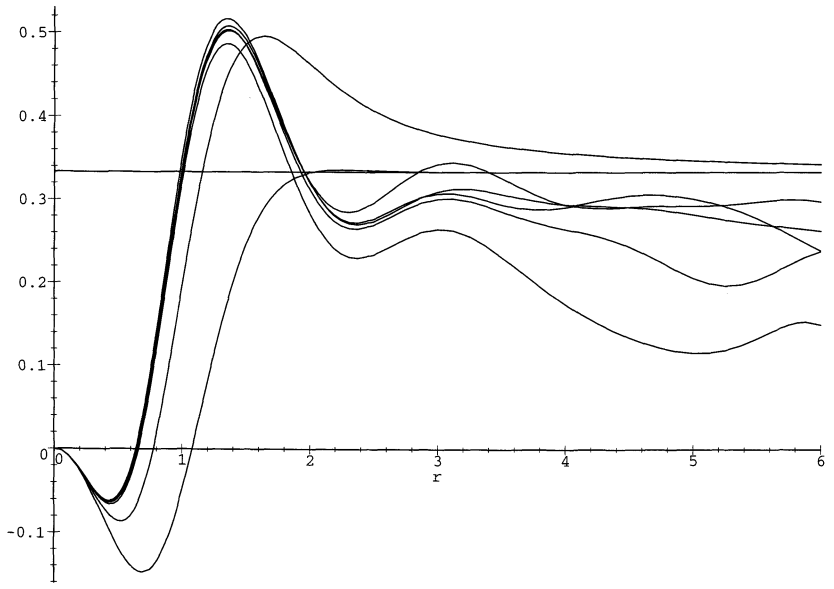


Fig. 21. The distance dependent statistics parameter for Laughlin quasi-electrons, compared to the two-anyon states. The five curves for 20, 50, 75, 100 and 200 electrons coincide for small ρ. The 200 electron curve overshoots the horizontal line $1/3$ at $\rho \approx 3$. The curve lying lowest for small ρ represents an anyon eigenstate projected onto the lowest Landau level, whereas the curve going highest for large ρ represents the coherent state of the $SU(1,1)$ algebra. From [286].

too few electrons, but at least one may conclude that there is a marked difference between the quality of the anyon description for the Laughlin quasi-hole states versus the quasi-electron states.

In these calculations the anyon model was compared with the Laughlin wave functions. A more interesting question is perhaps how it compares with experiment, or if not directly with experiment, at least with some less idealized theoretical model. One such model calculation for the quantum Hall system is that of reference [238], which did not rely on any specific trial wave function, but rather on state counting based on numerical simulations for interacting electrons on a sphere. The value of the one-dimensional exclusion statistics parameter [227] was found to be $1/3$ in the case of quasi-holes, and $2 - 1/3$ for the quasi-electrons, near to the magic filling fraction $1/3$.

The exclusion statistics parameter is in principle the same parameter as one reads from the Berry phase, although with an opposite sign for

the quasi-electrons, because their charge is negative. Thus the predictions for the anyon parameter, based on the numerical results for the exclusion statistics parameter, would be $1/3$ for quasi-holes and $-2+1/3$ for quasi-electrons. The values $1/3$ and $-2+1/3$ for the anyon parameter define of course the same particle statistics, but we distinguish between them here in the way we define the correspondence between quasi-particles and anyons. Thus, in the case of quasi-electrons, $1/3$ and $-2+1/3$ would represent the same species of anyons, but different anyon states, the $1/3$ state singular and the $-2+1/3$ state non-singular.

It is interesting to note that the numerical results for the exclusion statistics parameter of realistic quasi-holes and quasi-electrons are easily interpreted in terms of anyons of positive and negative charge, respectively, with non-singular wave functions, and with the same statistics in the two cases. That quasi-holes and quasi-electrons should have the same statistics, is also what one would expect if one regards them as antiparticles of each other.

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