Lecture Notes for Physics 110A

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Chapter 1

Introduction to Dynamics

1.1 Introduction and Review

Dynamics is the science of how things *move*. A complete solution to the motion of a system means that we know the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, this means we want to know its position vector $\boldsymbol{r}(t)$ as a function of time. If there are many particles, the motion is described by a set of functions $\boldsymbol{r}_i(t)$, where i labels which particle we are talking about. So generally speaking, solving for the motion means being able to predict where a particle will be at any given instant of time. Of course, knowing the function $\boldsymbol{r}_i(t)$ means we can take its derivative and obtain the velocity $\boldsymbol{v}_i(t) = d\boldsymbol{r}_i/dt$ at any time as well.

The complete motion for a system is not given to us outright, but rather is encoded in a set of differential equations, called the *equations of motion*. An example of an equation of motion is

$$m\frac{d^2x}{dt^2} = -mg \tag{1.1}$$

with the solution

$$x(t) = x_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{1.2}$$

where x_0 and v_0 are constants corresponding to the initial *boundary conditions* on the position and velocity: $x(0) = x_0$, $v(0) = v_0$. This particular solution describes the vertical motion of a particle of mass m moving near the earth's surface.

In this class, we shall discuss a general framework by which the equations of motion may be obtained, and methods for solving them. That "general framework" is Lagrangian Dynamics, which itself is really nothing more than an elegant restatement of Isaac Newton's Laws of Motion.

1.1.1 Newton's laws of motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens.

To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

I. A body remains in uniform motion unless acted on by a force.
II. Force equals rate of change of momentum: $\boldsymbol{F} = d\boldsymbol{p}/dt$.
III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t$. Now $\boldsymbol{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\boldsymbol{r}(t)$ in a different coordinate system whose origin $\boldsymbol{R}$ moves according to the function $\boldsymbol{R}(t)$, then in this new "frame of reference" the position of our particle will be

$$\begin{aligned} \boldsymbol{r}'(t) &= \boldsymbol{r}(t) - \boldsymbol{R}(t) \\ &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t - \boldsymbol{R}(t) \ . \end{aligned}$$

If the acceleration $d^2\boldsymbol{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton's Laws hold. A transformation from one frame $\mathcal{K}$ to another frame $\mathcal{K}'$ which moves at constant velocity $\boldsymbol{V}$ relative to $\mathcal{K}$ is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth's surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum $\boldsymbol{p}$, defined as the product $\boldsymbol{p} = m\boldsymbol{v}$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force $\boldsymbol{F}$ depends on the coordinates (or possibly velocities) themselves. This is known as a *force law.* Examples of force laws include:

$$\begin{aligned}
\text{Constant force:} &\qquad \boldsymbol{F} = -m\boldsymbol{g} \\
\text{Hooke's Law:} &\qquad F = -kx \\
\text{Gravitation:} &\qquad \boldsymbol{F} = -GMm\,\hat{\boldsymbol{r}}/r^2 \\
\text{Lorentz force:} &\qquad \boldsymbol{F} = q\,\boldsymbol{E} + q\,\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} \\
\text{Fluid friction } (v \text{ small}): &\qquad \boldsymbol{F} = -b\,\boldsymbol{v}\ .
\end{aligned}$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and hence the accelerations are parallel to each other, with

$$\frac{a_1}{a_2} = -\frac{m_2}{m_1}\ , \tag{1.3}$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is a constant, a result known as the *conservation of momentum.*

1.1.2 Aside : inertial *vs.* gravitational mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force $\boldsymbol{F}_{ij}$ exerted by a particle i by another particle j is

$$\boldsymbol{F}_{ij} = -Gm_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}\ , \tag{1.4}$$

where G, the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11}\mathrm{N}\cdot\mathrm{m}^2/\mathrm{kg}^2\ . \tag{1.5}$$

Notice Newton's Third Law in action: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_{\mathrm{e}}$, with $\boldsymbol{r}_i - \boldsymbol{r}_j \simeq R_{\mathrm{e}}\hat{\boldsymbol{r}}$ and obtain

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} \equiv -m\boldsymbol{g} \tag{1.6}$$

where $\hat{\boldsymbol{r}}$ is a radial unit vector pointing from the earth's center and $g = GM_e/R_e^2 \simeq 9.8\,\text{m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\boldsymbol{a} = -\boldsymbol{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$\boldsymbol{a} = -\frac{m'}{m}\,\boldsymbol{g} \tag{1.7}$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.2 Examples of Motion in One Dimension

To gain some experience with solving equations of motion in a physical setting, we consider some physically relevant examples of one-dimensional motion.

1.2.1 Uniform force

With $F = -mg$, appropriate for a particle falling under the influence of a uniform gravitational field, we have $m\,d^2x/dt^2 = -mg$, or $\ddot{x} = -g$. Notation:

$$\dot{x} \equiv \frac{dx}{dt}\ , \qquad \ddot{x} \equiv \frac{d^2x}{dt^2}\ , \qquad \overset{:::\cdot}{x} = \frac{d^7x}{dt^7}\ , \qquad etc. \tag{1.8}$$

With $v = \dot{x}$, we solve $dv/dt = -g$:

$$\begin{aligned} \int\limits_{v(0)}^{v(t)} dv &= \int\limits_0^t ds\,(-g) \\ v(t) - v(0) &= -gt\ . \end{aligned} \tag{1.9}$$

Note that there is a constant of integration, $v(0)$, which enters our solution.

We are now in position to solve $dx/dt = v$:

$$\begin{aligned} \int\limits_{x(0)}^{x(t)} dx &= \int\limits_0^t ds\, v(s) \\ x(t) &= x(0) + \int\limits_0^t ds\,\big[v(0) - gs\big] \\ &= x(0) + v(0)t - \tfrac{1}{2}gt^2\ . \end{aligned} \tag{1.10}$$

Note that a second constant of integration, $x(0)$, has appeared.

1.2.2 Uniform force with linear frictional damping

In this case,

$$m\frac{dv}{dt} = -mg - \gamma v \tag{1.11}$$

which may be rewritten

$$\begin{aligned} \frac{dv}{v + mg/\gamma} &= -\frac{\gamma}{m} dt \\ d\ln(v + mg/\gamma) &= -(\gamma/m)dt \ . \end{aligned} \tag{1.12}$$

Integrating then gives

$$\begin{aligned} \ln\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) &= -\gamma t/m \\ v(t) &= -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma}\right) e^{-\gamma t/m} \ . \end{aligned} \tag{1.13}$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$x(t) = x(0) + \frac{m}{\gamma}\left(v(0) + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma} t \ . \tag{1.14}$$

The solution to the *second* order ODE $m\ddot{x} = -mg - \gamma\dot{x}$ thus entails *two* constants of integration: $v(0)$ and $x(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \ . \tag{1.15}$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity})$$

$$\begin{aligned} j &= n \cdot (-e) \cdot (-v_\infty) \\ &= \frac{ne^2\tau}{m} E \ . \end{aligned} \tag{1.16}$$

The ratio j/E is called the *conductivity* of the metal, σ. According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations ("phonons"). In high purity copper at low temperatures ($T \lesssim 4\,\mathrm{K}$), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}\,\mathrm{s}$).

1.2.3 Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_{\rm f} = -cv^2 \, {\rm sgn}\,(v)$, where ${\rm sgn}\,(v)$ is the *sign* of v: ${\rm sgn}\,(v) = +1$ if $v > 0$ and ${\rm sgn}\,(v) = -1$ if $v < 0$. (Note one can also write ${\rm sgn}\,(v) = v/|v|$ where $|v|$ is the *absolute value.*) Why all this trouble with ${\rm sgn}\,(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that $F_{\rm f}$ be *oppositely directed* with respect to the velocity v. We will assume that $v < 0$ always, hence $F_{\rm f} = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \tag{1.17}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left[\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty}\right] \tag{1.18}$$

we obtain

$$\begin{aligned} \frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \\ &= \frac{1}{2v_\infty} d\ln\left(\frac{v_\infty - v}{v_\infty + v}\right) \\ &= \frac{g}{v_\infty^2}\, dt \ . \end{aligned} \tag{1.19}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\frac{1}{2v_\infty}\ln\left(\frac{v_\infty - v(t)}{v_\infty + v(t)}\right) = \frac{gt}{v_\infty^2} \tag{1.20}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \ . \tag{1.21}$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \ . \tag{1.22}$$

Again, as $t \to \infty$ one has $v(t) \to -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Advanced Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt, all the fluid molecules inside the volume $\Delta V = S \cdot v\,\Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v-u$ and it has suffered a change in velocity of $\Delta u = 2(v-u)$. The total momentum change is obtained by multiplying Δu by the total mass $M = \varrho\,\Delta V$, where ϱ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v-u)\cdot \varrho\, S\, v\, \Delta t \tag{1.23}$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S\,\varrho\, v(v-u)\ . \tag{1.24}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u\rangle = 0$, we obtain the result $\langle F\rangle = 2S\varrho v^2$, where $\langle\cdots\rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_{\rm f} = -\langle F\rangle = -cv^2$, where $c = 2S\varrho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu\varrho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.2.4 Crossed electric and magnetic fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We'll throw in gravity for good measure. We take $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. The equation of motion is Newton's 2nd Law again:

$$m\,\ddot{\boldsymbol{r}} = m\boldsymbol{g} + q\boldsymbol{E} + \tfrac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B}\ . \tag{1.25}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$\begin{aligned} m\ddot{x} &= qE + \frac{qB}{c}\,\dot{y} \\ m\ddot{y} &= -\frac{qB}{c}\,\dot{x} \\ m\ddot{z} &= -mg\ . \end{aligned} \tag{1.26}$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0)\,t - \tfrac{1}{2}gt^2\ . \tag{1.27}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\begin{aligned} \dot{v}_x &= \omega_{\rm c}(v_y + u_{\rm D}) \\ \dot{v}_y &= -\omega_{\rm c}\, v_x\ , \end{aligned} \tag{1.28}$$

where $\omega_{\rm c} = qB/mc$ is the *cyclotron frequency* and $u_{\rm D} = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$\begin{aligned} v_x(t) &= v_x(0)\,\cos(\omega_{\rm c}t) + \big(v_y(0) + u_{\rm D}\big)\,\sin(\omega_{\rm c}t) \\ v_y(t) &= -u_{\rm D} + \big(v_y(0) + u_{\rm D}\big)\cos(\omega_{\rm c}t) - v_x(0)\,\sin(\omega_{\rm c}t)\ . \end{aligned} \tag{1.29}$$

Integrating again, the full motion is given by:

$$\begin{aligned} x(t) &= x(0) + A\,\sin\delta + A\,\sin(\omega_{\rm c} t - \delta) \\ y(r) &= y(0) - u_{\rm D}\, t - A\cos\delta + A\,\cos(\omega_{\rm c} t - \delta)\ , \end{aligned} \tag{1.30}$$

where

$$A = \frac{1}{\omega_{\rm c}}\sqrt{\dot{x}^2(0) + \big(\dot{y}(0) + u_{\rm D}\big)^2} \quad , \quad \delta = \tan^{-1}\left(\frac{\dot{y}(0) + u_{\rm D}}{\dot{x}(0)}\right) . \tag{1.31}$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0)\ ,\ y(0)\ ,\ z(0)\ ,\ A\ ,\ \delta\ ,\ \dot{z}(0)\ . \tag{1.32}$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

1.3 Pause for Reflection

In mechanical systems, for each coordinate, or "degree of freedom," there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

Chapter 2

Systems of Particles

2.1 Work-Energy Theorem

Consider a system of many particles, with positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \ . \tag{2.1}$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \, \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i \ . \tag{2.2}$$

Here, we've used the relation

$$\frac{d}{dt} \big(\boldsymbol{A}^2 \big) = 2 \, \boldsymbol{A} \cdot \frac{d\boldsymbol{A}}{dt} \ . \tag{2.3}$$

We now invoke Newton's 2nd Law, $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, to write eqn. 2.2 as $\dot{T}_i = \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i$. We integrate this equation from time t_{A} to t_{B}:

$$\begin{aligned} T_i^{(\text{B})} - T_i^{(\text{A})} &= \int\limits_{t_{\text{A}}}^{t_{\text{B}}} \! dt \ \frac{dT_i}{dt} \\ &= \int\limits_{t_{\text{A}}}^{t_{\text{B}}} \! dt \, \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \equiv \sum_i W_i^{(\text{A}\to\text{B})} \ , \end{aligned} \tag{2.4}$$

where $W_i^{(\text{A}\to\text{B})}$ is the total *work done* on particle i during its motion from state A to state B, Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{(\text{A}\to\text{B})} = \sum_i W_i^{(\text{A}\to\text{B})}$. Eqn. 2.4 is known as the *work-energy theorem*. It says that

In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done: $T^{(\text{B})} - T^{(\text{A})} = W^{(\text{A}\to\text{B})}$.

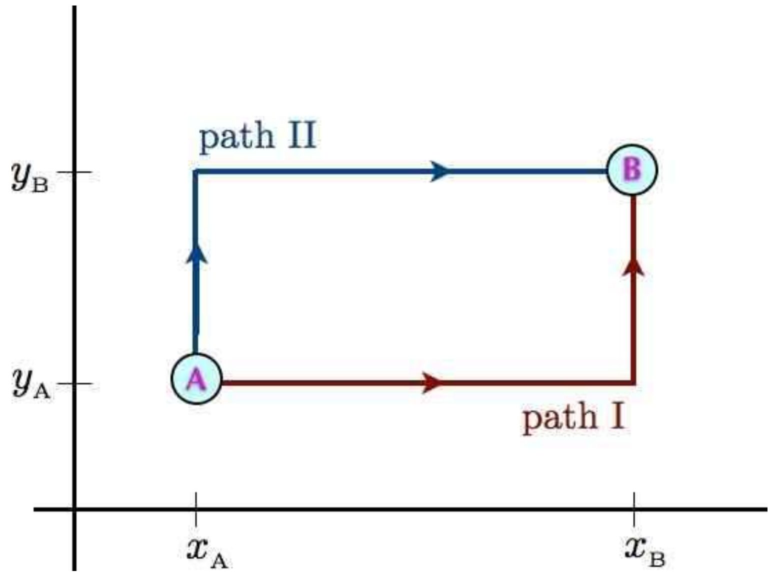

Figure 2.1: Two paths joining points A and B.

2.2 Conservative and Nonconservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{(\mathrm{A}\to\mathrm{B})} = \int_{t_\mathrm{A}}^{t_\mathrm{B}} dt\, \boldsymbol{F}\cdot\boldsymbol{v} \ , \tag{2.5}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. This is the most general expression for the work done. If the force $\boldsymbol{F}$ depends only on the particle's position $\boldsymbol{r}$, we may write $d\boldsymbol{r} = \boldsymbol{v}\, dt$, and then

$$W^{(\mathrm{A}\to\mathrm{B})} = \int_{\boldsymbol{r}_\mathrm{A}}^{\boldsymbol{r}_\mathrm{B}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) \ . \tag{2.6}$$

Consider now the force

$$\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}} \ , \tag{2.7}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 2.1:

$$\begin{aligned} W^{(\mathrm{I})} &= K_1 \int_{x_\mathrm{A}}^{x_\mathrm{B}} dx\, y_\mathrm{A} + K_2 \int_{y_\mathrm{A}}^{y_\mathrm{B}} dy\, x_\mathrm{B} = K_1\, y_\mathrm{A}\,(x_\mathrm{B} - x_\mathrm{A}) + K_2\, x_\mathrm{B}\,(y_\mathrm{B} - y_\mathrm{A}) \\ W^{(\mathrm{II})} &= K_1 \int_{x_\mathrm{A}}^{x_\mathrm{B}} dx\, y_\mathrm{B} + K_2 \int_{y_\mathrm{A}}^{y_\mathrm{B}} dy\, x_\mathrm{A} = K_1\, y_\mathrm{B}\,(x_\mathrm{B} - x_\mathrm{A}) + K_2\, x_\mathrm{A}\,(y_\mathrm{B} - y_\mathrm{A}) \ . \end{aligned} \tag{2.8}$$

Note that in general $W^{(\mathrm{I})} \neq W^{(\mathrm{II})}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(\mathrm{I})} - W^{(\mathrm{II})} = (K_2 - K_1)\,(x_{\mathrm{B}} - x_{\mathrm{A}})\,(y_{\mathrm{B}} - y_{\mathrm{A}}) \ . \tag{2.9}$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 2.1. The reason for this is Stokes' theorem:

$$\oint\limits_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F} = \int\limits_{\mathcal{C}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} \ . \tag{2.10}$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path[1], dS is the scalar differential area element, $\hat{\boldsymbol{n}}$ is the unit normal to that differential area element, and $\boldsymbol{\nabla} \times \boldsymbol{F}$ is the curl of $\boldsymbol{F}$:

$$\begin{aligned}\boldsymbol{\nabla} \times \boldsymbol{F} &= \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \\ &= \bigg(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\bigg)\hat{\boldsymbol{x}} + \bigg(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\bigg)\hat{\boldsymbol{y}} + \bigg(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\bigg)\hat{\boldsymbol{z}} \ .\end{aligned} \tag{2.11}$$

For the force under consideration, $\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}}$, the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = (K_2 - K_1)\,\hat{\boldsymbol{z}} \ , \tag{2.12}$$

which is a constant. The RHS of eqn. 2.10 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 2.9, we evaluate the integral $\oint\limits_{\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F}$ along the path $\gamma_{\mathrm{II}}^{-1} \circ \gamma_{\mathrm{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the integral of $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 2.9.

When $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ everywhere in space, we can always write $\boldsymbol{F} = -\boldsymbol{\nabla} U$, where $U(\boldsymbol{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$W^{(\mathrm{A}\to\mathrm{B})} = \int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) = -\int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r} \cdot \boldsymbol{\nabla} U = U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ . \tag{2.13}$$

The work-energy theorem then gives

$$T^{(\mathrm{B})} - T^{(\mathrm{A})} = U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ , \tag{2.14}$$

which says

$$E^{(\mathrm{B})} = T^{(\mathrm{B})} + U(\boldsymbol{r}_{\mathrm{B}}) = T^{(\mathrm{A})} + U(\boldsymbol{r}_{\mathrm{A}}) = E^{(\mathrm{A})} \ . \tag{2.15}$$

Thus, the total energy $E = T + U$ is conserved.

[1] If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

2.2.1 Example : integrating $\boldsymbol{F} = -\boldsymbol{\nabla} U$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can compute $U(\boldsymbol{r})$ by integrating, *viz.*

$$U(\boldsymbol{r}) = U(\boldsymbol{0}) - \int\limits_{\boldsymbol{0}}^{\boldsymbol{r}} d\boldsymbol{r}' \cdot \boldsymbol{F}(\boldsymbol{r}') \ . \tag{2.16}$$

The integral does not depend on the path chosen connecting $\boldsymbol{0}$ and $\boldsymbol{r}$. For example, we can take

$$U(x,y,z) = U(0,0,0) \ - \int\limits_{(0,0,0)}^{(x,0,0)} dx'\, F_x(x',0,0) \ - \int\limits_{(x,0,0)}^{(x,y,0)} dy'\, F_y(x,y',0) \ - \int\limits_{(z,y,0)}^{(x,y,z)} dz'\, F_z(x,y,z') \ . \tag{2.17}$$

The constant $U(0,0,0)$ is arbitrary and impossible to determine from $\boldsymbol{F}$ alone.

As an example, consider the force

$$\boldsymbol{F}(\boldsymbol{r}) = -ky\,\hat{\boldsymbol{x}} - kx\,\hat{\boldsymbol{y}} - 4bz^3\,\hat{\boldsymbol{z}} \ , \tag{2.18}$$

where k and b are constants. We have

$$\begin{aligned} \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_x &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_y &= \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_z &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \ , \end{aligned} \tag{2.19}$$

so $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ and $\boldsymbol{F}$ must be expressible as $\boldsymbol{F} = -\boldsymbol{\nabla} U$. Integrating using eqn. 2.17, we have

$$\begin{aligned} U(x,y,z) &= U(0,0,0) \ + \int\limits_{(0,0,0)}^{(x,0,0)} dx'\, k \cdot 0 \ + \int\limits_{(x,0,0)}^{(x,y,0)} dy'\, kxy' \ + \int\limits_{(z,y,0)}^{(x,y,z)} dz'\, 4bz'^3 \\ &= U(0,0,0) + kxy + bz^4 \ . \end{aligned} \tag{2.20}$$

Another approach is to integrate the partial differential equation $\boldsymbol{\nabla} U = -\boldsymbol{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\boldsymbol{x}}$-component,

$$\frac{\partial U}{\partial x} = ky \ . \tag{2.21}$$

Integrating, we obtain

$$U(x,y,z) = kxy + f(y,z) \ , \tag{2.22}$$

where $f(y,z)$ is at this point an *arbitrary function* of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \quad \Longrightarrow \quad U(x,y,z) = kxy + g(x,z) \ . \tag{2.23}$$

Finally, the z-component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \quad \Longrightarrow \quad U(x,y,z) = bz^4 + h(x,y) \ . \tag{2.24}$$

We now equate the first two expressions:

$$kxy + f(y,z) = kxy + g(x,z) \ . \tag{2.25}$$

Subtracting kxy from each side, we obtain the equation $f(y,z) = g(x,z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y,z) = g(x,z) = q(z) \ , \tag{2.26}$$

where $q(z)$ is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x,y) = kxy + q(z) \ . \tag{2.27}$$

The only possible solution is $h(x,y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x,y,z) = C + kxy + bz^4 \ . \tag{2.28}$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x,y,z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z. The fact that $f(y,z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y,z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y,z)$.

2.3 Conservative Forces in Many Particle Systems

$$\begin{aligned} T &= \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \\ U &= \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \end{aligned} \tag{2.29}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r} - \boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \, \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\text{ext})} + \boldsymbol{F}_i^{(\text{int})} \ , \tag{2.30}$$

with

$$\begin{aligned} \boldsymbol{F}_i^{(\text{ext})} &= -\frac{\partial V(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \\ \boldsymbol{F}_i^{(\text{int})} &= -\sum_j \frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\boldsymbol{r}_i} \equiv \sum_j \boldsymbol{F}_{ij}^{(\text{int})} \ . \end{aligned} \tag{2.31}$$

Here, $\boldsymbol{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j:

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\partial \boldsymbol{r}_i} = -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\, v'\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.32}$$

Note that $\boldsymbol{F}_{ij}^{(\text{int})} = -\boldsymbol{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$, in which case we may write the interparticle force as

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\hat{\boldsymbol{r}}_{ij}\, v'\big(r_{ij}\big) \ . \tag{2.33}$$

2.4 Linear and Angular Momentum

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$. Its rate of change is

$$\frac{d\boldsymbol{P}}{dt} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\sum_{i\neq j} \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{F}_{ij}^{(\text{int})}+\boldsymbol{F}_{ji}^{(\text{int})}=0} = \boldsymbol{F}_{\text{tot}}^{(\text{ext})} \ , \tag{2.34}$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\begin{aligned} \boldsymbol{P} &= \sum_i m_i \dot{\boldsymbol{r}}_i = M\dot{\boldsymbol{R}} \\ M &= \sum_i m_i \quad \text{(total mass)} \\ \boldsymbol{R} &= \frac{\sum_i m_i\, \boldsymbol{r}_i}{\sum_i m_i} \quad \text{(center-of-mass)} \ . \end{aligned} \tag{2.35}$$

Next, consider the total angular momentum,

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \ . \tag{2.36}$$

The rate of change of $\boldsymbol{L}$ is then

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \sum_i \big\{ m_i \dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i + m_i \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i \big\} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}^{(\text{int})} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\tfrac{1}{2}\sum_{i\neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{r}_{ij}\times\boldsymbol{F}_{ij}^{(\text{int})}=0} = \boldsymbol{N}_{\text{tot}}^{(\text{ext})} \ . \end{aligned} \tag{2.37}$$

Finally, it is useful to establish the result

$$T = \tfrac{1}{2} \sum_i m_i\, \dot{\boldsymbol{r}}_i^2 = \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \sum_i m_i \big(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{R}}\big)^2 \ , \tag{2.38}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$\begin{aligned} 0 &= \int\limits_{\text{initial}}^{\text{final}} dE = \int\limits_{\text{initial}}^{\text{final}} dT + \int\limits_{\text{initial}}^{\text{final}} dU \\ &= T^{(\text{B})} - T^{(\text{A})} - \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i \ , \end{aligned} \tag{2.39}$$

which is to say

$$\Delta T = T^{(\text{B})} - T^{(\text{A})} = \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i = -\Delta U \ . \tag{2.40}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.41}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i \, \phi(\boldsymbol{r}_i) \longrightarrow \int d^3\!r \, \rho(\boldsymbol{r}) \, \phi(\boldsymbol{r}) \ , \tag{2.42}$$

where $\rho(\boldsymbol{r})$ is the mass density, and $\phi(\boldsymbol{r})$ is any function.

2.5 Scaling of Solutions for Homogeneous Potentials

2.5.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U\big(\lambda \, \boldsymbol{r}_1, \ldots, \lambda \, \boldsymbol{r}_N\big) = \lambda^k \, U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \ . \tag{2.43}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) = -G \sum_{i<j} \frac{m_i \, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \quad ; \quad k = -1 \ , \tag{2.44}$$

and the harmonic oscillator,

$$U\big(q_1, \ldots, q_n\big) = \tfrac{1}{2} \sum_{\sigma,\sigma'} V_{\sigma\sigma'} \, q_\sigma \, q_{\sigma'} \quad ; \quad k = +2 \ . \tag{2.45}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1, \ldots, x_n)$ is homogeneous:

$$H(\lambda\, x_1, \ldots, \lambda\, x_n) = \lambda^k\, H(x_1, \ldots, x_n) \ . \tag{2.46}$$

Then

$$\boxed{\frac{d}{d\lambda}\bigg|_{\lambda=1} H\big(\lambda\, x_1, \ldots, \lambda\, x_n\big) = \sum_{i=1}^{n} x_i\, \frac{\partial H}{\partial x_i} = k\, H} \tag{2.47}$$

2.5.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\boldsymbol{r}_i = \alpha\, \tilde{\boldsymbol{r}}_i \qquad , \qquad t = \beta\, \tilde{t} \ . \tag{2.48}$$

Then

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\alpha}{\beta}\, \frac{d\tilde{\boldsymbol{r}}_i}{d\tilde{t}} \qquad , \qquad \frac{d^2\boldsymbol{r}_i}{dt^2} = \frac{\alpha}{\beta^2}\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} \ . \tag{2.49}$$

The force $\boldsymbol{F}_i$ is given by

$$\begin{aligned} \boldsymbol{F}_i &= -\frac{\partial}{\partial \boldsymbol{r}_i}\, U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \\ &= -\frac{\partial}{\partial(\alpha\tilde{\boldsymbol{r}}_i)}\, \alpha^k\, U\big(\tilde{\boldsymbol{r}}_1, \ldots, \tilde{\boldsymbol{r}}_N\big) = \alpha^{k-1}\, \tilde{\boldsymbol{F}}_i \ . \end{aligned} \tag{2.50}$$

Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2}\, m_i\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} = \alpha^{k-1}\, \tilde{\boldsymbol{F}}_i \ . \tag{2.51}$$

If we choose β such that

We now demand

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \ , \tag{2.52}$$

then the equation of motion is invariant under the rescaling transformation! This means that if $\boldsymbol{r}(t)$ is a solution to the equations of motion, then so is $\alpha\, \boldsymbol{r}\big(\alpha^{\frac{1}{2}k-1}\, t\big)$. This gives us an entire one-parameter family of solutions, for all real positive α.

If $\boldsymbol{r}(t)$ is periodic with period T, the $\boldsymbol{r}_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k}\, T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \ . \tag{2.53}$$

Here, $\alpha = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly:

$$\begin{aligned}
\big[v\big] &= \frac{L}{T} && \Rightarrow && \frac{v'}{v} = \frac{L'}{L}\bigg/\frac{T'}{T} = \alpha^{\frac{1}{2}k} \\
\big[E\big] &= \frac{ML^2}{T^2} && \Rightarrow && \frac{E'}{E} = \left(\frac{L'}{L}\right)^2\bigg/\left(\frac{T'}{T}\right)^2 = \alpha^k \\
\big[\boldsymbol{L}\big] &= \frac{ML^2}{T} && \Rightarrow && \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2\bigg/\frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \ .
\end{aligned} \tag{2.54}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\alpha) = \alpha \, q_\sigma(t) \ . \tag{2.55}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\alpha) = \alpha \, \boldsymbol{r}\big(\alpha^{-3/2}\, t\big) \ . \tag{2.56}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 \ , \tag{2.57}$$

also known as Kepler's Third Law.

2.6 Appendix I : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \ldots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \ldots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu\big(x_1, \ldots, x_d\big) \ . \tag{2.58}$$

In general these are nonlinear equations.

Let $\hat{\mathbf{e}}_i^0 = \hat{\boldsymbol{x}}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{\mathbf{e}}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\boldsymbol{s} = \sum_{i=1}^d \hat{\mathbf{e}}_i^0 \, dx_i = \sum_{\mu=1}^d \hat{\mathbf{e}}_\mu \, h_\mu(q)\, dq_\mu \ , \tag{2.59}$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν. Finding the coefficient of dq_μ then gives

$$h_\mu(q)\, \hat{\mathbf{e}}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu}\, \hat{\mathbf{e}}_i^0 \qquad \Rightarrow \qquad \hat{\mathbf{e}}_\mu = \sum_{i=1}^d M_{\mu i}\, \hat{\mathbf{e}}_i^0 \ , \tag{2.60}$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)}\,\frac{\partial x_i}{\partial q_\mu} \ . \tag{2.61}$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu = \big(MM^{\rm t}\big)_{\mu\nu} = \frac{1}{h_\mu(q)\,h_\nu(q)}\,\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu}\,\frac{\partial x_i}{\partial q_\nu} \ . \tag{2.62}$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu}\,\frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q)\,\delta_{\mu\nu} \ . \tag{2.63}$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \left(\frac{\partial x_i}{\partial q_\mu}\right)^2} \ . \tag{2.64}$$

Note that

$$(d\boldsymbol{s})^2 = \sum_{\mu=1}^d h_\mu^2(q)\,(dq_\mu)^2 \ . \tag{2.65}$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\boldsymbol{s})^2 = \sum_{\mu,\nu=1}^d g_{\mu\nu}(q)\,dq_\mu\,dq_\nu \ , \tag{2.66}$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

2.6.1 Example : spherical coordinates

Consider spherical coordinates (ρ,θ,ϕ):

$$x = \rho\,\sin\theta\,\cos\phi \quad , \quad y = \rho\,\sin\theta\,\sin\phi \quad , \quad z = \rho\,\cos\theta \ . \tag{2.67}$$

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2\,\sin^2\!\theta \ . \tag{2.68}$$

Thus,

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\,d\rho + \rho\,\hat{\boldsymbol{\theta}}\,d\theta + \rho\,\sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi \ . \tag{2.69}$$

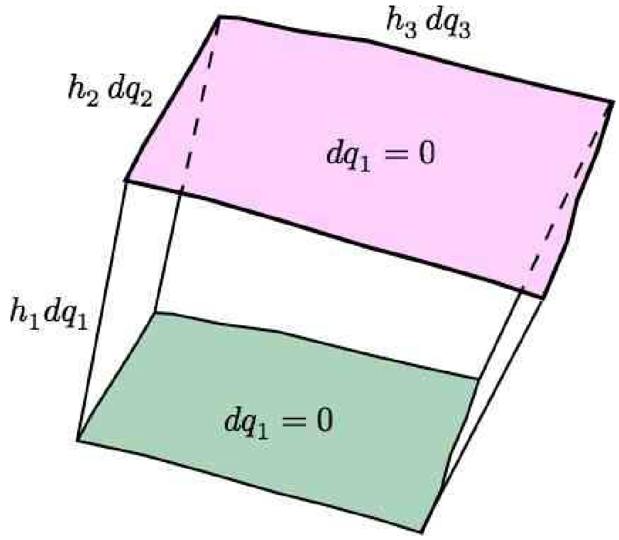

Figure 2.2: Volume element Ω for computing divergences.

2.6.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient $\boldsymbol{\nabla} U$ of a function $U(q)$ is defined by

$$
\begin{aligned}
dU &= \frac{\partial U}{\partial q_1}\, dq_1 + \frac{\partial U}{\partial q_2}\, dq_2 + \frac{\partial U}{\partial q_3}\, dq_3 \\
&\equiv \boldsymbol{\nabla} U \cdot d\boldsymbol{s} \ .
\end{aligned}
\tag{2.70}
$$

Thus,

$$
\boldsymbol{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1(q)}\, \frac{\partial}{\partial q_1} + \frac{\hat{\mathbf{e}}_2}{h_2(q)}\, \frac{\partial}{\partial q_2} + \frac{\hat{\mathbf{e}}_3}{h_3(q)}\, \frac{\partial}{\partial q_3} \ . \tag{2.71}
$$

For the divergence, we use the divergence theorem, and we appeal to fig. 2.2:

$$
\int\limits_{\Omega} dV\, \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int\limits_{\partial\Omega} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A} \ , \tag{2.72}
$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$
\text{LHS} = \boldsymbol{\nabla} \cdot \boldsymbol{A} \cdot (h_1\, dq_1)\, (h_2\, dq_2)\, (h_3\, dq_3) \ . \tag{2.73}
$$

The RHS is

$$
\begin{aligned}
\text{RHS} &= A_1\, h_2\, h_3 \Big|_{q_1}^{q_1+dq_1} dq_2\, dq_3 + A_2\, h_1\, h_3 \Big|_{q_2}^{q_2+dq_2} dq_1\, dq_3 + A_3\, h_1\, h_2 \Big|_{q_3}^{q_1+dq_3} dq_1\, dq_2 \\
&= \left[\frac{\partial}{\partial q_1} (A_1\, h_2\, h_3) + \frac{\partial}{\partial q_2} (A_2\, h_1\, h_3) + \frac{\partial}{\partial q_3} (A_3\, h_1\, h_2) \right] dq_1\, dq_2\, dq_3 \ .
\end{aligned}
\tag{2.74}
$$

We therefore conclude

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{h_1\,h_2\,h_3}\left[\frac{\partial}{\partial q_1}\big(A_1\,h_2\,h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\,h_1\,h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\,h_1\,h_2\big)\right]} \ . \tag{2.75}$$

To obtain the curl $\boldsymbol{\nabla}\times\boldsymbol{A}$, we use Stokes' theorem again,

$$\int\limits_{\Sigma} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \oint\limits_{\partial\Sigma} d\boldsymbol{\ell}\cdot\boldsymbol{A} \ , \tag{2.76}$$

where Σ is a two-dimensional region of space and $\partial\Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2\,dq_2$ and $h_3\,dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\mathbf{e}}_1\cdot\boldsymbol{\nabla}\times\boldsymbol{A}\,(h_2\,dq_2)\,(h_3\,dq_3) \ . \tag{2.77}$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_3\,h_3\,\Big|_{q_2}^{q_2+dq_2} dq_3 - A_2\,h_2\,\Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \left[\frac{\partial}{\partial q_2}\big(A_3\,h_3\big) - \frac{\partial}{\partial q_3}\big(A_2\,h_2\big)\right] dq_2\,dq_3 \ . \end{aligned} \tag{2.78}$$

Therefore

$$(\boldsymbol{\nabla}\times\boldsymbol{A})_1 = \frac{1}{h_2\,h_3}\left(\frac{\partial(h_3\,A_3)}{\partial q_2} - \frac{\partial(h_2\,A_2)}{\partial q_3}\right) \ . \tag{2.79}$$

This is one component of the full result

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{h_1\,h_2\,h_3}\,\det\begin{pmatrix} h_1\,\hat{\mathbf{e}}_1 & h_2\,\hat{\mathbf{e}}_2 & h_3\,\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\,A_1 & h_2\,A_2 & h_3\,A_3 \end{pmatrix} \ . \tag{2.80}$$

The Laplacian of a scalar function U is given by

$$\begin{aligned} \nabla^2 U &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}U \\ &= \frac{1}{h_1\,h_2\,h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2\,h_3}{h_1}\,\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1\,h_3}{h_2}\,\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1\,h_2}{h_3}\,\frac{\partial U}{\partial q_3}\right)\right\} \ . \end{aligned} \tag{2.81}$$

2.7 Common curvilinear orthogonal systems

2.7.1 Rectangular coordinates

In *rectangular* coordinates (x, y, z), we have

$$h_x = h_y = h_z = 1 \ . \tag{2.82}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz \tag{2.83}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \ . \tag{2.84}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{x}}\, \frac{\partial U}{\partial x} + \hat{\boldsymbol{y}}\, \frac{\partial U}{\partial y} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \ . \tag{2.85}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \ . \tag{2.86}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} \ . \tag{2.87}$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.88}$$

2.7.2 Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z), we have

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\, \cos\phi + \hat{\boldsymbol{y}}\, \sin\phi \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\, \cos\phi - \hat{\boldsymbol{\phi}}\, \sin\phi \qquad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\, d\phi \tag{2.89}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\, \sin\phi + \hat{\boldsymbol{y}}\, \cos\phi \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\, \sin\phi + \hat{\boldsymbol{\phi}}\, \cos\phi \qquad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}\, d\phi \ . \tag{2.90}$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \ . \tag{2.91}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}}\, \rho\, d\phi + \hat{\boldsymbol{z}}\, dz \tag{2.92}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \ . \tag{2.93}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{\rho}}\, \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\, \frac{\partial U}{\partial \phi} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \ . \tag{2.94}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{\rho}\, \frac{\partial(\rho\, A_\rho)}{\partial \rho} + \frac{1}{\rho}\, \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \ . \tag{2.95}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{1}{\rho}\, \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho}\, \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\, \frac{\partial A_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}} \ . \tag{2.96}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho}\, \frac{\partial}{\partial \rho}\left(\rho\, \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\, \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.97}$$

2.7.3 Spherical coordinates

In *spherical* coordinates (r, θ, ϕ), we have

$$\begin{aligned} \hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}} \sin\theta\cos\phi + \hat{\boldsymbol{y}} \sin\theta\sin\phi + \hat{\boldsymbol{z}} \sin\theta \\ \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}} \cos\theta\cos\phi + \hat{\boldsymbol{y}} \cos\theta\sin\phi - \hat{\boldsymbol{z}} \cos\theta \\ \hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi \ , \end{aligned} \tag{2.98}$$

for which

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}} \quad , \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \ . \tag{2.99}$$

The inverse is

$$\begin{aligned} \hat{\boldsymbol{x}} &= \hat{\boldsymbol{r}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \\ \hat{\boldsymbol{y}} &= \hat{\boldsymbol{r}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta \ . \end{aligned} \tag{2.100}$$

The differential relations are

$$\begin{aligned} d\hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}\, d\theta + \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \\ d\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{r}}\, d\theta + \cos\theta\, \hat{\boldsymbol{\phi}}\, d\phi \\ d\hat{\boldsymbol{\phi}} &= -\big(\sin\theta\, \hat{\boldsymbol{r}} + \cos\theta\, \hat{\boldsymbol{\theta}}\big)\, d\phi \end{aligned} \tag{2.101}$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r\, \sin\theta \ . \tag{2.102}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\theta}}\, r\, d\theta + \hat{\boldsymbol{\phi}}\, r \sin\theta\, d\phi \tag{2.103}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2 \ . \tag{2.104}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{r}}\, \frac{\partial U}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\, \frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\, \frac{\partial U}{\partial \phi} \ . \tag{2.105}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2}\, \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\, \frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\, \frac{\partial A_\phi}{\partial\phi} \ . \tag{2.106}$$

The curl is

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r\sin\theta} \left(\frac{\partial(\sin\theta\, A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi} \right) \hat{\boldsymbol{r}} + \frac{1}{r} \left(\frac{1}{\sin\theta}\, \frac{\partial A_r}{\partial\phi} - \frac{\partial (r A_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} \\ + \frac{1}{r} \left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta} \right) \hat{\boldsymbol{\phi}} \ . \end{aligned} \tag{2.107}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2}\, \frac{\partial}{\partial r} \left(r^2\, \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\, \frac{\partial}{\partial\theta} \left(\sin\theta\, \frac{\partial U}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\!\theta}\, \frac{\partial^2 U}{\partial\phi^2} \ . \tag{2.108}$$

2.7.4 Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \tfrac{1}{2}m\left(\frac{d\boldsymbol{s}}{dt}\right)^2 &= \tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2+\dot{z}^2\big) && \text{(3D Cartesian)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2+\rho^2\dot{\phi}^2\big) && \text{(2D polar)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{z}^2\big) && \text{(3D cylindrical)} \\ &= \tfrac{1}{2}m\big(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\!\theta\,\dot{\phi}^2\big) && \text{(3D polar)}\ . \end{aligned} \tag{2.109}$$

Chapter 3

One-Dimensional Conservative Systems

3.1 Description as a Dynamical System

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \ . \tag{3.1}$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x) \ , \tag{3.2}$$

is then conserved. This may be verified explicitly:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\Big[\tfrac{1}{2}m\dot{x}^2 + U(x)\Big] \\ &= \Big[m\ddot{x} + U'(x)\Big]\,\dot{x} = 0 \ . \end{aligned}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\bigg(E - U(x)\bigg)} \ . \tag{3.3}$$

Note that the constant E is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \ , \tag{3.4}$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E, the motion of the system is bounded by the turning points, and confined to the region(s) $U(x) \leq E$. We can integrate eqn. 3.3 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}}\int\limits_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}} \ . \tag{3.5}$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0. Since

$$E = E_0 = \tfrac{1}{2} m v_0^2 + U(x_0) \ , \tag{3.6}$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m} \int\limits_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} \ . \tag{3.7}$$

3.1.1 Example : harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \frac{1}{2} k x^2$, hence

$$\frac{dt}{dx} = \pm \sqrt{\frac{m}{2E - kx^2}} \ . \tag{3.8}$$

The turning points are $x_\pm(E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}} \, \sin\theta \ . \tag{3.9}$$

We then find

$$dt = \sqrt{\frac{m}{k}} \, d\theta \ , \tag{3.10}$$

with solution

$$\theta(t) = \theta_0 + \omega t \ , \tag{3.11}$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the complete motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \, \sin(\omega t + \theta_0) \ . \tag{3.12}$$

Note the two constants of integration, E and θ_0.

3.2 One-Dimensional Mechanics as a Dynamical System

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m} F(x) \ . \end{aligned} \tag{3.13}$$

This may be written in matrix-vector form, as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}\,F(x) \end{pmatrix} \ . \tag{3.14}$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \ , \tag{3.15}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ is an N-dimensional vector in *phase space*. For the model of eqn. 3.14, we evidently have $N = 2$. The object $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field*. It is itself a vector, existing at every point in phase space, R^N. Each of the components of $\boldsymbol{V}(\boldsymbol{\varphi})$ is a function (in general) of *all* the components of $\boldsymbol{\varphi}$:

$$V_j = V_j(\varphi_1, \ldots, \varphi_N) \qquad (j = 1, \ldots, N) \ . \tag{3.16}$$

Solutions to the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ are called *integral curves*. Each such integral curve $\boldsymbol{\varphi}(t)$ is uniquely determined by N constants of integration, which may be taken to be the initial value $\boldsymbol{\varphi}(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$. The phase curves are thus curves of constant energy.

3.2.1 Sketching phase curves

To plot the phase curves,

(i) Sketch the potential $U(x)$.

(ii) Below this plot, sketch $v(x;E) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

  (a) For E slightly above E_{min}, the phase curves are ellipses.

  (b) For E slightly below E_{max}, the phase curves are (locally) hyperbolae.

  (c) For $E = E_{\mathrm{max}}$ the phase curve is called a *separatrix*.

(iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

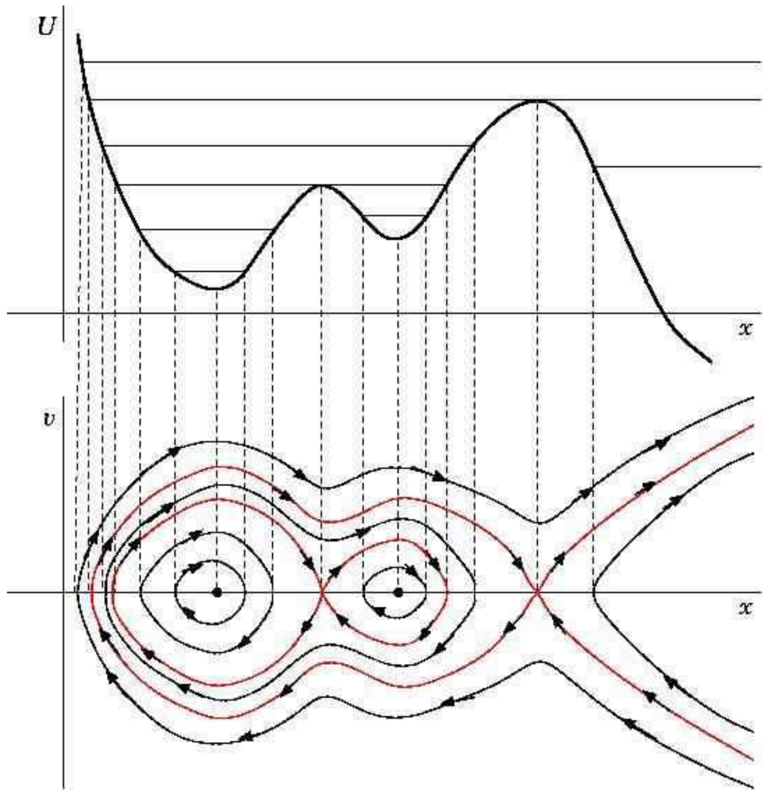

Figure 3.1: A potential $U(x)$ and the corresponding phase portraits. Separatrices are shown in red.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint\limits_E v\,dx = \sqrt{\frac{8}{m}} \int\limits_{x_-(E)}^{x_+(E)} dx'\,\sqrt{E - U(x')}\ . \tag{3.17}$$

Thus, the period is proportional to the rate of change of $A(E)$ with E:

$$T = m\,\frac{\partial\mathcal{A}}{\partial E}\ . \tag{3.18}$$

3.3 Fixed Points and their Vicinity

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*)\,(x - x^*) + \tfrac{1}{2}U''(x^*)\,(x - x^*)^2 + \tfrac{1}{6}U'''(x^*)\,(x - x^*)^3 + \ldots\ . \tag{3.19}$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \tfrac{1}{2}k(\delta x)^2\ , \tag{3.20}$$

where $U_0 = U(x^*)$ and $k = U''(x^*) > 0$. The solutions to the motion in this potential are:

$$\begin{aligned} U''(x^*) > 0 \ &: \ \delta x(t) = \delta x_0 \cos(\omega t) + \frac{\delta v_0}{\omega} \sin(\omega t) \\ U''(x^*) < 0 \ &: \ \delta x(t) = \delta x_0 \cosh(\gamma t) + \frac{\delta v_0}{\gamma} \sinh(\gamma t) \ , \end{aligned} \tag{3.21}$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \tfrac{1}{2} m \, (\delta v_0)^2 + \tfrac{1}{2} k \, (\delta x_0)^2 \ . \tag{3.22}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm \gamma \, \delta x_0$. Let's take $\delta v_0 = -\gamma \, \delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0 \exp(-\gamma t) \ . \tag{3.23}$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \ln \left(\frac{\delta x_0}{\delta x} \right) \ . \tag{3.24}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, *the period of motion along a separatrix is infinite*.

3.3.1 Linearized dynamics in the vicinity of a fixed point

Linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{m} \, U''(x^*) & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} + \ldots , \tag{3.25}$$

This is a *linear* equation, which we can solve completely.

Consider the general linear equation $\dot{\boldsymbol{\varphi}} = A \, \boldsymbol{\varphi}$, where A is a fixed real matrix. Now whenever we have a problem involving matrices, we should start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$A \, \boldsymbol{\psi}_\alpha = \lambda_\alpha \, \boldsymbol{\psi}_\alpha \ . \tag{3.26}$$

Here $\boldsymbol{\psi}_\alpha$ is the $\alpha^{\rm th}$ *right eigenvector*[1] of A. The eigenvalues are roots of the characteristic equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - A)$. Let's expand $\boldsymbol{\varphi}(t)$ in terms of the right eigenvectors of A:

$$\boldsymbol{\varphi}(t) = \sum_\alpha C_\alpha(t) \, \boldsymbol{\psi}_\alpha \ . \tag{3.27}$$

[1] If A is symmetric, the right and left eigenvectors are the same. If A is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same.

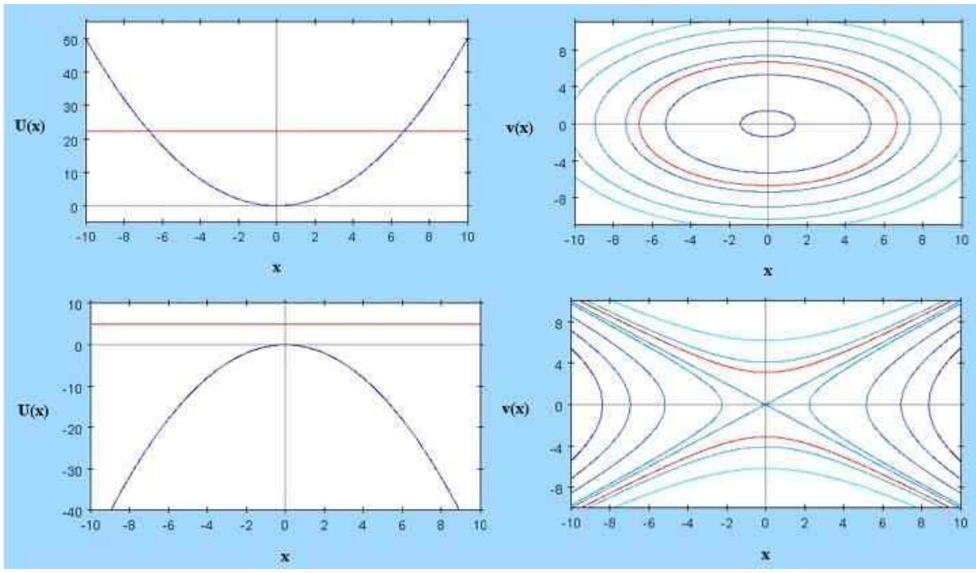

Figure 3.2: Phase curves in the vicinity of centers and saddles.

Assuming, for the purposes of this discussion, that A is nondegenerate, and its eigenvectors span R^N, the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha \, C_\alpha \ , \tag{3.28}$$

with solutions

$$C_\alpha(t) = C_\alpha(0) \, \exp(\lambda_\alpha t) \ . \tag{3.29}$$

If $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $\mathsf{Im}\,(\lambda_\alpha)$.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \ , \tag{3.30}$$

where $T = \mathrm{Tr}(A)$ and $D = \det(A)$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4D} \ . \tag{3.31}$$

We'll study the general case in Physics 110B. For now, we focus on our conservative mechanical system of eqn. 3.25. The trace and determinant of the above matrix are $T = 0$ and $D = \frac{1}{m}\,U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in Fig. 3.1.

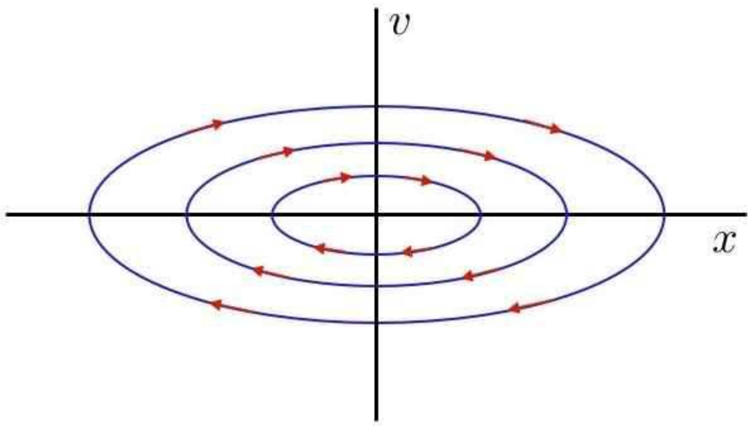

Figure 3.3: Phase curves for the harmonic oscillator.

3.4 Examples of Conservative One-Dimensional Systems

3.4.1 Harmonic oscillator

Recall the harmonic oscillator. The potential energy is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m \frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \ , \tag{3.32}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2\, x \end{pmatrix} \ , \tag{3.33}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$\begin{aligned} x(t) &= x_0 \, \cos(\omega t) + \frac{v_0}{\omega} \, \sin(\omega t) \\ v(t) &= v_0 \, \cos(\omega t) - \omega \, x_0 \, \sin(\omega t) \ . \end{aligned} \tag{3.34}$$

The phase curves are ellipses:

$$\omega_0 \, x^2(t) + \omega_0^{-1} \, v^2(t) = C \ , \tag{3.35}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in Fig. 3.3. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \ . \tag{3.36}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\rm max} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\rm max} = \sqrt{\frac{2E}{m}} \ . \tag{3.37}$$

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi\, x_{\rm max}\, v_{\rm max} = \frac{2\pi E}{\sqrt{mk}} \ . \tag{3.38}$$

The period of motion is therefore

$$T(E) = m\, \frac{\partial \mathcal{A}}{\partial E} = 2\pi \sqrt{\frac{m}{k}} \ , \tag{3.39}$$

which is independent of E.

3.4.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell\cos\theta$, hence

$$m\ell^2\, \ddot\theta = -\frac{dU}{d\theta} = -mg\ell\sin\theta \ . \tag{3.40}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega_0^2\, \sin\theta \end{pmatrix} \ , \tag{3.41}$$

where $\omega = \dot\theta$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations. The conserved energy is

$$E = \tfrac{1}{2}\, m\ell^2\, \dot\theta^2 + U(\theta) \ . \tag{3.42}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot\theta^2 - 2\omega_0^2\, \cos\theta = -2\omega_0^2\, \cos\theta_0 \ . \tag{3.43}$$

The period for motion of amplitude θ_0 is then

$$T\big(\theta_0\big) = \frac{\sqrt{8}}{\omega_0} \int\limits_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}\, {\rm K}\big(\sin^2 \tfrac{1}{2}\theta_0\big) \ , \tag{3.44}$$

where ${\rm K}(z)$ is the complete elliptic integral of the first kind. Expanding $K(z)$, we have

$$T\big(\theta_0\big) = \frac{2\pi}{\omega_0}\bigg\{ 1 + \tfrac{1}{4}\, \sin^2\big(\tfrac{1}{2}\theta_0\big) + \tfrac{9}{64}\, \sin^4\big(\tfrac{1}{2}\theta_0\big) + \ldots \bigg\} \ . \tag{3.45}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot\theta = -\omega_0^2\, \theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in Fig. 3.4. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These

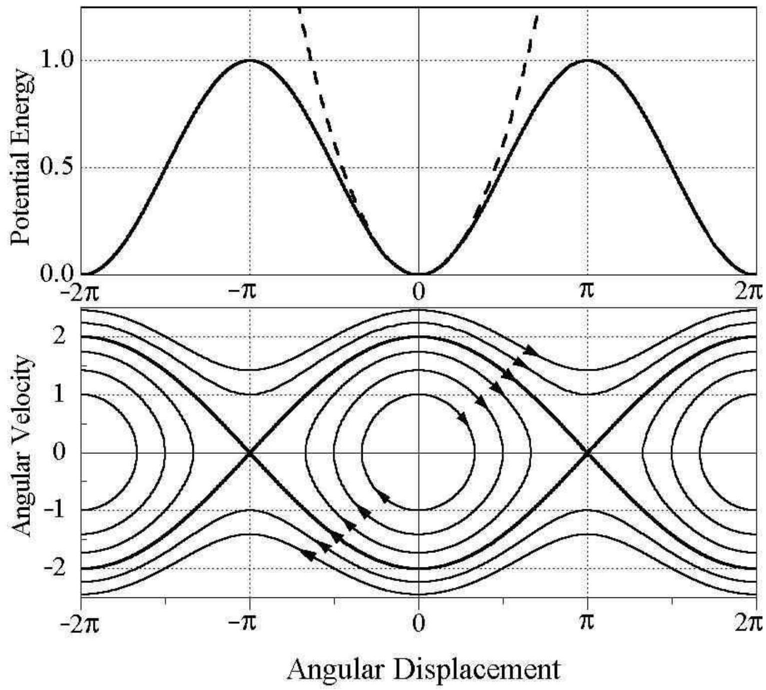

Figure 3.4: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

3.4.3 Other potentials

Using the phase plotter application written by Ben Schmidel, available on the Physics 110A course web page, it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\rm eff}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \ , \tag{3.46}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses $m_{1,2}$, $\mu = m_1 m_2/(m_1 + m_2)$ is the 'reduced mass', and $k = Gm_1 m_2$, where G is the Cavendish constant. We can then write

$$U_{\rm eff}(r) = U_0 \bigg\{ -\frac{1}{x} + \frac{1}{2x^2} \bigg\} \ , \tag{3.47}$$

where $r_0 = \ell^2/\mu k$ has the dimensions of length, and $x \equiv r/r_0$, and where $U_0 = k/r_0 = \mu k^2/\ell^2$. Thus, if distances are measured in units of r_0 and the potential in units of U_0, the potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \operatorname{sech}^2(x/a) \ , \tag{3.48}$$

which, in dimensionless form, is $\mathcal{U}(x) = -\operatorname{sech}^2(x)$, after measuring distances in units of a and potential in units of U_0.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} \ . \tag{3.49}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.

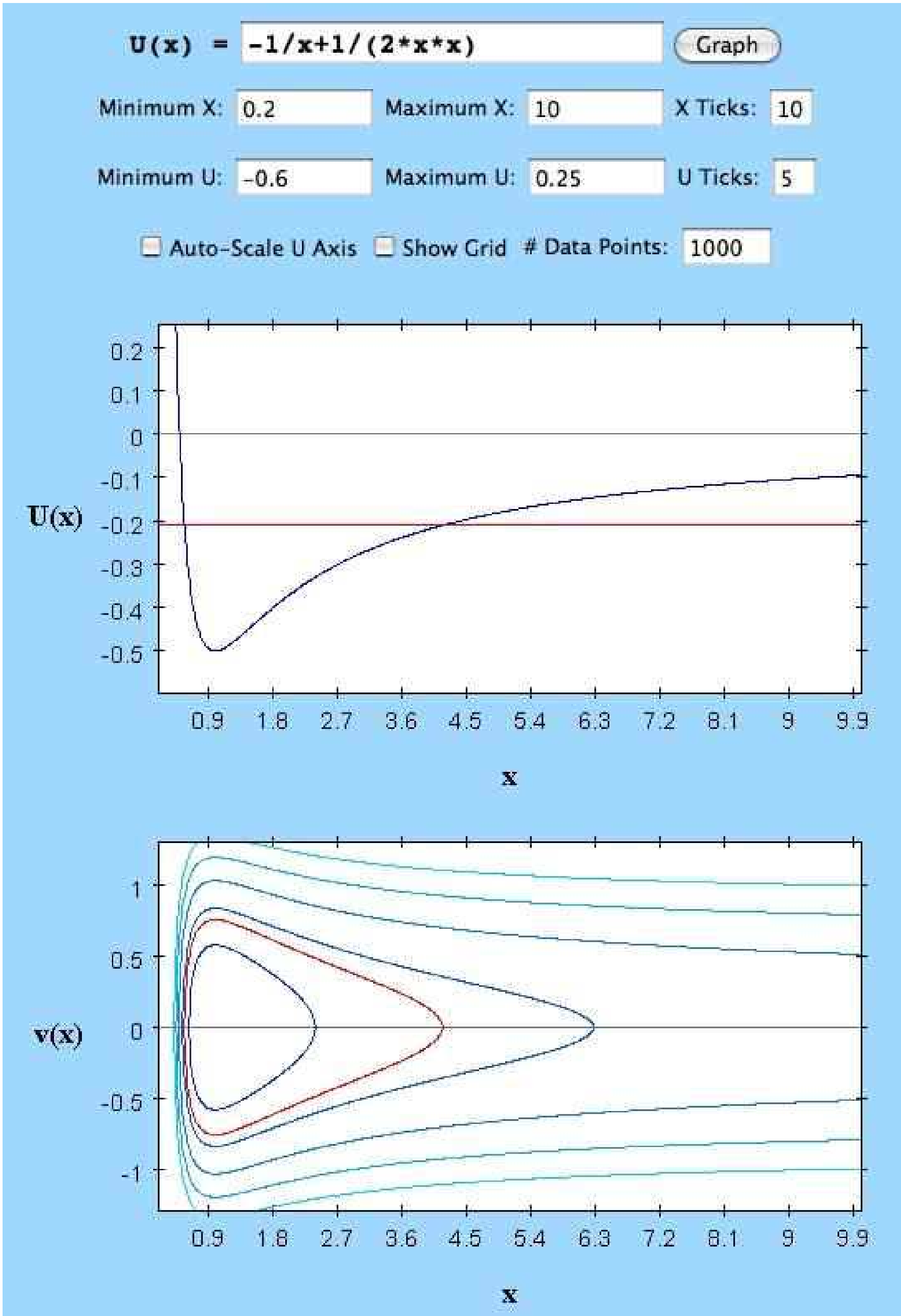

Figure 3.5: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

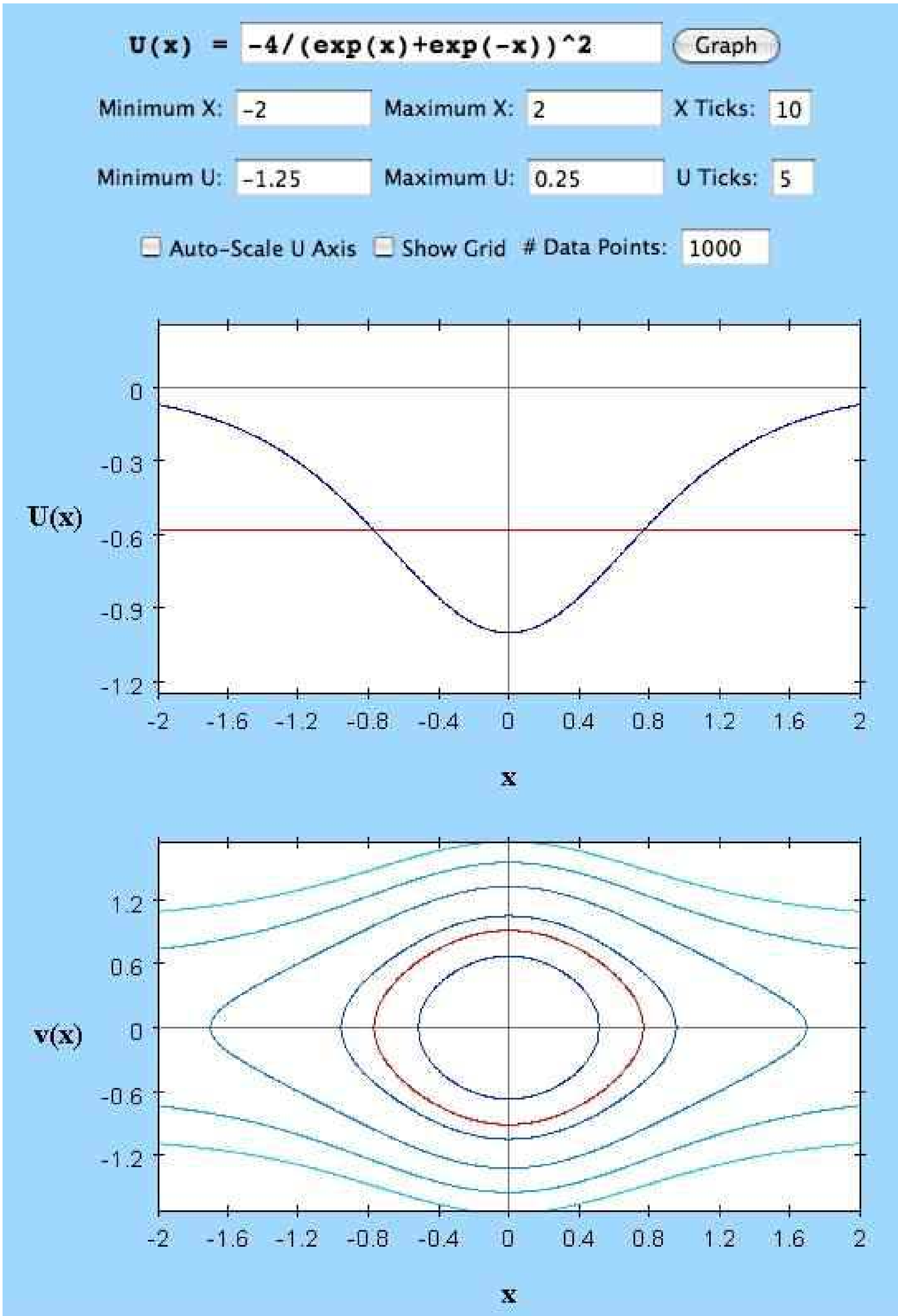

Figure 3.6: Phase curves for the potential $U(x) = -\text{sech}^2(x)$.

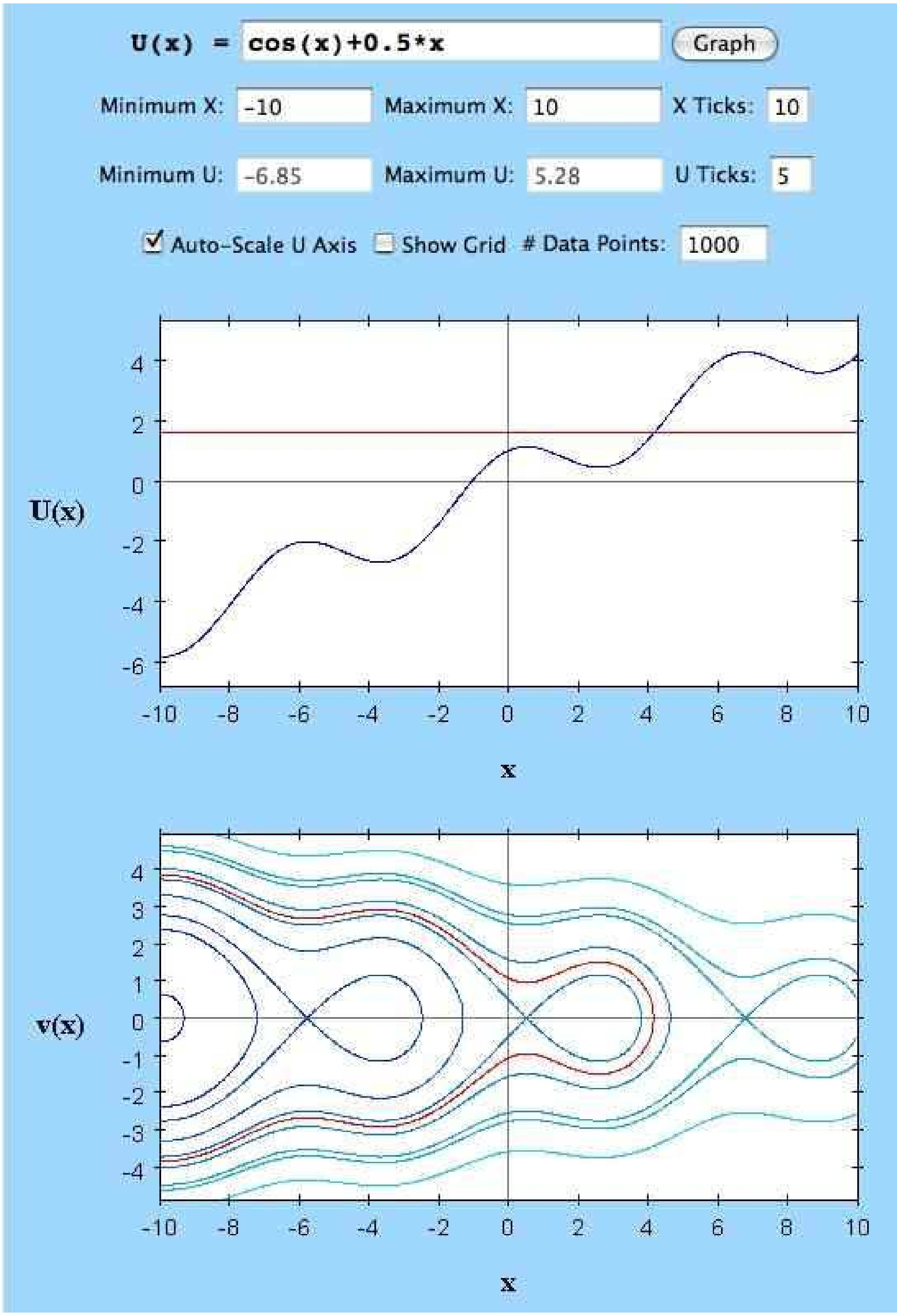

Figure 3.7: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.

Chapter 4

Linear Oscillations

Harmonic motion is ubiquitous in Physics. The reason is that any potential energy function, when expanded in a Taylor series in the vicinity of a local minimum, is a harmonic function:

$$U(\vec{q}) = U(\vec{q}^{\,*}) + \sum_{j=1}^{N} \overbrace{\frac{\partial U}{\partial q_j}\bigg|_{\vec{q}=\vec{q}^{\,*}}}^{\boldsymbol{\nabla} U(\vec{q}^{\,*})=0} (q_j - q_j^*) + \tfrac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 U}{\partial q_j \, \partial q_k}\bigg|_{\vec{q}=\vec{q}^{\,*}} (q_j - q_j^*)\,(q_k - q_k^*) + \ldots \ , \tag{4.1}$$

where the $\{q_j\}$ are *generalized coordinates* – more on this when we discuss Lagrangians. In one dimension, we have simply

$$U(x) = U(x^*) + \tfrac{1}{2}\, U''(x^*)\,(x - x^*)^2 + \ldots \ . \tag{4.2}$$

Provided the deviation $\eta = x - x^*$ is small enough in magnitude, the remaining terms in the Taylor expansion may be ignored. Newton's Second Law then gives

$$m\,\ddot{\eta} = -U''(x^*)\,\eta + \mathcal{O}(\eta^2) \ . \tag{4.3}$$

This, to lowest order, is the equation of motion for a harmonic oscillator. If $U''(x^*) > 0$, the equilibrium point $x = x^*$ is *stable*, since for small deviations from equilibrium the restoring force pushes the system back toward the equilibrium point. When $U''(x^*) < 0$, the equilibrium is *unstable*, and the forces push one further away from equilibrium.

4.1 Damped Harmonic Oscillator

In the real world, there are frictional forces, which we here will approximate by $F = -\gamma v$. We begin with the homogeneous equation for a damped harmonic oscillator,

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\, x = 0 \ , \tag{4.4}$$

where $\gamma = 2\beta m$. To solve, write $x(t) = \sum_n C_n\, e^{-i\omega_n t}$. This renders the differential equation 4.4 an *algebraic* equation for the two eigenfrequencies ω_i, each of which must satisfy

$$\omega^2 + 2i\beta\omega - \omega_0^2 = 0 \ , \tag{4.5}$$

hence

$$\omega_{\pm} = -i\beta \pm (\omega_0^2 - \beta^2)^{1/2} \ . \tag{4.6}$$

The most general solution to eqn. 4.4 is then

$$x(t) = C_+ \, e^{-i\omega_+ t} + C_- \, e^{-i\omega_- t} \tag{4.7}$$

where $C_\pm$ are arbitrary constants. Notice that the eigenfrequencies are in general complex, with a negative imaginary part (so long as the damping coefficient β is positive). Thus $e^{-i\omega_\pm t}$ decays to zero as $t \to \infty$.

4.1.1 Classes of damped harmonic motion

We identify three classes of motion:

$$\begin{aligned} &\text{(i) Underdamped } (\omega_0^2 > \beta^2) \\ &\text{(ii) Overdamped } (\omega_0^2 < \beta^2) \\ &\text{(iii) Critically Damped } (\omega_0^2 = \beta^2) \ . \end{aligned}$$

Underdamped motion

The solution for underdamped motion is

$$\begin{aligned} x(t) &= A\cos(\nu t + \phi)\, e^{-\beta t} \\ \dot x(t) &= -\omega_0 A \cos\big(\nu t + \phi + \sin^{-1}(\beta/\omega_0)\big)\, e^{-\beta t} \ , \end{aligned} \tag{4.8}$$

where $\nu = \sqrt{\omega_0^2 - \beta^2}$, and where A and ϕ are constants determined by initial conditions. From $x_0 = A\cos\phi$ and $\dot x_0 = -\beta A\cos\phi - \nu A \sin\phi$, we have $\dot x_0 + \beta x_0 = -\nu A \sin\phi$, and

$$A = \sqrt{x_0^2 + \left(\frac{\dot x_0 + \beta\, x_0}{\nu}\right)^2} \qquad , \qquad \phi = -\tan^{-1}\left(\frac{\dot x_0 + \beta\, x_0}{\nu\, x_0}\right) \ . \tag{4.9}$$

Overdamped motion

The solution in the case of overdamped motion is

$$\begin{aligned} x(t) &= C\, e^{-(\beta-\lambda)t} + D\, e^{-(\beta+\lambda)t} \\ \dot x(t) &= -(\beta-\lambda)\, C\, e^{-(\beta-\lambda)t} - (\beta+\lambda)\, D\, e^{-(\beta+\lambda)t} \ , \end{aligned} \tag{4.10}$$

where $\lambda = \sqrt{\beta^2 - \omega_0^2}$ and where C and D are constants determined by the initial conditions:

$$\begin{pmatrix} 1 & 1 \\ -(\beta-\lambda) & -(\beta+\lambda) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} x_0 \\ \dot x_0 \end{pmatrix} \ . \tag{4.11}$$

Inverting the above matrix, we have the solution

$$C = \frac{(\beta+\lambda)\, x_0}{2\lambda} + \frac{\dot x_0}{2\lambda} \qquad , \qquad D = -\frac{(\beta-\lambda)\, x_0}{2\lambda} - \frac{\dot x_0}{2\lambda} \ . \tag{4.12}$$

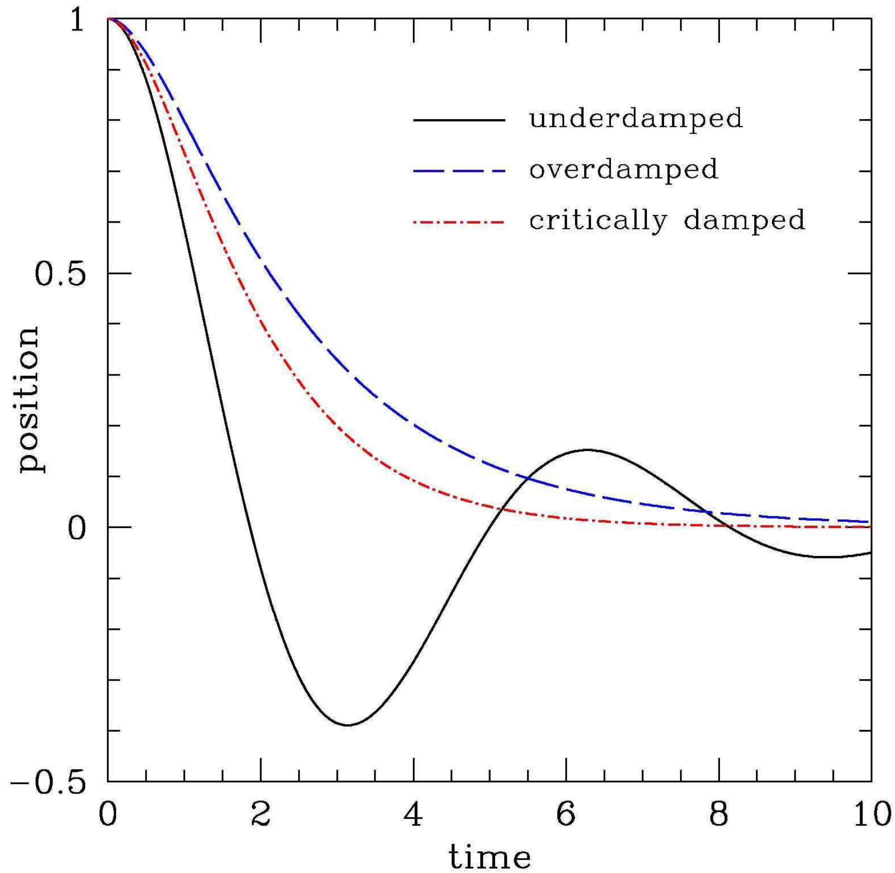

Figure 4.1: Three classifications of damped harmonic motion. The initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$.

Critically damped motion

The solution in the case of critically damped motion is

$$\begin{aligned} x(t) &= E\, e^{-\beta t} + Ft\, e^{-\beta t} \\ \dot{x}(t) &= -\Big(\beta E + (\beta t - 1)F\Big) e^{-\beta t} \ . \end{aligned} \tag{4.13}$$

Thus, $x_0 = E$ and $\dot{x}_0 = F - \beta E$, so

$$E = x_0 \qquad , \qquad F = \dot{x}_0 + \beta x_0 \ . \tag{4.14}$$

The screen door analogy

The three types of behavior are depicted in fig. 4.1. To concretize these cases in one's mind, it is helpful to think of the case of a screen door or a shock absorber. If the hinges on the door are underdamped, the door will swing back and forth (assuming it doesn't have a rim which smacks into the door frame) several times before coming to a stop. If the hinges are overdamped, the door may take a very long time to close. To see this, note that for $\beta \gg \omega_0$ we have

$$\begin{aligned}\sqrt{\beta^2-\omega_0^2} &= \beta\left(1-\frac{\omega_0^2}{\beta^2}\right)^{-1/2} \\ &= \beta\left(1-\frac{\omega_0^2}{2\beta^2}-\frac{\omega_0^4}{8\beta^4}+\ldots\right) ,\end{aligned} \tag{4.15}$$

which leads to

$$\begin{aligned}\beta-\sqrt{\beta^2-\omega_0^2} &= \frac{\omega_0^2}{2\beta}+\frac{\omega_0^4}{8\beta^3}+\ldots \\ \beta+\sqrt{\beta^2-\omega_0^2} &= 2\beta-\frac{\omega_0^2}{2\beta}-+\ldots \ .\end{aligned} \tag{4.16}$$

Thus, we can write

$$x(t) = C\,e^{-t/\tau_1} + D\,e^{-t/\tau_2} \ , \tag{4.17}$$

with

$$\begin{aligned}\tau_1 &= \frac{1}{\beta-\sqrt{\beta^2-\omega_0^2}} \approx \frac{2\beta}{\omega_0^2} \\ \tau_2 &= \frac{1}{\beta+\sqrt{\beta^2-\omega_0^2}} \approx \frac{1}{2\beta} \ .\end{aligned} \tag{4.18}$$

Thus $x(t)$ is a sum of exponentials, with decay times $\tau_{1,2}$. For $\beta \gg \omega_0$, we have that τ_1 is much larger than τ_2 – the ratio is $\tau_1/\tau_2 \approx 4\beta^2/\omega_0^2 \gg 1$. Thus, on time scales on the order of τ_1, the second term has completely damped away. The decay time τ_1, though, is very long, since β is so large. So a highly overdamped oscillator will take a very long time to come to equilbrium.

4.1.2 Remarks on the case of critical damping

Define the first order differential operator

$$\mathcal{D}_t = \frac{d}{dt} + \beta \ . \tag{4.19}$$

The solution to $\mathcal{D}_t\, x(t) = 0$ is $\tilde{x}(t) = A\, e^{-\beta t}$, where A is a constant. Note that the *commutator* of $\mathcal{D}_t$ and t is unity:

$$\big[\mathcal{D}_t\,, t\big] = 1 \ , \tag{4.20}$$

where $[A, B] \equiv AB - BA$. The simplest way to verify eqn. 4.20 is to compute its action upon an arbitrary function $f(t)$:

$$\begin{aligned}\big[\mathcal{D}_t\,,\, t\big]\, f(t) &= \left(\frac{d}{dt}+\beta\right) t\, f(t) - t\left(\frac{d}{dt}+\beta\right) f(t) \\ &= \frac{d}{dt}\big(t\, f(t)\big) - t\,\frac{d}{dt}\, f(t) = f(t) \ .\end{aligned} \tag{4.21}$$

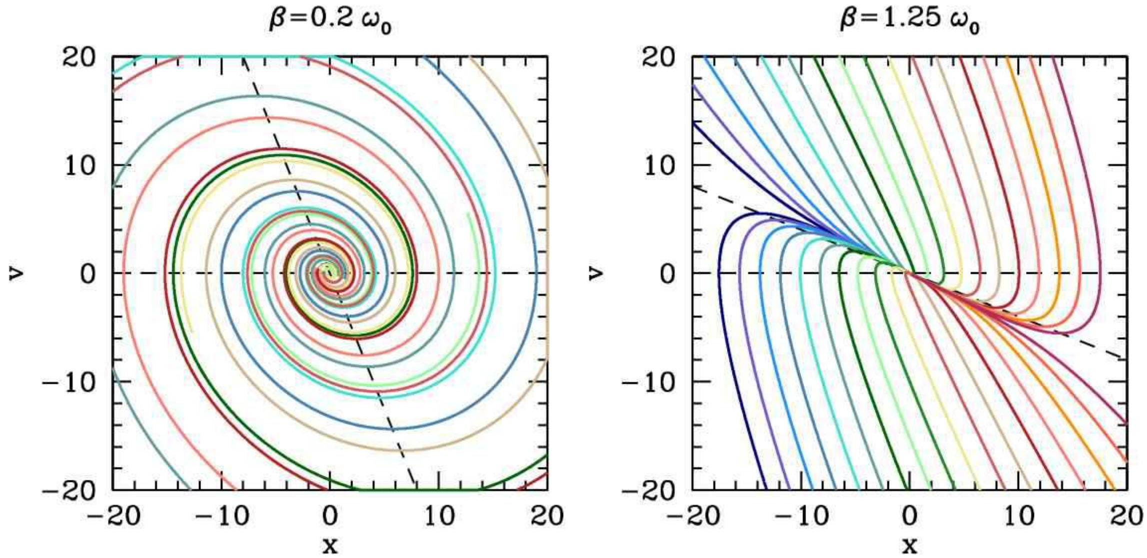

Figure 4.2: Phase curves for the damped harmonic oscillator. Left panel: underdamped motion. Right panel: overdamped motion. Note the *nullclines* along $v = 0$ and $v = -(\omega_0^2/2\beta)x$, which are shown as dashed lines.

We know that $x(t) = \tilde{x}(t) = A\,e^{-\beta t}$ satisfies $\mathcal{D}_t\, x(t) = 0$. Therefore

$$\begin{aligned} 0 &= \mathcal{D}_t \left[\mathcal{D}_t\,, t\right] \tilde{x}(t) \\ &= \mathcal{D}_t^2 \Big(t\, \tilde{x}(t) \Big) - \mathcal{D}_t\, t\, \overbrace{\mathcal{D}_t\, \tilde{x}(t)}^{0} \\ &= \mathcal{D}_t^2 \Big(t\, \tilde{x}(t) \Big) \ . \end{aligned} \tag{4.22}$$

We already know that $\mathcal{D}_t^2\, \tilde{x}(t) = \mathcal{D}_t\, \mathcal{D}_t\, \tilde{x}(t) = 0$. The above equation establishes that the second independent solution to the second order ODE $\mathcal{D}_t^2\, x(t) = 0$ is $x(t) = t\, \tilde{x}(t)$. Indeed, we can keep going, and show that

$$\mathcal{D}_t^n \Big(t^{n-1}\, \tilde{x}(t) \Big) = 0 \ . \tag{4.23}$$

Thus, the n independent solutions to the $n^{\rm th}$ order ODE

$$\left(\frac{d}{dt} + \beta \right)^{n} x(t) = 0 \tag{4.24}$$

are

$$x_k(t) = A\, t^k\, e^{-\beta t} \quad , \quad k = 0, 1, \ldots, n-1 \ . \tag{4.25}$$

4.1.3 Phase portraits for the damped harmonic oscillator

Expressed as a dynamical system, the equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ is written as two coupled first order ODEs, *viz.*

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega_0^2\, x - 2\beta v \ . \end{aligned} \tag{4.26}$$

In the theory of dynamical systems, a *nullcline* is a curve along which one component of the phase space velocity $\dot{\boldsymbol{\varphi}}$ vanishes. In our case, there are two nullclines: $\dot{x} = 0$ and $\dot{v} = 0$. The equation of the first nullcline, $\dot{x} = 0$, is simply $v = 0$, *i.e.* the first nullcline is the x-axis. The equation of the second nullcline, $\dot{v} = 0$, is $v = -(\omega_0^2/2\beta)\, x$. This is a line which runs through the origin and has negative slope. Everywhere along the first nullcline $\dot{x} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the v-axis. Similarly, everywhere along the second nullcline $\dot{v} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the x-axis. The situation is depicted in fig. 4.2.

4.2 Damped Harmonic Oscillator with Forcing

When forced, the equation for the damped oscillator becomes

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\, x = f(t) \ , \tag{4.27}$$

where $f(t) = F(t)/m$. Since this equation is linear in $x(t)$, we can, without loss of generality, restrict out attention to harmonic forcing terms of the form

$$f(t) = f_0 \cos(\Omega t + \varphi_0) = \text{Re}\Big[f_0\, e^{-i\varphi_0}\, e^{-i\Omega t}\Big] \tag{4.28}$$

where Re stands for "real part". Here, Ω is the forcing frequency.

Consider first the complex equation

$$\frac{d^2z}{dt^2} + 2\beta\,\frac{dz}{dt} + \omega_0^2\, z = f_0\, e^{-i\varphi_0}\, e^{-i\Omega t} \ . \tag{4.29}$$

We try a solution $z(t) = z_0\, e^{-i\Omega t}$. Plugging in, we obtain the algebraic equation

$$z_0 = \frac{f_0\, e^{-i\varphi_0}}{\omega_0^2 - 2i\beta\Omega - \Omega^2} \equiv A(\Omega)\, e^{i\delta(\Omega)}\, f_0 e^{-i\varphi_0} \ . \tag{4.30}$$

The amplitude $A(\Omega)$ and phase shift $\delta(\Omega)$ are given by the equation

$$A(\Omega)\, e^{i\delta(\Omega)} = \frac{1}{\omega_0^2 - 2i\beta\Omega - \Omega^2} \ . \tag{4.31}$$

A basic fact of complex numbers:

$$\frac{1}{a - ib} = \frac{a + ib}{a^2 + b^2} = \frac{e^{i\tan^{-1}(b/a)}}{\sqrt{a^2 + b^2}} \ . \tag{4.32}$$

Thus,

$$\begin{aligned} A(\Omega) &= \Big((\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\Big)^{-1/2} \\ \delta(\Omega) &= \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right) \ . \end{aligned} \tag{4.33}$$

Now since the coefficients β and ω_0^2 are real, we can take the complex conjugate of eqn. 4.29, and write

$$\begin{aligned} \ddot{z} + 2\beta\,\dot{z} + \omega_0^2\,z &= f_0\,e^{-i\varphi_0}\,e^{-i\Omega t} \\ \ddot{\bar{z}} + 2\beta\,\dot{\bar{z}} + \omega_0^2\,\bar{z} &= f_0\,e^{+i\varphi_0}\,e^{+i\Omega t} \ , \end{aligned} \tag{4.34}$$

where $\bar{z}$ is the complex conjugate of z. We now add these two equations and divide by two to arrive at

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f_0\,\cos(\Omega t + \varphi_0) \ . \tag{4.35}$$

Therefore, the real, physical solution we seek is

$$\begin{aligned} x_{\text{inh}}(t) &= \text{Re}\Big[A(\Omega)\,e^{i\delta(\Omega)}\cdot f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\Big] \\ &= A(\Omega)\,f_0\,\cos\big(\Omega t + \varphi_0 - \delta(\Omega)\big) \ . \end{aligned} \tag{4.36}$$

The quantity $A(\Omega)$ is the *amplitude* of the response (in units of f_0), while $\delta(\Omega)$ is the (dimensionless) *phase lag* (typically expressed in radians).

The maximum of the amplitude $A(\Omega)$ occurs when $A'(\Omega) = 0$. From

$$\frac{dA}{d\Omega} = -\frac{2\Omega}{\big[A(\Omega)\big]^3}\left(\Omega^2 - \omega_0^2 + 2\beta^2\right) \ , \tag{4.37}$$

we conclude that $A'(\Omega) = 0$ for $\Omega = 0$ and for $\Omega = \Omega_{\text{R}}$, where

$$\Omega_{\text{R}} = \sqrt{\omega_0^2 - 2\beta^2} \ . \tag{4.38}$$

The solution at $\Omega = \Omega_{\text{R}}$ pertains only if $\omega_0^2 > 2\beta^2$, of course, in which case $\Omega = 0$ is a local minimum and $\Omega = \Omega_{\text{R}}$ a local maximum. If $\omega_0^2 < 2\beta^2$ there is only a local maximum, at $\Omega = 0$. See Fig. 4.3.

Since equation 4.27 is linear, we can add a solution to the homogeneous equation to $x_{\text{inh}}(t)$ and we will still have a solution. Thus, the most general solution to eqn. 4.27 is

$$\begin{aligned} x(t) &= x_{\text{inh}}(t) + x_{\text{hom}}(t) \\ &= \text{Re}\Big[A(\Omega)\,e^{i\delta(\Omega)}\cdot f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\Big] + C_+\,e^{-i\omega_+ t} + C_-\,e^{-i\omega_- t} \\ &= \overbrace{A(\Omega)\,f_0\cos\big(\Omega t + \varphi_0 - \delta(\Omega)\big)}^{x_{\text{inh}}(t)} + \overbrace{C\,e^{-\beta t}\,\cos(\nu t) + D\,e^{-\beta t}\,\sin(\nu t)}^{x_{\text{hom}}(t)} \ , \end{aligned} \tag{4.39}$$

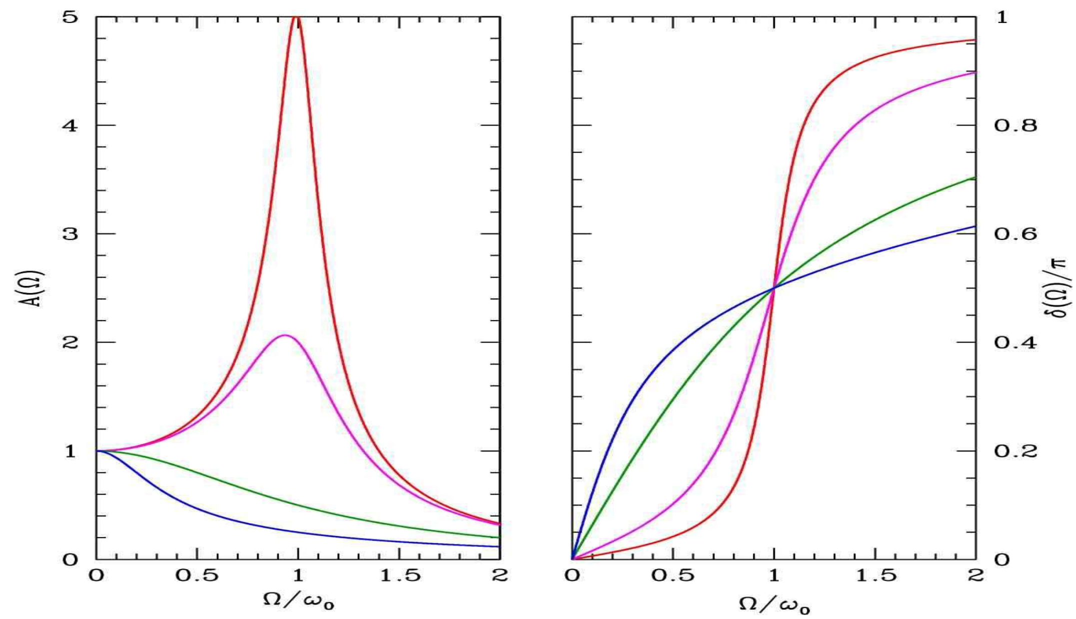

Figure 4.3: Amplitude and phase shift *versus* oscillator frequency (units of ω_0) for β/ω_0 values of 0.1 (red), 0.25 (magenta), 1.0 (green), and 2.0 (blue).

where $\nu = \sqrt{\omega_0^2 - \beta^2}$ as before.

The last two terms in eqn. 4.39 are the solution to the homogeneous equation, *i.e.* with $f(t) = 0$. They are necessary to include because they carry with them the two constants of integration which always arise in the solution of a second order ODE. That is, C and D are adjusted so as to satisfy $x(0) = x_0$ and $\dot{x}_0 = v_0$. However, due to their $e^{-\beta t}$ prefactor, these terms decay to zero once t reaches a relatively low multiple of β^{-1}. They are called *transients*, and may be set to zero if we are only interested in the long time behavior of the system. This means, incidentally, that the initial conditions are effectively forgotten over a time scale on the order of β^{-1}.

For $\varOmega_{\rm R} > 0$, one defines the *quality factor*, Q, of the oscillator by $Q = \varOmega_{\rm R}/2\beta$. Q is a rough measure of how many periods the unforced oscillator executes before its initial amplitude is damped down to a small value. For a forced oscillator driven near resonance, and for weak damping, Q is also related to the ratio of average energy in the oscillator to the energy lost per cycle by the external source. To see this, let us compute the energy lost per cycle,

$$\begin{aligned} \Delta E &= m \int\limits_0^{2\pi/\varOmega} dt\, \dot{x}\, f(t) \\ &= -m \int\limits_0^{2\pi/\varOmega} dt\, \varOmega\, A\, f_0^2\, \sin(\varOmega t + \varphi_0 - \delta)\, \cos(\varOmega t + \varphi_0) \\ &= \pi A\, f_0^2\, m \sin\delta = 2\pi\beta\, m\, \varOmega\, A^2(\varOmega)\, f_0^2 \ , \end{aligned} \tag{4.40}$$

since $\sin\delta(\Omega) = 2\beta\Omega\, A(\Omega)$. The oscillator energy, averaged over the cycle, is

$$\begin{aligned}\langle E\rangle &= \frac{\Omega}{2\pi}\int\limits_0^{2\pi/\Omega}\!dt\,\tfrac{1}{2}m\big(\dot{x}^2+\omega_0^2\,x^2\big) \\ &= \tfrac{1}{4}m\,(\Omega^2+\omega_0^2)\,A^2(\Omega)\,f_0^2\ .\end{aligned} \tag{4.41}$$

Thus, we have

$$\frac{2\pi\langle E\rangle}{\Delta E} = \frac{\Omega^2+\omega_0^2}{4\beta\Omega}\ . \tag{4.42}$$

Thus, for $\Omega \approx \Omega_{\rm R}$ and $\beta^2 \ll \omega_0^2$, we have

$$Q \approx \frac{2\pi\langle E\rangle}{\Delta E} \approx \frac{\omega_0}{2\beta}\ . \tag{4.43}$$

4.2.1 Resonant forcing

When the damping β vanishes, the response diverges at resonance. The solution to the resonantly forced oscillator

$$\ddot{x} + \omega_0^2\,x = f_0\cos(\omega_0\,t+\varphi_0) \tag{4.44}$$

is given by

$$x(t) = \frac{f_0}{2\omega_0}\,t\,\sin(\omega_0\,t+\varphi_0) + \overbrace{A\,\cos(\omega_0\,t) + B\,\sin(\omega_0\,t)}^{x_{\rm hom}(t)}\ . \tag{4.45}$$

The amplitude of this solution grows linearly due to the energy pumped into the oscillator by the resonant external forcing. In the real world, nonlinearities can mitigate this unphysical, unbounded response.

4.2.2 R-L-C circuits

Consider the R-L-C circuit of Fig. 4.4. When the switch is to the left, the capacitor is charged, eventually to a steady state value $Q = CV$. At $t = 0$ the switch is thrown to the right, completing the R-L-C circuit. Recall that the sum of the voltage drops across the three elements must be zero:

$$L\,\frac{dI}{dt} + IR + \frac{Q}{C} = 0\ . \tag{4.46}$$

We also have $\dot{Q} = I$, hence

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\,\frac{dQ}{dt} + \frac{1}{LC}\,Q = 0\ , \tag{4.47}$$

which is the equation for a damped harmonic oscillator, with $\omega_0 = (LC)^{-1/2}$ and $\beta = R/2L$.

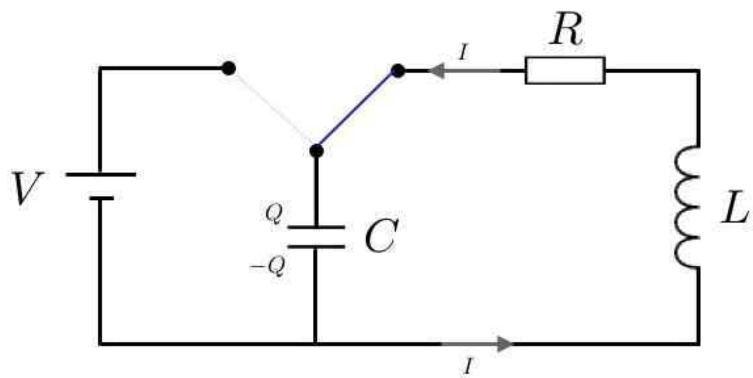

Figure 4.4: An R-L-C circuit which behaves as a damped harmonic oscillator.

The boundary conditions at $t = 0$ are $Q(0) = CV$ and $\dot{Q}(0) = 0$. Under these conditions, the full solution at all times is

$$\begin{aligned} Q(t) &= CV\, e^{-\beta t} \left(\cos \nu t + \frac{\beta}{\nu} \sin \nu t \right) \\ I(t) &= -CV\, \frac{\omega_0^2}{\nu}\, e^{-\beta t} \sin \nu t \ , \end{aligned} \tag{4.48}$$

again with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

If we put a time-dependent voltage source in series with the resistor, capacitor, and inductor, we would have

$$L \frac{dI}{dt} + IR + \frac{Q}{C} = V(t) \ , \tag{4.49}$$

which is the equation of a *forced* damped harmonic oscillator.

4.2.3 Examples

Third order linear ODE with forcing

The problem is to solve the equation

$$\mathcal{L}_t\, x \equiv \dddot{x} + (a+b+c)\, \ddot{x} + (ab+ac+bc)\, \dot{x} + abc\, x = f_0 \cos(\Omega t) \ . \tag{4.50}$$

The key to solving this is to note that the differential operator $\mathcal{L}_t$ factorizes:

$$\begin{aligned} \mathcal{L}_t &= \frac{d^3}{dt^3} + (a+b+c)\, \frac{d^2}{dt^2} + (ab+ac+bc)\, \frac{d}{dt} + abc \\ &= \left(\frac{d}{dt} + a \right) \left(\frac{d}{dt} + b \right) \left(\frac{d}{dt} + c \right) \ , \end{aligned} \tag{4.51}$$

which says that the third order differential operator appearing in the ODE is in fact a product of first order differential operators. Since

$$\frac{dx}{dt} + \alpha x = 0 \quad \Longrightarrow \quad x(t) = A\, e^{-\alpha x} \ , \tag{4.52}$$

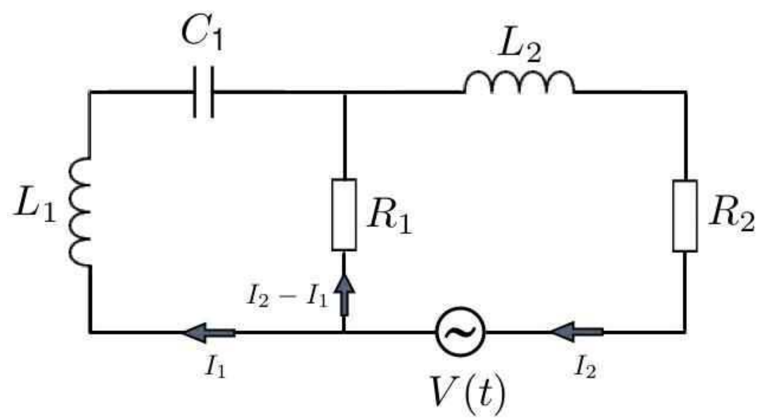

Figure 4.5: A driven L-C-R circuit, with $V(t) = V_0 \cos(\omega t)$.

we see that the homogeneous solution takes the form

$$x_{\rm h}(t) = A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct} \ , \tag{4.53}$$

where A, B, and C are constants.

To find the inhomogeneous solution, we solve $L_t\, x = f_0\, e^{-i\Omega t}$ and take the real part. Writing $x(t) = x_0\, e^{-i\Omega t}$, we have

$$\mathcal{L}_t\, x_0\, e^{-i\Omega t} = (a - i\Omega)\,(b - i\Omega)\,(c - i\Omega)\, x_0\, e^{-i\Omega t} \tag{4.54}$$

and thus

$$x_0 = \frac{f_0\, e^{-i\Omega t}}{(a - i\Omega)(b - i\Omega)(c - i\Omega)} \equiv A(\Omega)\, e^{i\delta(\Omega)}\, f_0\, e^{-i\Omega t} \ , \tag{4.55}$$

where

$$\begin{aligned} A(\Omega) &= \Big[(a^2 + \Omega^2)\,(b^2 + \Omega^2)\,(c^2 + \Omega^2)\Big]^{-1/2} \\ \delta(\Omega) &= \tan^{-1}\Big(\frac{\Omega}{a}\Big) + \tan^{-1}\Big(\frac{\Omega}{b}\Big) + \tan^{-1}\Big(\frac{\Omega}{c}\Big) \ . \end{aligned} \tag{4.56}$$

Thus, the most general solution to $L_t\, x(t) = f_0 \cos(\Omega t)$ is

$$x(t) = A(\Omega)\, f_0\, \cos\big(\Omega t - \delta(\Omega)\big) + A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct} \ . \tag{4.57}$$

Note that the phase shift increases monotonically from $\delta(0) = 0$ to $\delta(\infty) = \frac{3}{2}\pi$.

Mechanical analog of RLC circuit

Consider the electrical circuit in fig. 4.5. Our task is to construct its mechanical analog. To do so, we invoke Kirchoff's laws around the left and right loops:

$$\begin{aligned} L_1\, \dot I_1 + \frac{Q_1}{C_1} + R_1\,(I_1 - I_2) &= 0 \\ L_2\, \dot I_2 + R_2\, I_2 + R_1\,(I_2 - I_1) &= V(t) \ . \end{aligned} \tag{4.58}$$

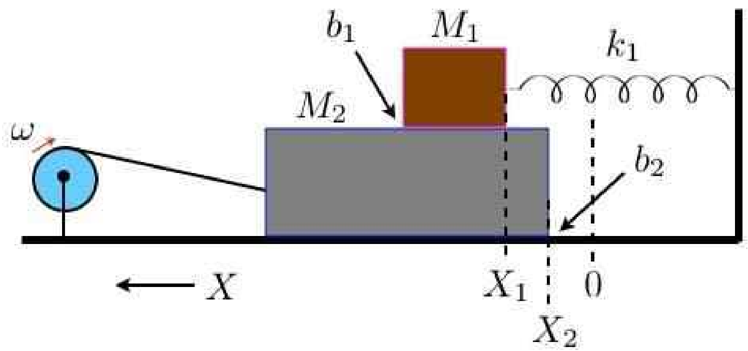

Figure 4.6: The equivalent mechanical circuit for fig. 4.5.

Let $Q_1(t)$ be the charge on the left plate of capacitor C_1, and define

$$Q_2(t) = \int_0^t dt'\, I_2(t') \ . \tag{4.59}$$

Then Kirchoff's laws may be written

$$\begin{aligned} \ddot{Q}_1 + \frac{R_1}{L_1}\,(\dot{Q}_1 - \dot{Q}_2) + \frac{1}{L_1 C_1}\,Q_1 &= 0 \\ \ddot{Q}_2 + \frac{R_2}{L_2}\,\dot{Q}_2 + \frac{R_1}{L_2}\,(\dot{Q}_2 - \dot{Q}_1) &= \frac{V(t)}{L_2} \ . \end{aligned} \tag{4.60}$$

Now consider the mechanical system in Fig. 4.6. The blocks have masses M_1 and M_2. The friction coefficient between blocks 1 and 2 is b_1, and the friction coefficient between block 2 and the floor is b_2. Here we assume a velocity-dependent frictional force $F_{\rm f} = -b\dot{x}$, rather than the more conventional constant $F_{\rm f} = -\mu\, W$, where W is the weight of an object. Velocity-dependent friction is applicable when the relative velocity of an object and a surface is sufficiently large. There is a spring of spring constant k_1 which connects block 1 to the wall. Finally, block 2 is driven by a periodic acceleration $f_0 \cos(\omega t)$. We now identify

$$X_1 \leftrightarrow Q_1 \quad , \quad X_2 \leftrightarrow Q_2 \quad , \quad b_1 \leftrightarrow \frac{R_1}{L_1} \quad , \quad b_2 \leftrightarrow \frac{R_2}{L_2} \quad , \quad k_1 \leftrightarrow \frac{1}{L_1 C_1} \ , \tag{4.61}$$

as well as $f(t) \leftrightarrow V(t)/L_2$.

The solution again proceeds by Fourier transform. We write

$$V(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{V}(\omega)\, e^{-i\omega t} \tag{4.62}$$

and

$$\begin{Bmatrix} Q_1(t) \\ \hat{I}_2(t) \end{Bmatrix} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \begin{Bmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{Bmatrix} e^{-i\omega t} \tag{4.63}$$

The frequency space version of Kirchoff's laws for this problem is

$$\overbrace{\begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1\, C_1 & R_1/L_1 \\ i\omega\, R_1/L_2 & -i\omega + (R_1+R_2)/L_2 \end{pmatrix}}^{\hat{G}(\omega)} \begin{pmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{V}(\omega)/L_2 \end{pmatrix} \tag{4.64}$$

The homogeneous equation has eigenfrequencies given by the solution to $\det \hat{G}(\omega) = 0$, which is a cubic equation. Correspondingly, there are three initial conditions to account for: $Q_1(0)$, $I_1(0)$, and $I_2(0)$. As in the case of the single damped harmonic oscillator, these transients are damped, and for large times may be ignored. The solution then is

$$\begin{pmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1\, C_1 & R_1/L_1 \\ i\omega\, R_1/L_2 & -i\omega + (R_1+R_2)/L_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{V}(\omega)/L_2 \end{pmatrix} . \tag{4.65}$$

To obtain the time-dependent $Q_1(t)$ and $I_2(t)$, we must compute the Fourier transform back to the time domain.

4.3 General solution by Green's function method

For a general forcing function $f(t)$, we solve by Fourier transform. Recall that a function $F(t)$ in the time domain has a Fourier transform $\hat{F}(\omega)$ in the frequency domain. The relation between the two is:[1]

$$F(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{F}(\omega) \qquad \Longleftrightarrow \qquad \hat{F}(\omega) = \int\limits_{-\infty}^{\infty} dt\, e^{+i\omega t}\, F(t) \ . \tag{4.66}$$

We can convert the differential equation 4.3 to an algebraic equation in the frequency domain, $\hat{x}(\omega) = \hat{G}(\omega)\, \hat{f}(\omega)$, where

$$\hat{G}(\omega) = \frac{1}{\omega_0^2 - 2i\beta\omega - \omega^2} \tag{4.67}$$

is the *Green's function* in the frequency domain. The general solution is written

$$x(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{G}(\omega)\, \hat{f}(\omega) + x_{\rm h}(t) \ , \tag{4.68}$$

[1] Different texts often use different conventions for Fourier and inverse Fourier transforms. Sometimes the factor of $(2\pi)^{-1}$ is associated with the time integral, and sometimes a factor of $(2\pi)^{-1/2}$ is assigned to both frequency and time integrals. The convention I use is obviously the best.

where $x_{\rm h}(t) = \sum_i C_i \, e^{-i\omega_i t}$ is a solution to the homogeneous equation. We may also write the above integral over the time domain:

$$\begin{aligned} x(t) &= \int\limits_{-\infty}^{\infty} dt' \, G(t-t') \, f(t') + x_{\rm h}(t) \\ G(s) &= \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega s} \, \hat{G}(\omega) \\ &= \nu^{-1} \, \exp(-\beta s) \, \sin(\nu s) \, \Theta(s) \end{aligned} \tag{4.69}$$

where $\Theta(s)$ is the *step function*,

$$\Theta(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases} \tag{4.70}$$

where once again $\nu \equiv \sqrt{\omega_0^2 - \beta^2}$.

Example: force pulse

Consider a pulse force

$$f(t) = f_0 \, \Theta(t) \, \Theta(T-t) = \begin{cases} f_0 & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases} \tag{4.71}$$

In the underdamped regime, for example, we find the solution

$$x(t) = \frac{f_0}{\omega_0^2} \left\{ 1 - e^{-\beta t} \cos \nu t - \frac{\beta}{\nu} e^{-\beta t} \sin \nu t \right\} \tag{4.72}$$

if $0 \leq t \leq T$ and

$$\begin{aligned} x(t) = \frac{f_0}{\omega_0^2} \Bigg\{ & \left(e^{-\beta(t-T)} \cos \nu(t-T) - e^{-\beta t} \cos \nu t \right) \\ & + \frac{\beta}{\nu} \left(e^{-\beta(t-T)} \sin \nu(t-T) - e^{-\beta t} \sin \nu t \right) \Bigg\} \end{aligned} \tag{4.73}$$

if $t > T$.

4.4 General Linear Autonomous Inhomogeneous ODEs

This method immediately generalizes to the case of general autonomous linear inhomogeneous ODEs of the form

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 \, x = f(t) \; . \tag{4.74}$$

We can write this as

$$\mathcal{L}_t \, x(t) = f(t) \; , \tag{4.75}$$

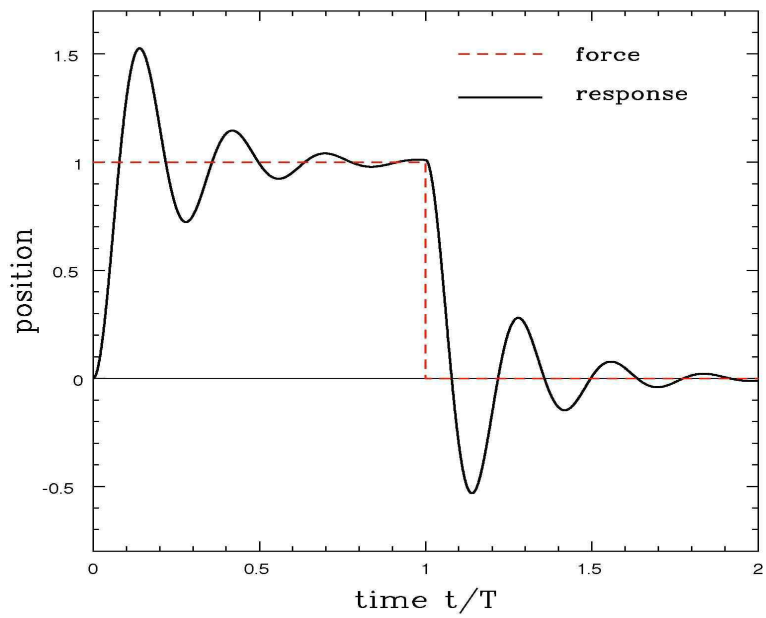

Figure 4.7: Response of an underdamped oscillator to a pulse force.

where $\mathcal{L}_t$ is the $n^{\rm th}$ order differential operator

$$\mathcal{L}_t = \frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\,\frac{d}{dt} + a_0 \ . \tag{4.76}$$

The general solution to the inhomogeneous equation is given by

$$x(t) = x_{\rm h}(t) + \int\limits_{-\infty}^{\infty}\!dt'\,G(t,t')\,f(t') \ , \tag{4.77}$$

where $G(t,t')$ is the Green's function. Note that $\mathcal{L}_t\,x_{\rm h}(t) = 0$. Thus, in order for eqns. 4.75 and 4.77 to be true, we must have

$$\mathcal{L}_t\,x(t) = \overbrace{\mathcal{L}_t\,x_{\rm h}(t)}^{\rm this\ vanishes} + \int\limits_{-\infty}^{\infty}\!dt'\,\mathcal{L}_t\,G(t,t')\,f(t') = f(t) \ , \tag{4.78}$$

which means that

$$\mathcal{L}_t\,G(t,t') = \delta(t-t') \ , \tag{4.79}$$

where $\delta(t-t')$ is the Dirac δ-function. Some properties of $\delta(x)$:

$$\int_a^b dx\, f(x)\,\delta(x-y) = \begin{cases} f(y) & \text{if } a<y<b \\ \\ 0 & \text{if } y<a \text{ or } y>b \ . \end{cases} \tag{4.80}$$

$$\delta\big(g(x)\big) = \sum_{\substack{x_i \text{ with} \\ g(x_i)=0}} \frac{\delta(x-x_i)}{\big|g'(x_i)\big|} \ , \tag{4.81}$$

valid for any functions $f(x)$ and $g(x)$. The sum in the second equation is over the zeros x_i of $g(x)$.

Incidentally, the Dirac δ-function enters into the relation between a function and its Fourier transform, in the following sense. We have

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{f}(\omega) \\ \hat{f}(\omega) &= \int_{-\infty}^{\infty} dt\, e^{+i\omega t}\, f(t) \ . \end{aligned} \tag{4.82}$$

Substituting the second equation into the first, we have

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\, f(t') \\ &= \int_{-\infty}^{\infty} dt' \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t'-t)} \right\} f(t') \ , \end{aligned} \tag{4.83}$$

which is indeed correct because the term in brackets is a representation of $\delta(t-t')$:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega s} = \delta(s) \ . \tag{4.84}$$

If the differential equation $\mathcal{L}_t\, x(t) = f(t)$ is defined over some finite t interval with prescribed boundary conditions on $x(t)$ at the endpoints, then $G(t,t')$ will depend on t and t' separately. For the case we are considering, the interval is the entire real line $t \in (-\infty,\infty)$, and $G(t,t') = G(t-t')$ is a function of the single variable $t-t'$.

Note that $\mathcal{L}_t = \mathcal{L}\big(\frac{d}{dt}\big)$ may be considered a function of the differential operator $\frac{d}{dt}$. If we now Fourier transform the equation $\mathcal{L}_t \, x(t) = f(t)$, we obtain

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \! dt \, e^{i\omega t} \, f(t) &= \int\limits_{-\infty}^{\infty} \! dt \, e^{i\omega t} \left\{ \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0 \right\} x(t) \\ &= \int\limits_{-\infty}^{\infty} \! dt \, e^{i\omega t} \left\{ (-i\omega)^n + a_{n-1} \, (-i\omega)^{n-1} + \ldots + a_1 \, (-i\omega) + a_0 \right\} x(t) \ ,\end{aligned} \tag{4.85}$$

where we integrate by parts on t, assuming the boundary terms at $t = \pm\infty$ vanish, *i.e.* $x(\pm\infty) = 0$, so that, inside the t integral,

$$e^{i\omega t} \left(\frac{d}{dt} \right)^{\! k} x(t) \to \left[\left(-\frac{d}{dt} \right)^{\! k} e^{i\omega t} \right] x(t) = (-i\omega)^k \, e^{i\omega t} \, x(t) \ . \tag{4.86}$$

Thus, if we define

$$\hat{\mathcal{L}}(\omega) = \sum_{k=0}^{n} a_k \, (-i\omega)^k \ , \tag{4.87}$$

then we have

$$\hat{\mathcal{L}}(\omega) \, \hat{\boldsymbol{x}}(\omega) = \hat{f}(\omega) \ , \tag{4.88}$$

where $a_n \equiv 1$. According to the Fundamental Theorem of Algebra, the n^{th} degree polynomial $\hat{\mathcal{L}}(\omega)$ may be uniquely factored over the complex ω plane into a product over n roots:

$$\hat{\mathcal{L}}(\omega) = (-i)^n \, (\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n) \ . \tag{4.89}$$

If the $\{a_k\}$ are all real, then $\big[\hat{\mathcal{L}}(\omega)\big]^* = \hat{\mathcal{L}}(-\omega^*)$, hence if Ω is a root then so is $-\Omega^*$. Thus, the roots appear in pairs which are symmetric about the imaginary axis. *I.e.* if $\Omega = a + ib$ is a root, then so is $-\Omega^* = -a + ib$.

The general solution to the homogeneous equation is

$$x_{\text{h}}(t) = \sum_{i=1}^{n} A_i \, e^{-i\omega_i t} \ , \tag{4.90}$$

which involves n arbitrary complex constants A_i. The susceptibility, or Green's function in Fourier space, $\hat{G}(\omega)$ is then

$$\hat{G}(\omega) = \frac{1}{\hat{\mathcal{L}}(\omega)} = \frac{i^n}{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)} \ , \tag{4.91}$$

and the general solution to the inhomogeneous equation is again given by

$$x(t) = x_{\text{h}}(t) + \int\limits_{-\infty}^{\infty} \! dt' \, G(t - t') \, f(t') \ , \tag{4.92}$$

where $x_{\rm h}(t)$ is the solution to the homogeneous equation, *i.e.* with zero forcing, and where

$$\begin{aligned} G(s) &= \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega s}\, \hat{G}(\omega) \\ &= i^n \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega s}}{(\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)} \\ &= \sum_{j=1}^{n} \frac{e^{-i\omega_j s}}{i\,\mathcal{L}'(\omega_j)}\, \Theta(s)\ , \end{aligned} \tag{4.93}$$

where we assume that $\text{Im}\,\omega_j < 0$ for all j. The integral above was done using Cauchy's theorem and the calculus of residues – a beautiful result from the theory of complex functions.

As an example, consider the familiar case

$$\begin{aligned} \hat{\mathcal{L}}(\omega) &= \omega_0^2 - 2i\beta\omega - \omega^2 \\ &= -(\omega-\omega_+)\,(\omega-\omega_-)\ , \end{aligned} \tag{4.94}$$

with $\omega_\pm = -i\beta \pm \nu$, and $\nu = (\omega_0^2 - \beta^2)^{1/2}$. This yields

$$\mathcal{L}'(\omega_\pm) = \mp(\omega_+ - \omega_-) = \mp 2\nu\ . \tag{4.95}$$

Then according to equation 4.93,

$$\begin{aligned} G(s) &= \left\{ \frac{e^{-i\omega_+ s}}{i\mathcal{L}'(\omega_+)} + \frac{e^{-i\omega_- s}}{i\mathcal{L}'(\omega_-)} \right\} \Theta(s) \\ &= \left\{ \frac{e^{-\beta s}\, e^{-i\nu s}}{-2i\nu} + \frac{e^{-\beta s}\, e^{i\nu s}}{2i\nu} \right\} \Theta(s) \\ &= \nu^{-1}\, e^{-\beta s}\, \sin(\nu s)\, \Theta(s)\ , \end{aligned} \tag{4.96}$$

exactly as before.

4.5 Kramers-Krönig Relations (advanced material)

Suppose $\hat{\chi}(\omega) \equiv \hat{G}(\omega)$ is analytic in the UHP[2]. Then for all ν, we must have

$$\int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi}\, \frac{\hat{\chi}(\nu)}{\nu - \omega + i\epsilon} = 0\ , \tag{4.97}$$

[2] In this section, we use the notation $\hat{\chi}(\omega)$ for the susceptibility, rather than $\hat{G}(\omega)$

where ϵ is a positive infinitesimal. The reason is simple: just close the contour in the UHP, assuming $\hat{\chi}(\omega)$ vanishes sufficiently rapidly that Jordan's lemma can be applied. Clearly this is an extremely weak restriction on $\hat{\chi}(\omega)$, given the fact that the denominator already causes the integrand to vanish as $|\omega|^{-1}$.

Let us examine the function

$$\frac{1}{\nu-\omega+i\epsilon}=\frac{\nu-\omega}{(\nu-\omega)^2+\epsilon^2}-\frac{i\epsilon}{(\nu-\omega)^2+\epsilon^2}\ . \tag{4.98}$$

which we have separated into real and imaginary parts. Under an integral sign, the first term, in the limit $\epsilon\to 0$, is equivalent to taking a *principal part* of the integral. That is, for any function $F(\nu)$ which is regular at $\nu=\omega$,

$$\lim_{\epsilon\to 0}\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\frac{\nu-\omega}{(\nu-\omega)^2+\epsilon^2}\,F(\nu)\equiv\mathcal{P}\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\frac{F(\nu)}{\nu-\omega}\ . \tag{4.99}$$

The principal part symbol $\mathcal{P}$ means that the singularity at $\nu=\omega$ is elided, either by smoothing out the function $1/(\nu-\epsilon)$ as above, or by simply cutting out a region of integration of width ϵ on either side of $\nu=\omega$.

The imaginary part is more interesting. Let us write

$$h(u)\equiv\frac{\epsilon}{u^2+\epsilon^2}\ . \tag{4.100}$$

For $|u|\gg\epsilon$, $h(u)\simeq\epsilon/u^2$, which vanishes as $\epsilon\to 0$. For $u=0$, $h(0)=1/\epsilon$ which diverges as $\epsilon\to 0$. Thus, $h(u)$ has a huge peak at $u=0$ and rapidly decays to 0 as one moves off the peak in either direction a distance greater that ϵ. Finally, note that

$$\int\limits_{-\infty}^{\infty}du\,h(u)=\pi\ , \tag{4.101}$$

a result which itself is easy to show using contour integration. Putting it all together, this tells us that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{u^2+\epsilon^2}=\pi\delta(u)\ . \tag{4.102}$$

Thus, for positive infinitesimal ϵ,

$$\frac{1}{u\pm i\epsilon}=\mathcal{P}\,\frac{1}{u}\mp i\pi\delta(u)\ , \tag{4.103}$$

a most useful result.

We now return to our initial result 4.97, and we separate $\hat{\chi}(\omega)$ into real and imaginary parts:

$$\hat{\chi}(\omega)=\hat{\chi}'(\omega)+i\hat{\chi}''(\omega)\ . \tag{4.104}$$

(In this equation, the primes do not indicate differentiation with respect to argument.) We therefore have, for every real value of ω,

$$0=\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\Big[\chi'(\nu)+i\chi''(\nu)\Big]\Big[\mathcal{P}\,\frac{1}{\nu-\omega}-i\pi\delta(\nu-\omega)\Big]\ . \tag{4.105}$$

Taking the real and imaginary parts of this equation, we derive the *Kramers-Krönig relations*:

$$\begin{aligned} \chi'(\omega) &= +\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{d\nu}{\pi}\,\frac{\hat{\chi}''(\nu)}{\nu-\omega} \\ \chi''(\omega) &= -\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{d\nu}{\pi}\,\frac{\hat{\chi}'(\nu)}{\nu-\omega}\ . \end{aligned} \tag{4.106}$$

Chapter 5

Calculus of Variations

5.1 Snell's Law

Warm-up problem: You are standing at point (x_1, y_1) on the beach and you want to get to a point (x_2, y_2) in the water, a few meters offshore. The interface between the beach and the water lies at $x = 0$. What path results in the shortest travel time? It is not a straight line! This is because your speed v_1 on the sand is greater than your speed v_2 in the water. The optimal path actually consists of two line segments, as shown in Fig. 5.1. Let the path pass through the point $(0, y)$ on the interface. Then the time T is a function of y:

$$T(y) = \frac{1}{v_1}\sqrt{x_1^2 + (y - y_1)^2} + \frac{1}{v_2}\sqrt{x_2^2 + (y_2 - y)^2} \ . \tag{5.1}$$

To find the minimum time, we set

$$\begin{aligned} \frac{dT}{dy} = 0 &= \frac{1}{v_1}\frac{y - y_1}{\sqrt{x_1^2 + (y - y_1)^2}} - \frac{1}{v_2}\frac{y_2 - y}{\sqrt{x_2^2 + (y_2 - y)^2}} \\ &= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \ . \end{aligned} \tag{5.2}$$

Thus, the optimal path satisfies

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \ , \tag{5.3}$$

which is known as *Snell's Law.*

Snell's Law is familiar from optics, where the speed of light in a polarizable medium is written $v = c/n$, where n is the index of refraction. In terms of n,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ . \tag{5.4}$$

If there are several interfaces, Snell's law holds at each one, so that

$$n_i \sin\theta_i = n_{i+1} \sin\theta_{i+1} \ , \tag{5.5}$$

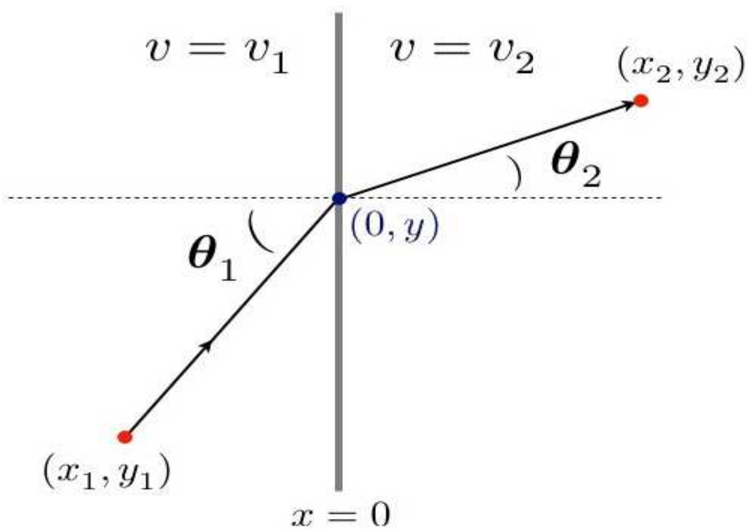

Figure 5.1: The shortest path between (x_1, y_1) and (x_2, y_2) is not a straight line, but rather two successive line segments of different slope.

at the interface between media i and $i + 1$.

In the limit where the number of slabs goes to infinity but their thickness is infinitesimal, we can regard n and θ as functions of a continuous variable x. One then has

$$\frac{\sin\theta(x)}{v(x)} = \frac{y'}{v\sqrt{1+y'^2}} = P \ , \tag{5.6}$$

where P is a constant. Here wve have used the result $\sin\theta = y'/\sqrt{1+y'^2}$, which follows from drawing a right triangle with side lengths dx, dy, and $\sqrt{dx^2 + dy^2}$. If we differentiate the above equation with respect to x, we eliminate the constant and obtain the second order ODE

$$\frac{1}{1+y'^2}\,\frac{y''}{y'} = \frac{v'}{v} \ . \tag{5.7}$$

This is a differential equation that $y(x)$ must satisfy if the *functional*

$$T\big[y(x)\big] = \int \frac{ds}{v} = \int\limits_{x_1}^{x_2} \! dx\, \frac{\sqrt{1+y'^2}}{v(x)} \tag{5.8}$$

is to be minimized.

5.2 Functions and Functionals

A *function* is a mathematical object which takes a real (or complex) variable, or several such variables, and returns a real (or complex) number. A *functional* is a mathematical object which takes an entire

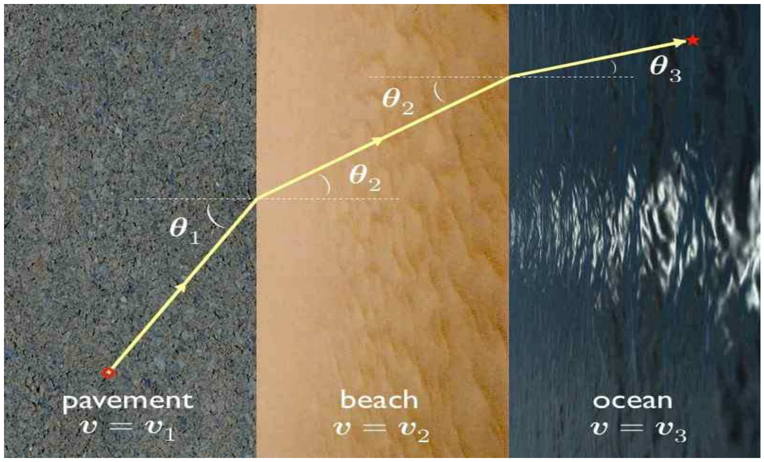

Figure 5.2: The path of shortest length is composed of three line segments. The relation between the angles at each interface is governed by Snell's Law.

function and returns a number. In the case at hand, we have

$$T\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\, L(y, y', x) \ , \tag{5.9}$$

where the function $L(y, y', x)$ is given by

$$L(y, y', x) = \frac{1}{v(x)} \sqrt{1 + y'^2} \ . \tag{5.10}$$

Here $v(x)$ is a given function characterizing the medium, and $y(x)$ is the path whose time is to be evaluated.

In ordinary calculus, we extremize a function $f(x)$ by demanding that f not change to lowest order when we change $x \to x + dx$:

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)\, (dx)^2 + \ldots \ . \tag{5.11}$$

We say that $x = x^*$ is an extremum when $f'(x^*) = 0$.

For a functional, the first *functional variation* is obtained by sending $y(x) \to y(x) + \delta y(x)$, and extracting

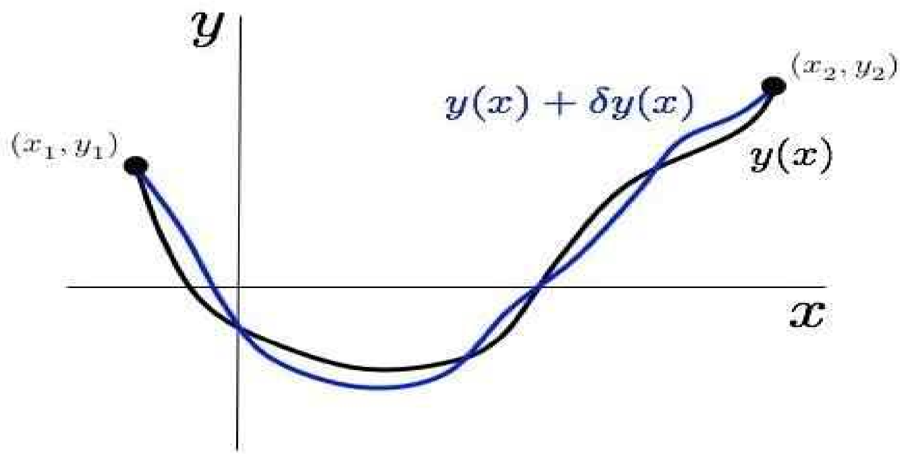

Figure 5.3: A path $y(x)$ and its variation $y(x) + \delta y(x)$.

the variation in the functional to order δy. Thus, we compute

$$\begin{aligned} T\big[y(x) + \delta y(x)\big] &= \int\limits_{x_1}^{x_2} dx\, L(y + \delta y, y' + \delta y', x) \\ &= \int\limits_{x_1}^{x_2} dx \left\{ L + \frac{\partial L}{\partial y}\,\delta y + \frac{\partial L}{\partial y'}\,\delta y' + \mathcal{O}\big((\delta y)^2\big) \right\} \\ &= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left\{ \frac{\partial L}{\partial y}\,\delta y + \frac{\partial L}{\partial y'}\,\frac{d}{dx}\,\delta y \right\} \\ &= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right] \delta y + \frac{\partial L}{\partial y'}\,\delta y \,\Bigg|_{x_1}^{x_2} . \end{aligned} \tag{5.12}$$

Now one very important thing about the variation $\delta y(x)$ is that it must vanish at the endpoints: $\delta y(x_1) = \delta y(x_2) = 0$. This is because the space of functions under consideration satisfy fixed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Thus, the last term in the above equation vanishes, and we have

$$\delta T = \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right] \delta y \ . \tag{5.13}$$

We say that the first functional derivative of T with respect to $y(x)$ is

$$\frac{\delta T}{\delta y(x)} = \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right]_x \ , \tag{5.14}$$

where the subscript indicates that the expression inside the square brackets is to be evaluated at x. The functional $T\big[y(x)\big]$ is *extremized* when its first functional derivative vanishes, which results in a differential equation for $y(x)$,

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 \ , \tag{5.15}$$

known as the *Euler-Lagrange* equation.

$L(y, y', x)$ independent of y

Suppose $L(y, y', x)$ is independent of y. Then from the Euler-Lagrange equations we have that

$$P \equiv \frac{\partial L}{\partial y'} \tag{5.16}$$

is a constant. In classical mechanics, this will turn out to be a *generalized momentum.* For $L = \frac{1}{v}\sqrt{1+y'^2}$, we have

$$P = \frac{y'}{v\sqrt{1+y'^2}} \ . \tag{5.17}$$

Setting $dP/dx = 0$, we recover the second order ODE of eqn. 5.7. Solving for y',

$$\frac{dy}{dx} = \pm \frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \ , \tag{5.18}$$

where $v_0 = 1/P$.

$L(y, y', x)$ independent of x

When $L(y, y', x)$ is independent of x, we can again integrate the equation of motion. Consider the quantity

$$H = y' \frac{\partial L}{\partial y'} - L \ . \tag{5.19}$$

Then

$$\begin{aligned} \frac{dH}{dx} = \frac{d}{dx}\left[y' \frac{\partial L}{\partial y'} - L\right] &= y'' \frac{\partial L}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y'}\, y'' - \frac{\partial L}{\partial y} y' - \frac{\partial L}{\partial x} \\ &= y' \left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] - \frac{\partial L}{\partial x} \ , \end{aligned} \tag{5.20}$$

where we have used the Euler-Lagrange equations to write $\frac{d}{dx}\big(\frac{\partial L}{\partial y'}\big) = \frac{\partial L}{\partial y}$. So if $\partial L/\partial x = 0$, we have $dH/dx = 0$, *i.e.* H is a constant.

5.2.1 Functional Taylor series

In general, we may expand a functional $F[y + \delta y]$ in a *functional Taylor series*,

$$\begin{aligned} F[y+\delta y] = F[y] + \int\!dx_1\, K_1(x_1)\, \delta y(x_1) + \tfrac{1}{2!}\int\!dx_1\int\!dx_2\, K_2(x_1,x_2)\, \delta y(x_1)\, \delta y(x_2) \\ + \tfrac{1}{3!}\int\!dx_1\int\!dx_2\int\!dx_3\, K_3(x_1,x_2,x_3)\, \delta y(x_1)\, \delta y(x_2)\, \delta y(x_3) + \ldots \end{aligned} \tag{5.21}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta y(x_1)\cdots\delta y(x_n)} \tag{5.22}$$

for the $n^{\rm th}$ functional derivative.

5.3 Examples from the Calculus of Variations

Here we present three useful examples of variational calculus as applied to problems in mathematics and physics.

5.3.1 Example 1 : minimal surface of revolution

Consider a surface formed by rotating the function $y(x)$ about the x-axis. The area is then

$$A\big[y(x)\big] = \int\limits_{x_1}^{x_2}\!dx\, 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ , \tag{5.23}$$

and is a functional of the curve $y(x)$. Thus we can define $L(y,y') = 2\pi y\sqrt{1+y'^2}$ and make the identification $y(x) \leftrightarrow q(t)$. Since $L(y,y',x)$ is independent of x, we have

$$H = y'\,\frac{\partial L}{\partial y'} - L \quad \Rightarrow \quad \frac{dH}{dx} = -\frac{\partial L}{\partial x}\ , \tag{5.24}$$

and when L has no explicit x-dependence, H is conserved. One finds

$$H = 2\pi y \cdot \frac{y'^2}{\sqrt{1+y'^2}} - 2\pi y\,\sqrt{1+y'^2} = -\frac{2\pi y}{\sqrt{1+y'^2}}\ . \tag{5.25}$$

Solving for y',

$$\frac{dy}{dx} = \pm\sqrt{\left(\frac{2\pi y}{H}\right)^2 - 1}\ , \tag{5.26}$$

which may be integrated with the substitution $y = \frac{H}{2\pi}\cosh u$, yielding

$$y(x) = b\,\cosh\left(\frac{x-a}{b}\right)\ , \tag{5.27}$$

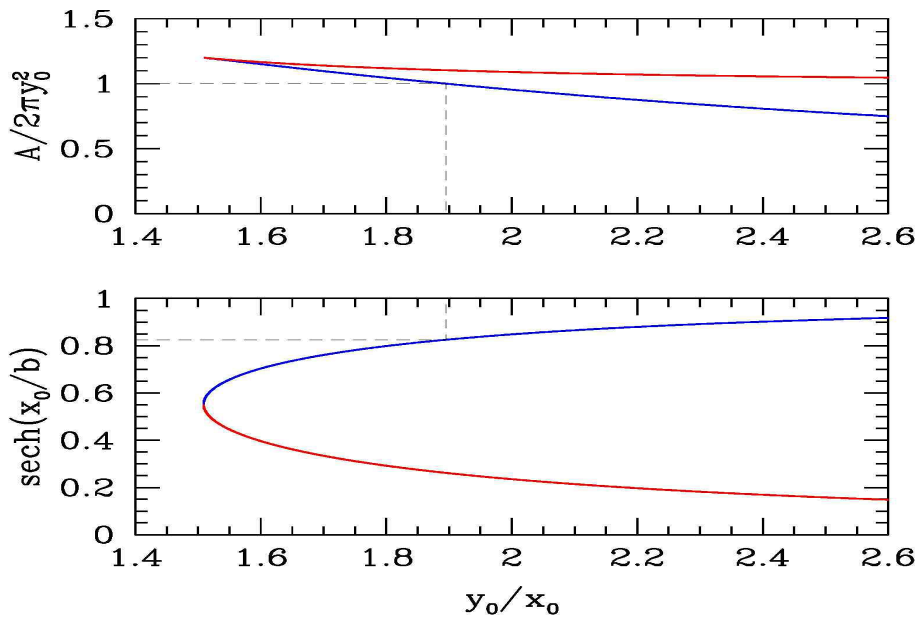

Figure 5.4: Minimal surface solution, with $y(x) = b\cosh(x/b)$ and $y(x_0) = y_0$. Top panel: $A/2\pi y_0^2$ *vs.* y_0/x_0. Bottom panel: $\mathrm{sech}(x_0/b)$ *vs.* y_0/x_0. The blue curve corresponds to a global minimum of $A[y(x)]$, and the red curve to a local minimum or saddle point.

where a and $b = \frac{H}{2\pi}$ are constants of integration. Note there are two such constants, as the original equation was second order. This shape is called a *catenary.* As we shall later find, it is also the shape of a uniformly dense rope hanging between two supports, under the influence of gravity. To fix the constants a and b, we invoke the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Consider the case where $-x_1 = x_2 \equiv x_0$ and $y_1 = y_2 \equiv y_0$. Then clearly $a = 0$, and we have

$$y_0 = b\,\cosh\left(\frac{x_0}{b}\right) \quad \Rightarrow \quad \gamma = \kappa^{-1}\,\cosh\kappa\ , \tag{5.28}$$

with $\gamma \equiv y_0/x_0$ and $\kappa \equiv x_0/b$. One finds that for any $\gamma > 1.5089$ there are two solutions, one of which is a global minimum and one of which is a local minimum or saddle of $A[y(x)]$. The solution with the smaller value of κ (*i.e.* the larger value of $\mathrm{sech}\,\kappa$) yields the smaller value of A, as shown in Fig. 5.4. Note that

$$\frac{y}{y_0} = \frac{\cosh(x/b)}{\cosh(x_0/b)}\ , \tag{5.29}$$

so $y(x=0) = y_0\,\mathrm{sech}(x_0/b)$.

When extremizing functions that are defined over a finite or semi-infinite interval, one must take care to evaluate the function at the boundary, for it may be that the boundary yields a global extremum even though the derivative may not vanish there. Similarly, when extremizing functionals, one must investigate the functions at the boundary of function space. In this case, such a function would be the discontinuous solution, with

$$y(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ \\ 0 & \text{if } x_1 < x < x_2 \\ \\ y_2 & \text{if } x = x_2 \ . \end{cases} \tag{5.30}$$

This solution corresponds to a surface consisting of two discs of radii y_1 and y_2, joined by an infinitesimally thin thread. The area functional evaluated for this particular $y(x)$ is clearly $A = \pi(y_1^2 + y_2^2)$. In Fig. 5.4, we plot $A/2\pi y_0^2$ *versus* the parameter $\gamma = y_0/x_0$. For $\gamma > \gamma_c \approx 1.564$, one of the catenary solutions is the global minimum. For $\gamma < \gamma_c$, the minimum area is achieved by the discontinuous solution.

Note that the functional derivative,

$$K_1(x) = \frac{\delta A}{\delta y(x)} = \left\{ \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right\} = \frac{2\pi\big(1 + y'^2 - yy''\big)}{(1 + y'^2)^{3/2}} \ , \tag{5.31}$$

indeed vanishes for the catenary solutions, but does not vanish for the discontinuous solution, where $K_1(x) = 2\pi$ throughout the interval $(-x_0, x_0)$. Since $y = 0$ on this interval, y cannot be decreased. The fact that $K_1(x) > 0$ means that increasing y will result in an increase in A, so the boundary value for A, which is $2\pi y_0^2$, is indeed a local minimum.

We furthermore see in Fig. 5.4 that for $\gamma < \gamma_* \approx 1.5089$ the local minimum and saddle are no longer present. This is the familiar saddle-node bifurcation, here in function space. Thus, for $\gamma \in [0, \gamma_*)$ there are no extrema of $A[y(x)]$, and the minimum area occurs for the discontinuous $y(x)$ lying at the boundary of function space. For $\gamma \in (\gamma_*, \gamma_c)$, two extrema exist, one of which is a local minimum and the other a saddle point. Still, the area is minimized for the discontinuous solution. For $\gamma \in (\gamma_c, \infty)$, the local minimum is the global minimum, and has smaller area than for the discontinuous solution.

5.3.2 Example 2 : geodesic on a surface of revolution

We use cylindrical coordinates (ρ, ϕ, z) on the surface $z = z(\rho)$. Thus,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2\, d\phi^2 + dx^2 \\ &= \Big\{ 1 + \big[z'(\rho)\big]^2 \Big\}\, d\rho + \rho^2\, d\phi^2 \ , \end{aligned} \tag{5.32}$$

and the distance functional $D\big[\phi(\rho)\big]$ is

$$D\big[\phi(\rho)\big] = \int\limits_{\rho_1}^{\rho_2} d\rho\, L(\phi, \phi', \rho) \ , \tag{5.33}$$

where

$$L(\phi, \phi', \rho) = \sqrt{1 + z'^2(\rho) + \rho^2\, \phi'^2(\rho)} \ . \tag{5.34}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial \phi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial \phi'}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \phi'} = \text{const.} \tag{5.35}$$

Thus,

$$\frac{\partial L}{\partial \phi'} = \frac{\rho^2\,\phi'}{\sqrt{1+z'^2+\rho^2\,\phi'^2}} = a \ , \tag{5.36}$$

where a is a constant. Solving for ϕ', we obtain

$$d\phi = \frac{a\sqrt{1+\big[z'(\rho)\big]^2}}{\rho\sqrt{\rho^2-a^2}}\,d\rho \ , \tag{5.37}$$

which we must integrate to find $\phi(\rho)$, subject to boundary conditions $\phi(\rho_i) = \phi_i$, with $i = 1, 2$.

On a cone, $z(\rho) = \lambda\rho$, and we have

$$d\phi = a\sqrt{1+\lambda^2}\,\frac{d\rho}{\rho\sqrt{\rho^2-a^2}} = \sqrt{1+\lambda^2}\,d\tan^{-1}\sqrt{\frac{\rho^2}{a^2}-1} \ , \tag{5.38}$$

which yields

$$\phi(\rho) = \beta + \sqrt{1+\lambda^2}\ \tan^{-1}\sqrt{\frac{\rho^2}{a^2}-1} \ , \tag{5.39}$$

which is equivalent to

$$\rho\,\cos\left(\frac{\phi-\beta}{\sqrt{1+\lambda^2}}\right) = a \ . \tag{5.40}$$

The constants β and a are determined from $\phi(\rho_i) = \phi_i$.

5.3.3 Example 3 : brachistochrone

Problem: find the path between (x_1, y_1) and (x_2, y_2) which a particle sliding frictionlessly and under constant gravitational acceleration will traverse in the shortest time. To solve this we first must invoke some elementary mechanics. Assuming the particle is released from (x_1, y_1) at rest, energy conservation says

$$\tfrac{1}{2}mv^2 + mgy = mgy_1 \ . \tag{5.41}$$

Then the time, which is a functional of the curve $y(x)$, is

$$\begin{aligned} T\big[y(x)\big] &= \int\limits_{x_1}^{x_2} \frac{ds}{v} = \frac{1}{\sqrt{2g}}\int\limits_{x_1}^{x_2} dx\,\sqrt{\frac{1+y'^2}{y_1-y}} \\ &\equiv \int\limits_{x_1}^{x_2} dx\, L(y,y',x) \ , \end{aligned} \tag{5.42}$$

with

$$L(y, y', x) = \sqrt{\frac{1 + y'^2}{2g(y_1 - y)}} \ . \tag{5.43}$$

Since L is independent of x, eqn. 5.20, we have that

$$H = y' \, \frac{\partial L}{\partial y'} - L = -\Big[2g\,(y_1 - y)\,\big(1 + y'^2\big)\Big]^{-1/2} \tag{5.44}$$

is conserved. This yields

$$dx = -\sqrt{\frac{y_1 - y}{2a - y_1 + y}} \; dy \ , \tag{5.45}$$

with $a = (4gH^2)^{-1}$. This may be integrated parametrically, writing

$$y_1 - y = 2a\sin^2(\tfrac{1}{2}\theta) \quad \Rightarrow \quad dx = 2a\sin^2(\tfrac{1}{2}\theta)\, d\theta \ , \tag{5.46}$$

which results in the parametric equations

$$\begin{aligned} x - x_1 &= a\,\big(\theta - \sin\theta\big) \\ y - y_1 &= -a\,(1 - \cos\theta) \ . \end{aligned} \tag{5.47}$$

This curve is known as a *cycloid.*

5.3.4 Ocean waves

Surface waves in fluids propagate with a definite relation between their angular frequency ω and their wavevector $k = 2\pi/\lambda$, where λ is the wavelength. The *dispersion relation* is a function $\omega = \omega(k)$. The *group velocity* of the waves is then $v(k) = d\omega/dk$.

In a fluid with a flat bottom at depth h, the dispersion relation turns out to be

$$\omega(k) = \sqrt{gk\,\tanh kh} \approx \begin{cases} \sqrt{gh}\,k & \text{shallow } (kh \ll 1) \\ \\ \sqrt{gk} & \text{deep } (kh \gg 1) \ . \end{cases} \tag{5.48}$$

Suppose we are in the shallow case, where the wavelength λ is significantly greater than the depth h of the fluid. This is the case for ocean waves which break at the shore. The phase velocity and group velocity are then identical, and equal to $v(h) = \sqrt{gh}$. The waves propagate more slowly as they approach the shore.

Let us choose the following coordinate system: x represents the distance parallel to the shoreline, y the distance perpendicular to the shore (which lies at $y = 0$), and $h(y)$ is the depth profile of the bottom. We assume $h(y)$ to be a slowly varying function of y which satisfies $h(0) = 0$. Suppose a disturbance in the ocean at position (x_2, y_2) propagates until it reaches the shore at $(x_1, y_1 = 0)$. The time of propagation is

$$T\big[y(x)\big] = \int \frac{ds}{v} = \int\limits_{x_1}^{x_2} \! dx \, \sqrt{\frac{1 + y'^2}{g\,h(y)}} \ . \tag{5.49}$$

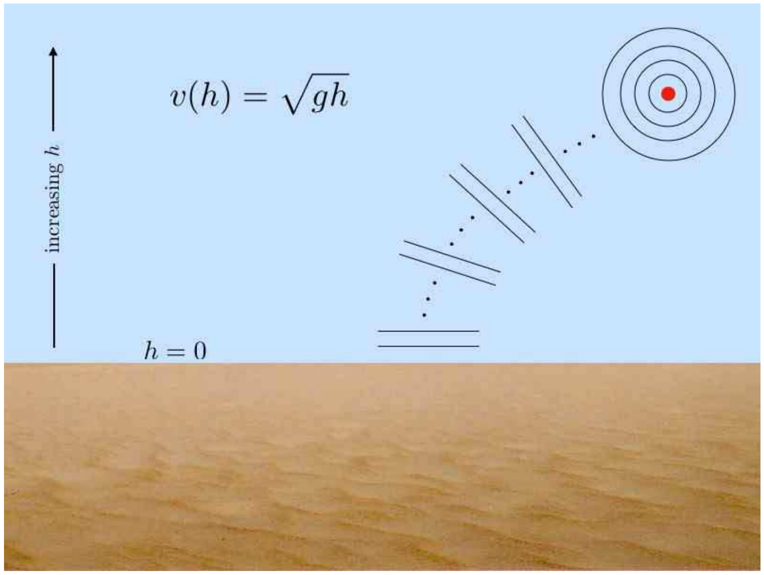

Figure 5.5: For shallow water waves, $v = \sqrt{gh}$. To minimize the propagation time from a source to the shore, the waves break parallel to the shoreline.

We thus identify the integrand

$$L(y, y', x) = \sqrt{\frac{1 + y'^2}{g\,h(y)}} \ . \tag{5.50}$$

As with the brachistochrone problem, to which this bears an obvious resemblance, L is cyclic in the independent variable x, hence

$$H = y'\,\frac{\partial L}{\partial y'} - L = -\Big[g\,h(y)\,\big(1 + y'^2\big)\Big]^{-1/2} \tag{5.51}$$

is constant. Solving for $y'(x)$, we have

$$\tan\theta = \frac{dy}{dx} = \sqrt{\frac{a}{h(y)} - 1} \ , \tag{5.52}$$

where $a = (gH)^{-1}$ is a constant, and where θ is the local slope of the function $y(x)$. Thus, we conclude that near $y = 0$, where $h(y) \to 0$, the waves come in *parallel to the shoreline.* If $h(y) = \alpha y$ has a linear profile, the solution is again a cycloid, with

$$\begin{aligned} x(\theta) &= b\,(\theta - \sin\theta) \\ y(\theta) &= b\,(1 - \cos\theta) \ , \end{aligned} \tag{5.53}$$

where $b = 2a/\alpha$ and where the shore lies at $\theta = 0$. Expanding in a Taylor series in θ for small θ, we may eliminate θ and obtain $y(x)$ as

$$y(x) = \big(\tfrac{9}{2}\big)^{1/3}\, b^{1/3}\, x^{2/3} + \ldots \ . \tag{5.54}$$

A *tsunami* is a shallow water wave that propagates in deep water. This requires $\lambda > h$, as we've seen, which means the disturbance must have a very long spatial extent out in the open ocean, where $h \sim 10\,\text{km}$.

An undersea earthquake is the only possible source; the characteristic length of earthquake fault lines can be hundreds of kilometers. If we take $h = 10\,\text{km}$, we obtain $v = \sqrt{gh} \approx 310\,\text{m/s}$ or $1100\,\text{km/hr}$. At these speeds, a tsunami can cross the Pacific Ocean in less than a day.

As the wave approaches the shore, it must slow down, since $v = \sqrt{gh}$ is diminishing. But energy is conserved, which means that the amplitude must concomitantly rise. In extreme cases, the water level rise at shore may be 20 meters or more.

5.4 Appendix : More on Functionals

We remarked in section 5.2 that a function f is an animal which gets fed a real number x and excretes a real number $f(x)$. We say f maps the reals to the reals, or

$$f\colon \mathbb{R} \to \mathbb{R} \tag{5.55}$$

Of course we also have functions $g\colon \mathbf{C} \to \mathbf{C}$ which eat and excrete complex numbers, multivariable functions $h\colon \mathbb{R}^N \to \mathbb{R}$ which eat N-tuples of numbers and excrete a single number, *etc.*

A *functional* $F[f(x)]$ eats entire functions (!) and excretes numbers. That is,

$$F\colon \Big\{ f(x) \;\Big|\; x \in \mathbb{R} \Big\} \to \mathbb{R} \tag{5.56}$$

This says that F operates on the set of real-valued functions of a single real variable, yielding a real number. Some examples:

$$\begin{aligned}
F[f(x)] &= \tfrac{1}{2}\int\limits_{-\infty}^{\infty}\!dx\,\big[f(x)\big]^2 \\
F[f(x)] &= \tfrac{1}{2}\int\limits_{-\infty}^{\infty}\!dx\int\limits_{-\infty}^{\infty}\!dx'\,K(x,x')\,f(x)\,f(x') \\
F[f(x)] &= \int\limits_{-\infty}^{\infty}\!dx\,\bigg\{\tfrac{1}{2}A\,f^2(x) + \tfrac{1}{2}B\left(\frac{df}{dx}\right)^{\!2}\bigg\}\ .
\end{aligned} \tag{5.57}$$

In classical mechanics, the action S is a functional of the path $q(t)$:

$$S[q(t)] = \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}}\!dt\,\Big\{\tfrac{1}{2}m\dot{q}^2 - U(q)\Big\}\ . \tag{5.58}$$

We can also have functionals which feed on functions of more than one independent variable, such as

$$S[y(x,t)] = \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}}\!dt\int\limits_{x_\mathrm{a}}^{x_\mathrm{b}}\!dx\,\bigg\{\tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^{\!2} - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^{\!2}\bigg\}\ , \tag{5.59}$$

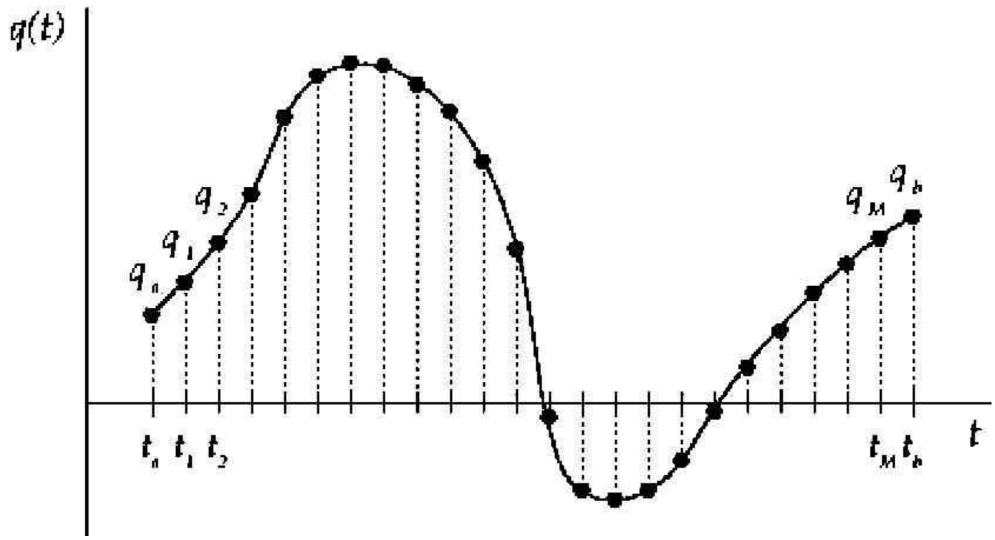

Figure 5.6: A functional $S[q(t)]$ is the continuum limit of a function of a large number of variables, $S(q_1, \ldots, q_M)$.

which happens to be the functional for a string of mass density μ under uniform tension τ. Another example comes from electrodynamics:

$$S[A^\mu(\boldsymbol{x}, t)] = -\int d^3\!x \int dt \left\{ \frac{1}{16\pi} F_{\mu\nu}\, F^{\mu\nu} + \frac{1}{c} j_\mu\, A^\mu \right\} , \tag{5.60}$$

which is a functional of the four fields $\{A^0, A^1, A^2, A^3\}$, where $A^0 = c\phi$. These are the components of the 4-potential, each of which is itself a function of four independent variables (x^0, x^1, x^2, x^3), with $x^0 = ct$. The field strength tensor is written in terms of derivatives of the A^μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where we use a metric $g_{\mu\nu} = \text{diag}(+, -, -, -)$ to raise and lower indices. The 4-potential couples linearly to the source term J_μ, which is the electric 4-current $(c\rho, \boldsymbol{J})$.

We extremize functions by sending the independent variable x to $x + dx$ and demanding that the variation $df = 0$ to first order in dx. That is,

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)(dx)^2 + \ldots\ , \tag{5.61}$$

whence $df = f'(x)\, dx + \mathcal{O}\big((dx)^2\big)$ and thus

$$f'(x^*) = 0 \quad \Longleftrightarrow \quad x^* \text{ an extremum.} \tag{5.62}$$

We extremize *functionals* by sending

$$f(x) \to f(x) + \delta f(x) \tag{5.63}$$

and demanding that the variation δF in the functional $F[f(x)]$ vanish to first order in $\delta f(x)$. The variation $\delta f(x)$ must sometimes satisfy certain boundary conditions. For example, if $F[f(x)]$ only operates on functions which vanish at a pair of endpoints, *i.e.* $f(x_a) = f(x_b) = 0$, then when we extremize the functional F we must do so *within the space of allowed functions.* Thus, we would in this case require $\delta f(x_a) = \delta f(x_b) = 0$. We may expand the functional $F[f + \delta f]$ in a *functional Taylor series*,

$$\begin{aligned} F[f+\delta f] &= F[f] + \int\! dx_1\, K_1(x_1)\,\delta f(x_1) + \tfrac{1}{2!}\int\! dx_1 \int\! dx_2\, K_2(x_1,x_2)\,\delta f(x_1)\,\delta f(x_2) \\ &\quad + \tfrac{1}{3!}\int\! dx_1 \int\! dx_2 \int\! dx_3\, K_3(x_1,x_2,x_3)\,\delta f(x_1)\,\delta f(x_2)\,\delta f(x_3) + \ldots \end{aligned} \tag{5.64}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta f(x_1)\cdots\delta f(x_n)} \ . \tag{5.65}$$

In a more general case, $F = F\big[\{f_i(\boldsymbol{x})\}\big]$ is a functional of several functions, each of which is a function of several independent variables.[1] We then write

$$\begin{aligned} F[\{f_i+\delta f_i\}] = F[\{f_i\}] &+ \int\! d\boldsymbol{x}_1\, K_1^i(\boldsymbol{x}_1)\,\delta f_i(\boldsymbol{x}_1) \\ &+ \tfrac{1}{2!}\int\! d\boldsymbol{x}_1\!\int\! d\boldsymbol{x}_2\, K_2^{ij}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,\delta f_i(\boldsymbol{x}_1)\,\delta f_j(\boldsymbol{x}_2) \\ &+ \tfrac{1}{3!}\int\! d\boldsymbol{x}_1\!\int\! d\boldsymbol{x}_2\!\int\! dx_3\ K_3^{ijk}(\boldsymbol{x}_1,\boldsymbol{x}_2,x_3)\,\delta f_i(\boldsymbol{x}_1)\,\delta f_j(\boldsymbol{x}_2)\,\delta f_k(\boldsymbol{x}_3) + \ldots\ , \end{aligned} \tag{5.66}$$

with

$$K_n^{i_1 i_2 \cdots i_n}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n) = \frac{\delta^n F}{\delta f_{i_1}(\boldsymbol{x}_1)\,\delta f_{i_2}(\boldsymbol{x}_2)\,\delta f_{i_n}(\boldsymbol{x}_n)} \ . \tag{5.67}$$

Another way to compute functional derivatives is to send

$$f(x) \to f(x) + \epsilon_1\,\delta(x-x_1) + \ldots + \epsilon_n\,\delta(x-x_n) \tag{5.68}$$

and then differentiate n times with respect to ϵ_1 through ϵ_n. That is,

$$\frac{\delta^n F}{\delta f(x_1)\cdots\delta f(x_n)} = \frac{\partial^n}{\partial\epsilon_1\cdots\partial\epsilon_n}\bigg|_{\epsilon_1=\epsilon_2=\cdots\epsilon_n=0} F\big[f(x)+\epsilon_1\,\delta(x-x_1)+\ldots+\epsilon_n\,\delta(x-x_n)\big] \ . \tag{5.69}$$

Let's see how this works. As an example, we'll take the action functional from classical mechanics,

$$S[q(t)] = \int\limits_{t_a}^{t_b}\! dt\,\Big\{\tfrac{1}{2}m\dot{q}^2 - U(q)\Big\} \ . \tag{5.70}$$

To compute the first functional derivative, we replace the function $q(t)$ with $q(t)+\epsilon\,\delta(t-t_1)$, and expand in powers of ϵ:

$$\begin{aligned} S\big[q(t)+\epsilon\delta(t-t_1)\big] &= S[q(t)] + \epsilon\int\limits_{t_a}^{t_b}\! dt\,\Big\{m\,\dot{q}\,\delta'(t-t_1) - U'(q)\,\delta(t-t_1)\Big\} \\ &= -\epsilon\,\Big\{m\,\ddot{q}(t_1) + U'\big(q(t_1)\big)\Big\} \ , \end{aligned} \tag{5.71}$$

[1] It may be also be that different functions depend on a different number of independent variables. *E.g.* $F = F[f(x),\, g(x,y),\, h(x,y,z)]$.

hence

$$\frac{\delta S}{\delta q(t)} = -\Big\{ m\,\ddot{q}(t) + U'\big(q(t)\big)\Big\} \tag{5.72}$$

and setting the first functional derivative to zero yields Newton's Second Law, $m\ddot{q} = -U'(q)$, for all $t \in [t_{\rm a}, t_{\rm b}]$. Note that we have used the result

$$\int\limits_{-\infty}^{\infty} \!dt\,\delta'(t-t_1)\,h(t) = -h'(t_1)\ , \tag{5.73}$$

which is easily established upon integration by parts.

To compute the second functional derivative, we replace

$$q(t) \to q(t) + \epsilon_1\,\delta(t-t_1) + \epsilon_2\,\delta(t-t_2) \tag{5.74}$$

and extract the term of order $\epsilon_1\,\epsilon_2$ in the double Taylor expansion. One finds this term to be

$$\epsilon_1\,\epsilon_2 \int\limits_{t_{\rm a}}^{t_{\rm b}} \!dt\,\Big\{ m\,\delta'(t-t_1)\,\delta'(t-t_2) - U''(q)\,\delta(t-t_1)\,\delta(t-t_2)\Big\}\ . \tag{5.75}$$

Note that we needn't bother with terms proportional to ϵ_1^2 or ϵ_2^2 since the recipe is to differentiate once with respect to each of ϵ_1 and ϵ_2 and then to set $\epsilon_1 = \epsilon_2 = 0$. This procedure uniquely selects the term proportional to $\epsilon_1\,\epsilon_2$, and yields

$$\frac{\delta^2 S}{\delta q(t_1)\,\delta q(t_2)} = -\Big\{ m\,\delta''(t_1-t_2) + U''\big(q(t_1)\big)\,\delta(t_1-t_2)\Big\}\ . \tag{5.76}$$

In multivariable calculus, the stability of an extremum is assessed by computing the matrix of second derivatives at the extremal point, known as the Hessian matrix. One has

$$\frac{\partial f}{\partial x_i}\bigg|_{\boldsymbol{x}^*} = 0 \quad \forall\ i \qquad ; \qquad H_{ij} = \frac{\partial^2 f}{\partial x_i\,\partial x_j}\bigg|_{\boldsymbol{x}^*}\ . \tag{5.77}$$

The eigenvalues of the Hessian H_{ij} determine the stability of the extremum. Since H_{ij} is a symmetric matrix, its eigenvectors η^α may be chosen to be orthogonal. The associated eigenvalues λ_α, defined by the equation

$$H_{ij}\,\eta^\alpha_j = \lambda_\alpha\,\eta^\alpha_i\ , \tag{5.78}$$

are the respective curvatures in the directions η^α, where $\alpha \in \{1,\ldots,n\}$ where n is the number of variables. The extremum is a local minimum if all the eigenvalues λ_α are positive, a maximum if all are negative, and otherwise is a saddle point. Near a saddle point, there are some directions in which the function increases and some in which it decreases.

In the case of functionals, the second functional derivative $K_2(x_1,x_2)$ defines an eigenvalue problem for $\delta f(x)$:

$$\int\limits_{x_a}^{x_b} \!dx_2\,K_2(x_1,x_2)\,\delta f(x_2) = \lambda\,\delta f(x_1)\ . \tag{5.79}$$

In general there are an infinite number of solutions to this equation which form a basis in function space, subject to appropriate boundary conditions at $x_{\rm a}$ and $x_{\rm b}$. For example, in the case of the action functional from classical mechanics, the above eigenvalue equation becomes a differential equation,

$$-\left\{m\,\frac{d^2}{dt^2} + U''\big(q^*(t)\big)\right\}\delta q(t) = \lambda\,\delta q(t)\ , \tag{5.80}$$

where $q^*(t)$ is the solution to the Euler-Lagrange equations. As with the case of ordinary multivariable functions, the functional extremum is a local minimum (in function space) if every eigenvalue λ_α is positive, a local maximum if every eigenvalue is negative, and a saddle point otherwise.

Consider the simple harmonic oscillator, for which $U(q) = \frac{1}{2}\,m\omega_0^2\,q^2$. Then $U''\big(q^*(t)\big) = m\,\omega_0^2$; note that we don't even need to know the solution $q^*(t)$ to obtain the second functional derivative in this special case. The eigenvectors obey $m(\delta\ddot{q} + \omega_0^2\,\delta q) = -\lambda\,\delta q$, hence

$$\delta q(t) = A\cos\Big(\sqrt{\omega_0^2 + (\lambda/m)}\;t + \varphi\Big)\ , \tag{5.81}$$

where A and φ are constants. Demanding $\delta q(t_{\rm a}) = \delta q(t_{\rm b}) = 0$ requires

$$\sqrt{\omega_0^2 + (\lambda/m)}\;(t_{\rm b} - t_{\rm a}) = n\pi\ , \tag{5.82}$$

where n is an integer. Thus, the eigenfunctions are

$$\delta q_n(t) = A\sin\left(n\pi\cdot\frac{t - t_{\rm a}}{t_{\rm b} - t_{\rm a}}\right)\ , \tag{5.83}$$

and the eigenvalues are

$$\lambda_n = m\Big(\frac{n\pi}{T}\Big)^2 - m\omega_0^2\ , \tag{5.84}$$

where $T = t_{\rm b} - t_{\rm a}$. Thus, so long as $T > \pi/\omega_0$, there is at least one negative eigenvalue. Indeed, for $\frac{n\pi}{\omega_0} < T < \frac{(n+1)\pi}{\omega_0}$ there will be n negative eigenvalues. This means the action is generally not a minimum, but rather lies at a *saddle point* in the (infinite-dimensional) function space.

To test this explicitly, consider a harmonic oscillator with the boundary conditions $q(0) = 0$ and $q(T) = Q$. The equations of motion, $\ddot{q} + \omega_0^2\,q = 0$, along with the boundary conditions, determine the motion,

$$q^*(t) = \frac{Q\,\sin(\omega_0 t)}{\sin(\omega_0 T)}\ . \tag{5.85}$$

The action for this path is then

$$\begin{aligned} S[q^*(t)] &= \int\limits_0^T\! dt\,\Big\{\tfrac{1}{2}m\,\dot{q}^{*2} - \tfrac{1}{2}m\omega_0^2\,q^{*2}\Big\} \\ &= \frac{m\,\omega_0^2\,Q^2}{2\sin^2\!\omega_0 T}\int\limits_0^T\! dt\,\Big\{\cos^2\!\omega_0 t - \sin^2\!\omega_0 t\Big\} \\ &= \tfrac{1}{2}m\omega_0\,Q^2\ \mathrm{ctn}\,(\omega_0 T)\ . \end{aligned} \tag{5.86}$$

Next consider the path $q(t) = Q\,t/T$ which satisfies the boundary conditions but does not satisfy the equations of motion (it proceeds with constant velocity). One finds the action for this path is

$$S[q(t)] = \tfrac{1}{2} m\omega_0\, Q^2 \left(\frac{1}{\omega_0 T} - \tfrac{1}{3}\omega_0 T \right) . \tag{5.87}$$

Thus, provided $\omega_0 T \neq n\pi$, in the limit $T \to \infty$ we find that the constant velocity path has lower action.

Finally, consider the general mechanical action,

$$S\big[q(t)\big] = \int\limits_{t_a}^{t_b} \!dt\, L(q, \dot{q}, t) \ . \tag{5.88}$$

We now evaluate the first few terms in the functional Taylor series:

$$\begin{aligned} S\big[q^*(t) + \delta q(t)\big] = \int\limits_{t_a}^{t_b}\! dt \,\Bigg\{ & L(q^*, \dot{q}^*, t) + \frac{\partial L}{\partial q_i}\bigg|_{q^*} \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{q^*} \delta \dot{q}_i \\ & + \frac{1}{2}\, \frac{\partial^2 L}{\partial q_i\, \partial q_j}\bigg|_{q^*} \delta q_i\, \delta q_j + \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_j}\bigg|_{q^*} \delta q_i\, \delta \dot{q}_j + \frac{1}{2}\, \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\bigg|_{q^*} \delta \dot{q}_i\, \delta \dot{q}_j + \ldots \Bigg\} \ . \end{aligned} \tag{5.89}$$

To identify the functional derivatives, we integrate by parts. Let $\Phi_{\ldots}(t)$ be an arbitrary function of time. Then

$$\begin{aligned} \int\limits_{t_a}^{t_b}\! dt\, \Phi_i(t)\, \delta\dot{q}_i(t) &= - \int\limits_{t_a}^{t_b}\! dt\, \dot{\Phi}_i(t)\, \delta q_i(t) \\ \int\limits_{t_a}^{t_b}\! dt\, \Phi_{ij}(t)\, \delta q_i(t)\, \delta\dot{q}_j(t) &= \int\limits_{t_a}^{t_b}\! dt \int\limits_{t_a}^{t_b}\! dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= \int\limits_{t_a}^{t_b}\! dt \int\limits_{t_a}^{t_b}\! dt'\, \Phi_{ij}(t))\, \delta'(t-t')\, \delta q_i(t)\, \delta q_j(t') \\ \int\limits_{t_a}^{t_b}\! dt\, \Phi_{ij}(t)\, d\dot{q}_i(t)\, \delta\dot{q}_j(t) &= \int\limits_{t_a}^{t_b}\! dt \int\limits_{t_a}^{t_b}\! dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt}\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= - \int\limits_{t_a}^{t_b}\! dt \int\limits_{t_a}^{t_b}\! dt' \,\Big[\dot{\Phi}_{ij}(t)\, \delta'(t-t') + \Phi_{ij}(t)\, \delta''(t-t') \Big]\, \delta q_i(t)\, \delta q_j(t') \ . \end{aligned} \tag{5.90}$$

Thus,

$$\begin{aligned}
\frac{\delta S}{\delta q_i(t)} &= \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]_{q^*(t)} \\
\frac{\delta^2 S}{\delta q_i(t)\,\delta q_j(t')} &= \left\{ \frac{\partial^2 L}{\partial q_i\,\partial q_j}\bigg|_{q^*(t)} \delta(t-t') - \frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\bigg|_{q^*(t)} \delta''(t-t') \right. \\
&\qquad \left. + \left[2\,\frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\right)\right]_{q^*(t)} \delta'(t-t') \right\} \ .
\end{aligned} \tag{5.91}$$

Chapter 6

Lagrangian Mechanics

6.1 Generalized Coordinates

A set of *generalized coordinates* $q_1, \ldots, q_n$ completely describes the positions of all particles in a mechanical system. In a system with $d_{\rm f}$ degrees of freedom and k constraints, $n = d_{\rm f} - k$ independent generalized coordinates are needed to completely specify all the positions. A constraint is a relation among coordinates, such as $x^2 + y^2 + z^2 = a^2$ for a particle moving on a sphere of radius a. In this case, $d_{\rm f} = 3$ and $k = 1$. In this case, we could eliminate z in favor of x and y, *i.e.* by writing $z = \pm\sqrt{a^2 - x^2 - y^2}$, or we could choose as coordinates the polar and azimuthal angles θ and ϕ.

For the moment we will assume that $n = d_{\rm f} - k$, and that the generalized coordinates are independent, satisfying no additional constraints among them. Later on we will learn how to deal with any remaining constraints among the $\{q_1, \ldots, q_n\}$.

The generalized coordinates may have units of length, or angle, or perhaps something totally different. In the theory of small oscillations, the normal coordinates are conventionally chosen to have units of $({\rm mass})^{1/2} \times ({\rm length})$. However, once a choice of generalized coordinate is made, with a concomitant set of units, the units of the conjugate momentum and force are determined:

$$\big[p_\sigma\big] = \frac{ML^2}{T} \cdot \frac{1}{\big[q_\sigma\big]} \qquad , \qquad \big[F_\sigma\big] = \frac{ML^2}{T^2} \cdot \frac{1}{\big[q_\sigma\big]} \ , \tag{6.1}$$

where $\big[A\big]$ means 'the units of A', and where M, L, and T stand for mass, length, and time, respectively. Thus, if q_σ has dimensions of length, then p_σ has dimensions of momentum and F_σ has dimensions of force. If q_σ is dimensionless, as is the case for an angle, p_σ has dimensions of angular momentum (ML^2/T) and F_σ has dimensions of torque (ML^2/T^2).

6.2 Hamilton's Principle

The equations of motion of classical mechanics are embodied in a variational principle, called *Hamilton's principle.* Hamilton's principle states that the motion of a system is such that the *action functional*

$$S\big[q(t)\big] = \int_{t_1}^{t_2} dt\, L(q,\dot{q},t) \tag{6.2}$$

is an extremum, *i.e.* $\delta S = 0$. Here, $q = \{q_1, \ldots, q_n\}$ is a complete set of *generalized coordinates* for our mechanical system, and

$$L = T - U \tag{6.3}$$

is the *Lagrangian*, where T is the kinetic energy and U is the potential energy. Setting the first variation of the action to zero gives the Euler-Lagrange equations,

$$\frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{momentum } p_\sigma} = \overbrace{\frac{\partial L}{\partial q_\sigma}}^{\text{force } F_\sigma} \quad . \tag{6.4}$$

Thus, we have the familiar $\dot{p}_\sigma = F_\sigma$, also known as Newton's second law. Note, however, that the $\{q_\sigma\}$ are *generalized coordinates*, so p_σ may not have dimensions of momentum, nor F_σ of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

6.2.1 Invariance of the equations of motion

Suppose

$$\tilde{L}(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}\,G(q,t) \ . \tag{6.5}$$

Then

$$\tilde{S}[q(t)] = S[q(t)] + G(q_b,t_b) - G(q_a,t_a) \ . \tag{6.6}$$

Since the difference $\tilde{S} - S$ is a function only of the endpoint values $\{q_a, q_b\}$, their variations are identical: $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Thus, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time.

6.2.2 Remarks on the order of the equations of motion

The equations of motion are second order in time. This follows from the fact that $L = L(q,\dot{q},t)$. Using the chain rule,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial \dot{q}_{\sigma'}}\,\ddot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial q_{\sigma'}}\,\dot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial t} \ . \tag{6.7}$$

That the equations are second order in time can be regarded as an empirical fact. It follows, as we have just seen, from the fact that L depends on q and on $\dot{q}$, but on no higher time derivative terms. Suppose the Lagrangian did depend on the generalized accelerations $\ddot{q}$ as well. What would the equations of motion look like?

Taking the variation of S,

$$\begin{aligned}\delta\int_{t_a}^{t_b} dt\, L(q,\dot{q},\ddot{q},t) &= \left[\frac{\partial L}{\partial \dot{q}_\sigma}\,\delta q_\sigma + \frac{\partial L}{\partial \ddot{q}_\sigma}\,\delta \dot{q}_\sigma - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\delta q_\sigma\right]_{t_a}^{t_b} \\ &\quad + \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\right\}\delta q_\sigma \ .\end{aligned} \tag{6.8}$$

The boundary term vanishes if we require $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = \delta \dot{q}_\sigma(t_a) = \delta \dot{q}_\sigma(t_b) = 0 \;\forall\; \sigma$. The equations of motion would then be *fourth order* in time.

6.2.3 Lagrangian for a free particle

For a free particle, we can use Cartesian coordinates for each particle as our system of generalized coordinates. For a single particle, the Lagrangian $L(\boldsymbol{x},\boldsymbol{v},t)$ must be a function solely of $\boldsymbol{v}^2$. This is because homogeneity with respect to space and time preclude any dependence of L on $\boldsymbol{x}$ or on t, and isotropy of space means L must depend on $\boldsymbol{v}^2$. We next invoke Galilean relativity, which says that the equations of motion are invariant under transformation to a reference frame moving with constant velocity. Let $\boldsymbol{V}$ be the velocity of the new reference frame $\mathcal{K}'$ relative to our initial reference frame $\mathcal{K}$. Then $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t$, and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$. In order that the equations of motion be invariant under the change in reference frame, we demand

$$L'(\boldsymbol{v}') = L(\boldsymbol{v}) + \frac{d}{dt}\,G(\boldsymbol{x},t) \ . \tag{6.9}$$

The only possibility is $L = \frac{1}{2}m\boldsymbol{v}^2$, where the constant m is the mass of the particle. Note:

$$L' = \tfrac{1}{2}m(\boldsymbol{v}-\boldsymbol{V})^2 = \tfrac{1}{2}m\boldsymbol{v}^2 + \frac{d}{dt}\Big(\tfrac{1}{2}m\boldsymbol{V}^2\,t - m\boldsymbol{V}\cdot\boldsymbol{x}\Big) = L + \frac{dG}{dt} \ . \tag{6.10}$$

For N interacting particles,

$$L = \tfrac{1}{2}\sum_{a=1}^{N} m_a\left(\frac{d\boldsymbol{x}_a}{dt}\right)^2 - U\big(\{\boldsymbol{x}_a\},\{\dot{\boldsymbol{x}}_a\}\big) \ . \tag{6.11}$$

Here, U is the *potential energy*. Generally, U is of the form

$$U = \sum_a U_1(\boldsymbol{x}_a) + \sum_{a<a'} v(\boldsymbol{x}_a - \boldsymbol{x}_{a'}) \ , \tag{6.12}$$

however, as we shall see, velocity-dependent potentials appear in the case of charged particles interacting with electromagnetic fields. In general, though,

$$L = T - U \ , \tag{6.13}$$

where T is the kinetic energy, and U is the potential energy.

6.3 Conserved Quantities

A conserved quantity $\Lambda(q, \dot{q}, t)$ is one which does not vary throughout the motion of the system. This means

$$\frac{d\Lambda}{dt}\bigg|_{q=q(t)} = 0 \ . \tag{6.14}$$

We shall discuss conserved quantities in detail in the chapter on Noether's Theorem, which follows.

6.3.1 Momentum conservation

The simplest case of a conserved quantity occurs when the Lagrangian does not explicitly depend on one or more of the generalized coordinates, *i.e.* when

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = 0 \ . \tag{6.15}$$

We then say that L is *cyclic* in the coordinate q_σ. In this case, the Euler-Lagrange equations $\dot{p}_\sigma = F_\sigma$ say that the conjugate momentum p_σ is conserved. Consider, for example, the motion of a particle of mass m near the surface of the earth. Let (x, y) be coordinates parallel to the surface and z the height. We then have

$$\begin{aligned} T &= \tfrac{1}{2} m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) \\ U &= mgz \\ L &= T - U = \tfrac{1}{2} m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) - mgz \ . \end{aligned} \tag{6.16}$$

Since

$$F_x = \frac{\partial L}{\partial x} = 0 \quad \text{and} \quad F_y = \frac{\partial L}{\partial y} = 0 \ , \tag{6.17}$$

we have that p_x and p_y are conserved, with

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ . \tag{6.18}$$

These first order equations can be integrated to yield

$$x(t) = x(0) + \frac{p_x}{m}\, t \quad , \quad y(t) = y(0) + \frac{p_y}{m}\, t \ . \tag{6.19}$$

The z equation is of course

$$\dot{p}_z = m\ddot{z} = -mg = F_z \ , \tag{6.20}$$

with solution

$$z(t) = z(0) + \dot{z}(0)\, t - \tfrac{1}{2} g t^2 \ . \tag{6.21}$$

As another example, consider a particle moving in the (x, y) plane under the influence of a potential $U(x, y) = U\big(\sqrt{x^2 + y^2}\big)$ which depends only on the particle's distance from the origin $\rho = \sqrt{x^2 + y^2}$. The Lagrangian, expressed in two-dimensional polar coordinates (ρ, ϕ), is

$$L = \tfrac{1}{2} m\big(\dot{\rho}^2 + \rho^2 \dot{\phi}^2\big) - U(\rho) \ . \tag{6.22}$$

We see that L is cyclic in the angle ϕ, hence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} \tag{6.23}$$

is conserved. p_ϕ is the angular momentum of the particle about the $\hat{\boldsymbol{z}}$ axis. In the language of the calculus of variations, momentum conservation is what follows when the integrand of a functional is independent of the *independent variable*.

6.3.2 Energy conservation

When the integrand of a functional is independent of the *dependent* variable, another conservation law follows. For Lagrangian mechanics, consider the expression

$$H(q,\dot{q},t) = \sum_{\sigma=1}^{n} p_\sigma \, \dot{q}_\sigma - L \ . \tag{6.24}$$

Now we take the total time derivative of H:

$$\frac{dH}{dt} = \sum_{\sigma=1}^{n} \bigg\{ p_\sigma \, \ddot{q}_\sigma + \dot{p}_\sigma \, \dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} \, \dot{q}_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} \, \ddot{q}_\sigma \bigg\} - \frac{\partial L}{\partial t} \ . \tag{6.25}$$

We evaluate $\dot{H}$ along the motion of the system, which entails that the terms in the curly brackets above cancel for each σ:

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad \dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma} \ . \tag{6.26}$$

Thus, we find

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{6.27}$$

which means that H is conserved *whenever the Lagrangian contains no explicit time dependence*. For a Lagrangian of the form

$$L = \sum_a \tfrac{1}{2} \, m_a \dot{\boldsymbol{r}}_a^2 - U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \ , \tag{6.28}$$

we have that $\boldsymbol{p}_a = m_a \, \dot{\boldsymbol{r}}_a$, and

$$H = T + U = \sum_a \tfrac{1}{2} \, m_a \dot{\boldsymbol{r}}_a^2 + U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \ . \tag{6.29}$$

However, it is not always the case that $H = T + U$ is the total energy, as we shall see in the next chapter.

6.4 Choosing Generalized Coordinates

Any choice of generalized coordinates will yield an equivalent set of equations of motion. However, some choices result in an apparently simpler set than others. This is often true with respect to the form of the potential energy. Additionally, certain constraints that may be present are more amenable to treatment using a particular set of generalized coordinates.

The kinetic energy T is always simple to write in Cartesian coordinates, and it is good practice, at least when one is first learning the method, to write T in Cartesian coordinates and then convert to generalized coordinates. In Cartesian coordinates, the kinetic energy of a single particle of mass m is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) . \tag{6.30}$$

If the motion is two-dimensional, and confined to the plane $z = \text{const.}$, one of course has $T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)$.

Two other commonly used coordinate systems are the cylindrical and spherical systems. In cylindrical coordinates (ρ, ϕ, z), ρ is the radial coordinate in the (x, y) plane and ϕ is the azimuthal angle:

$$\begin{aligned} x &= \rho\,\cos\phi \quad , \quad & \dot{x} &= \cos\phi\,\dot{\rho} - \rho\sin\phi\,\dot{\phi} \\ y &= \rho\,\sin\phi \quad , \quad & \dot{y} &= \sin\phi\,\dot{\rho} + \rho\cos\phi\,\dot{\phi} \ , \end{aligned} \tag{6.31}$$

and the third, orthogonal coordinate is of course z. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2 + \dot{z}^2\right) . \end{aligned} \tag{6.32}$$

When the motion is confined to a plane with $z = \text{const.}$, this coordinate system is often referred to as 'two-dimensional polar' coordinates.

In spherical coordinates (r, θ, ϕ), r is the radius, θ is the polar angle, and ϕ is the azimuthal angle. On the globe, θ would be the 'colatitude', which is $\theta = \frac{\pi}{2} - \lambda$, where λ is the latitude. *I.e.* $\theta = 0$ at the north pole. In spherical polar coordinates,

$$\begin{aligned} x &= r\,\sin\theta\,\cos\phi \quad , \quad & \dot{x} &= \sin\theta\,\cos\phi\,\dot{r} + r\,\cos\theta\,\cos\phi\,\dot{\theta} - r\,\sin\theta\sin\phi\,\dot{\phi} \\ y &= r\,\sin\theta\,\sin\phi \quad , \quad & \dot{y} &= \sin\theta\,\sin\phi\,\dot{r} + r\,\cos\theta\,\sin\phi\,\dot{\theta} + r\,\sin\theta\cos\phi\,\dot{\phi} \\ z &= r\,\cos\theta \quad , \quad & \dot{z} &= \cos\theta\,\dot{r} - r\,\sin\theta\,\dot{\theta} \ . \end{aligned} \tag{6.33}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\!\theta\,\dot{\phi}^2\right) . \end{aligned} \tag{6.34}$$

6.5 How to Solve Mechanics Problems

Here are some simple steps you can follow toward obtaining the equations of motion:

1. Choose a set of generalized coordinates $\{q_1, \ldots, q_n\}$.

2. Find the kinetic energy $T(q, \dot{q}, t)$, the potential energy $U(q, t)$, and the Lagrangian $L(q, \dot{q}, t) = T - U$. It is often helpful to first write the kinetic energy in Cartesian coordinates for each particle before converting to generalized coordinates.

3. Find the canonical momenta $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$ and the generalized forces $F_\sigma = \frac{\partial L}{\partial q_\sigma}$.
4. Evaluate the time derivatives $\dot{p}_\sigma$ and write the equations of motion $\dot{p}_\sigma = F_\sigma$. Be careful to differentiate properly, using the chain rule and the Leibniz rule where appropriate.
5. Identify any conserved quantities (more about this later).

6.6 Examples

6.6.1 One-dimensional motion

For a one-dimensional mechanical system with potential energy $U(x)$,

$$L = T - U = \tfrac{1}{2} m \dot{x}^2 - U(x) \ . \tag{6.35}$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{6.36}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Rightarrow \qquad m\ddot{x} = -U'(x) \ , \tag{6.37}$$

which is of course $F = ma$.

Note that we can multiply the equation of motion by $\dot{x}$ to get

$$0 = \dot{x}\left\{m\ddot{x} + U'(x)\right\} = \frac{d}{dt}\left\{\tfrac{1}{2}m\dot{x}^2 + U(x)\right\} = \frac{dE}{dt} \ , \tag{6.38}$$

where $E = T + U$.

6.6.2 Central force in two dimensions

Consider next a particle of mass m moving in two dimensions under the influence of a potential $U(\rho)$ which is a function of the distance from the origin $\rho = \sqrt{x^2 + y^2}$. Clearly cylindrical ($2d$ polar) coordinates are called for:

$$L = \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \, \dot{\phi}^2\right) - U(\rho) \ . \tag{6.39}$$

The equations of motion are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) &= \frac{\partial L}{\partial \rho} \qquad \Rightarrow \qquad m\ddot{\rho} = m\rho\,\dot{\phi}^2 - U'(\rho) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \qquad \Rightarrow \qquad \frac{d}{dt}\left(m\rho^2\,\dot{\phi}\right) = 0 \ . \end{aligned} \tag{6.40}$$

Note that the canonical momentum conjugate to ϕ, which is to say the angular momentum, is conserved:

$$p_\phi = m\rho^2\,\dot{\phi} = \text{const.} \tag{6.41}$$

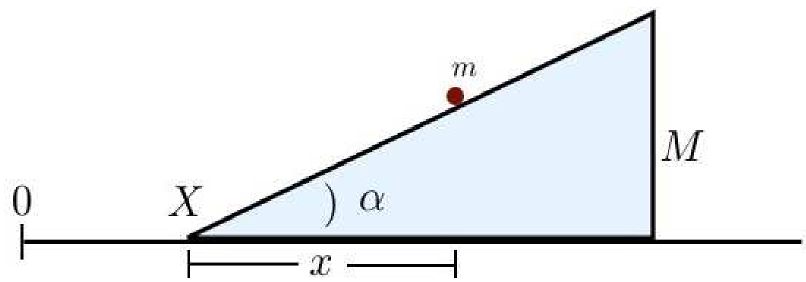

Figure 6.1: A wedge of mass M and opening angle α slides frictionlessly along a horizontal surface, while a small object of mass m slides frictionlessly along the wedge.

We can use this to eliminate $\dot{\phi}$ from the first Euler-Lagrange equation, obtaining

$$m\ddot{\rho} = \frac{p_\phi^2}{m\rho^3} - U'(\rho) \ . \tag{6.42}$$

We can also write the total energy as

$$\begin{aligned} E &= \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) + U(\rho) \\ &= \tfrac{1}{2}m\,\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} + U(\rho) \ , \end{aligned} \tag{6.43}$$

from which it may be shown that E is also a constant:

$$\frac{dE}{dt} = \Big(m\,\ddot{\rho} - \frac{p_\phi^2}{m\rho^3} + U'(\rho)\Big)\,\dot{\rho} = 0 \ . \tag{6.44}$$

We shall discuss this case in much greater detail in the coming weeks.

6.6.3 A sliding point mass on a sliding wedge

Consider the situation depicted in Fig. 6.1, in which a point object of mass m slides frictionlessly along a wedge of opening angle α. The wedge itself slides frictionlessly along a horizontal surface, and its mass is M. We choose as generalized coordinates the horizontal position X of the left corner of the wedge, and the horizontal distance x from the left corner to the sliding point mass. The vertical coordinate of the sliding mass is then $y = x\,\tan\alpha$, where the horizontal surface lies at $y = 0$. With these generalized coordinates, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\,(\dot{X} + \dot{x})^2 + \tfrac{1}{2}m\dot{y}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1 + \tan^2\!\alpha)\,\dot{x}^2 \ . \end{aligned} \tag{6.45}$$

The potential energy is simply

$$U = mgy = mg\,x\,\tan\alpha \ . \tag{6.46}$$

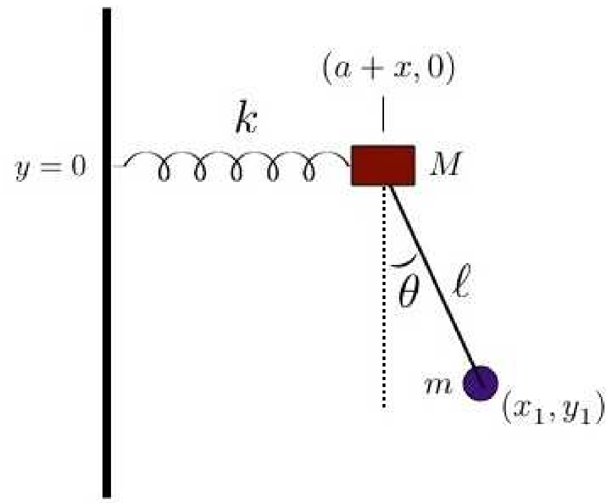

Figure 6.2: The spring–pendulum system.

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha\ , \tag{6.47}$$

and the equations of motion are

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{X}}\bigg) &= \frac{\partial L}{\partial X} \quad\Rightarrow\quad (M+m)\ddot{X} + m\,\ddot{x} = 0 \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{x}}\bigg) &= \frac{\partial L}{\partial x} \quad\Rightarrow\quad m\ddot{X} + m\,(1+\tan^2\!\alpha)\,\ddot{x} = -mg\tan\alpha\ . \end{aligned} \tag{6.48}$$

At this point we can use the first of these equations to write

$$\ddot{X} = -\frac{m}{M+m}\,\ddot{x}\ . \tag{6.49}$$

Substituting this into the second equation, we obtain the constant accelerations

$$\ddot{x} = -\frac{(M+m)g\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha} \quad , \quad \ddot{X} = \frac{mg\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha}\ . \tag{6.50}$$

6.6.4 A pendulum attached to a mass on a spring

Consider next the system depicted in Fig. 6.2 in which a mass M moves horizontally while attached to a spring of spring constant k. Hanging from this mass is a pendulum of arm length ℓ and bob mass m.

A convenient set of generalized coordinates is (x, θ), where x is the displacement of the mass M relative to the equilibrium extension a of the spring, and θ is the angle the pendulum arm makes with respect to the vertical. Let the Cartesian coordinates of the pendulum bob be (x_1, y_1). Then

$$x_1 = a + x + \ell\,\sin\theta \quad , \quad y_1 = -l\cos\theta\ . \tag{6.51}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2) \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[(\dot{x}+\ell\cos\theta\,\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2\right] \\ &= \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta}\ , \end{aligned} \tag{6.52}$$

and the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kx^2 + mgy_1 \\ &= \tfrac{1}{2}kx^2 - mg\ell\cos\theta\ . \end{aligned} \tag{6.53}$$

Thus,

$$L = \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} - \tfrac{1}{2}kx^2 + mg\ell\cos\theta\ . \tag{6.54}$$

The canonical momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial\dot{x}} = (M+m)\,\dot{x} + m\ell\cos\theta\,\dot{\theta} \\ p_\theta &= \frac{\partial L}{\partial\dot{\theta}} = m\ell\cos\theta\,\dot{x} + m\ell^2\,\dot{\theta}\ , \end{aligned} \tag{6.55}$$

and the canonical forces are

$$\begin{aligned} F_x &= \frac{\partial L}{\partial x} = -kx \\ F_\theta &= \frac{\partial L}{\partial\theta} = -m\ell\sin\theta\,\dot{x}\,\dot{\theta} - mg\ell\,\sin\theta\ . \end{aligned} \tag{6.56}$$

The equations of motion then yield

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\cos\theta\,\ddot{\theta} - m\ell\,\sin\theta\,\dot{\theta}^2 &= -kx \\ m\ell\,\cos\theta\,\ddot{x} + m\ell^2\,\ddot{\theta} &= -mg\ell\,\sin\theta\ . \end{aligned} \tag{6.57}$$

Small Oscillations : If we assume both x and θ are small, we may write $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, in which case the equations of motion may be linearized to

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\ddot{\theta} + kx &= 0 \\ m\ell\,\ddot{x} + m\ell^2\,\ddot{\theta} + mg\ell\,\theta &= 0\ . \end{aligned} \tag{6.58}$$

If we define

$$u \equiv \frac{x}{\ell} \quad , \quad \alpha \equiv \frac{m}{M} \quad , \quad \omega_0^2 \equiv \frac{k}{M} \quad , \quad \omega_1^2 \equiv \frac{g}{\ell}\ , \tag{6.59}$$

then

$$\begin{aligned} (1+\alpha)\,\ddot{u} + \alpha\,\ddot{\theta} + \omega_0^2\,u &= 0 \\ \ddot{u} + \ddot{\theta} + \omega_1^2\,\theta &= 0\ . \end{aligned} \tag{6.60}$$

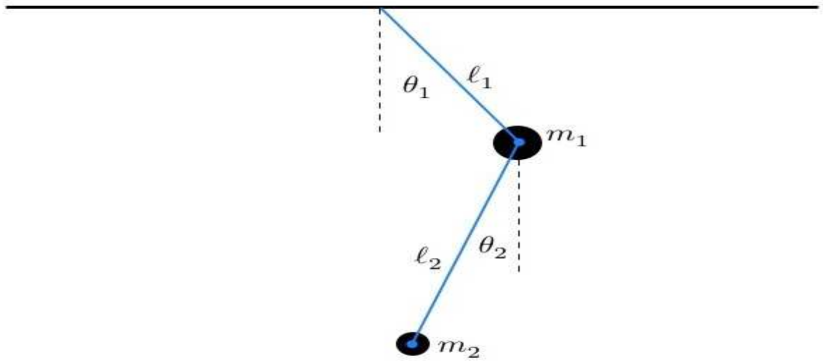

Figure 6.3: The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

We can solve by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-i\omega t} \ , \tag{6.61}$$

in which case

$$\begin{pmatrix} \omega_0^2 - (1+\alpha)\,\omega^2 & -\alpha\,\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ . \tag{6.62}$$

In order to have a nontrivial solution (*i.e.* without $a = b = 0$), the determinant of the above 2×2 matrix must vanish. This gives a condition on ω^2, with solutions

$$\omega_\pm^2 = \tfrac{1}{2}\big(\omega_0^2 + (1+\alpha)\,\omega_1^2\big) \pm \tfrac{1}{2}\sqrt{\big(\omega_0^2 - \omega_1^2\big)^2 + 2\alpha\,(\omega_0^2 + \omega_1^2)\,\omega_1^2} \ . \tag{6.63}$$

6.6.5 The double pendulum

As yet another example of the generalized coordinate approach to Lagrangian dynamics, consider the double pendulum system, sketched in Fig. 6.3. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of the generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\dot\theta_1, \dot\theta_2\}$.

In Cartesian coordinates,

$$\begin{aligned} T &= \tfrac{1}{2} m_1\,(\dot x_1^2 + \dot y_1^2) + \tfrac{1}{2} m_2\,(\dot x_2^2 + \dot y_2^2) \\ U &= m_1\,g\,y_1 + m_2\,g\,y_2 \ . \end{aligned} \tag{6.64}$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of the generalized coordinates $\{\theta_1, \theta_2\}$:

$$\begin{aligned} x_1 &= \ell_1\,\sin\theta_1 \qquad , \qquad x_2 = \ell_1\,\sin\theta_1 + \ell_2\,\sin\theta_2 \\ y_1 &= -\ell_1\,\cos\theta_1 \qquad , \qquad y_2 = -\ell_1\,\cos\theta_1 - \ell_2\,\cos\theta_2 \ . \end{aligned} \tag{6.65}$$

Thus, the velocities are

$$\begin{aligned} \dot{x}_1 &= \ell_1\,\dot\theta_1\,\cos\theta_1 \quad , \quad \dot{x}_2 = \ell_1\,\dot\theta_1\,\cos\theta_1 + \ell_2\,\dot\theta_2\,\cos\theta_2 \\ \dot{y}_1 &= \ell_1\,\dot\theta_1\,\sin\theta_1 \quad , \quad \dot{y}_2 = \ell_1\,\dot\theta_1\,\sin\theta_1 + \ell_2\,\dot\theta_2\,\sin\theta_2 \ . \end{aligned} \tag{6.66}$$

Thus,

$$\begin{aligned} T &= \tfrac{1}{2}m_1\,\ell_1^2\,\dot\theta_1^2 + \tfrac{1}{2}m_2\Big\{\ell_1^2\,\dot\theta_1^2 + 2\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 + \ell_2^2\,\dot\theta_2^2\Big\} \\ U &= -m_1\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_2\,\cos\theta_2\ , \end{aligned} \tag{6.67}$$

and

$$\begin{aligned} L = T - U = \tfrac{1}{2}(m_1+m_2)\,\ell_1^2\,\dot\theta_1^2 &+ m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 + \tfrac{1}{2}m_2\,\ell_2^2\,\dot\theta_2^2 \\ &+ (m_1+m_2)\,g\,\ell_1\,\cos\theta_1 + m_2\,g\,\ell_2\,\cos\theta_2\ . \end{aligned} \tag{6.68}$$

The generalized (canonical) momenta are

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial\dot\theta_1} = (m_1+m_2)\,\ell_1^2\,\dot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\dot\theta_2 \\ p_2 &= \frac{\partial L}{\partial\dot\theta_2} = m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\dot\theta_2\ , \end{aligned} \tag{6.69}$$

and the equations of motion are

$$\begin{aligned} \dot{p}_1 &= (m_1+m_2)\,\ell_1^2\,\ddot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\ddot\theta_2 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,(\dot\theta_1-\dot\theta_2)\,\dot\theta_2 \\ &= -(m_1+m_2)\,g\,\ell_1\,\sin\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_1} \end{aligned} \tag{6.70}$$

and

$$\begin{aligned} \dot{p}_2 &= m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\ddot\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,(\dot\theta_1-\dot\theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\ddot\theta_2 \\ &= -m_2\,g\,\ell_2\,\sin\theta_2 + m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_2}\ . \end{aligned} \tag{6.71}$$

We therefore find

$$\begin{aligned} \ell_1\,\ddot\theta_1 + \frac{m_2\,\ell_2}{m_1+m_2}\,\cos(\theta_1-\theta_2)\,\ddot\theta_2 + \frac{m_2\,\ell_2}{m_1+m_2}\,\sin(\theta_1-\theta_2)\,\dot\theta_2^2 + g\,\sin\theta_1 &= 0 \\ \ell_1\,\cos(\theta_1-\theta_2)\,\ddot\theta_1 + \ell_2\,\ddot\theta_2 - \ell_1\,\sin(\theta_1-\theta_2)\,\dot\theta_1^2 + g\,\sin\theta_2 &= 0. \end{aligned} \tag{6.72}$$

Small Oscillations : The equations of motion are coupled, nonlinear second order ODEs. When the system is close to equilibrium, the amplitudes of the motion are small, and we may expand in powers of the θ_1 and θ_2. The linearized equations of motion are then

$$\begin{aligned} \ddot\theta_1 + \alpha\beta\,\ddot\theta_2 + \omega_0^2\,\theta_1 &= 0 \\ \ddot\theta_1 + \beta\,\ddot\theta_2 + \omega_0^2\,\theta_2 &= 0\ , \end{aligned} \tag{6.73}$$

where we have defined

$$\alpha \equiv \frac{m_2}{m_1 + m_2} \quad , \quad \beta \equiv \frac{\ell_2}{\ell_1} \quad , \quad \omega_0^2 \equiv \frac{g}{\ell_1} \ . \tag{6.74}$$

We can solve this coupled set of equations by a nifty trick. Let's take a linear combination of the first equation plus an undetermined coefficient, r, times the second:

$$(1+r)\,\ddot{\theta}_1 + (\alpha + r)\beta\,\ddot{\theta}_2 + \omega_0^2\,(\theta_1 + r\,\theta_2) = 0 \ . \tag{6.75}$$

We now demand that the ratio of the coefficients of θ_2 and θ_1 is the same as the ratio of the coefficients of $\ddot{\theta}_2$ and $\ddot{\theta}_1$:

$$\frac{(\alpha+r)\,\beta}{1+r} = r \quad \Rightarrow \quad r_\pm = \tfrac{1}{2}(\beta - 1) \pm \tfrac{1}{2}\sqrt{(1-\beta)^2 + 4\alpha\beta} \tag{6.76}$$

When $r = r_\pm$, the equation of motion may be written

$$\frac{d^2}{dt^2}\big(\theta_1 + r_\pm\,\theta_2\big) = -\frac{\omega_0^2}{1+r_\pm}\,\big(\theta_1 + r_\pm\,\theta_2\big) \tag{6.77}$$

and defining the (unnormalized) *normal modes*

$$\xi_\pm \equiv \big(\theta_1 + r_\pm\,\theta_2\big) \ , \tag{6.78}$$

we find

$$\ddot{\xi}_\pm + \omega_\pm^2\,\xi_\pm = 0 \ , \tag{6.79}$$

with

$$\omega_\pm = \frac{\omega_0}{\sqrt{1+r_\pm}} \ . \tag{6.80}$$

Thus, by switching to the normal coordinates, we decoupled the equations of motion, and identified the two *normal frequencies of oscillation*. We shall have much more to say about small oscillations further below.

For example, with $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$, we have $\alpha = \frac{1}{2}$, and $\beta = 1$, in which case

$$r_\pm = \pm\tfrac{1}{\sqrt{2}} \quad , \quad \xi_\pm = \theta_1 \pm \tfrac{1}{\sqrt{2}}\,\theta_2 \quad , \quad \omega_\pm = \sqrt{2 \mp \sqrt{2}}\,\sqrt{\frac{g}{\ell}} \ . \tag{6.81}$$

Note that the oscillation frequency for the 'in-phase' mode ξ_+ is low, and that for the 'out of phase' mode ξ_- is high.

6.6.6 The thingy

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached to each vertex. The opposite corners are joined by springs of spring constant k. In the square configuration, the strings are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about equilibrium.

Solution

The rhombus is depicted in figure 6.4. Let a be the equilibrium length of the springs; clearly $L = \frac{a}{\sqrt{2}}$. Let ϕ be half of one of the opening angles, as shown. Then the masses are located at $(\pm X, 0)$ and $(0, \pm Y)$, with $X = \frac{a}{\sqrt{2}} \cos\phi$ and $Y = \frac{a}{\sqrt{2}} \sin\phi$. The spring extensions are $\delta X = 2X - a$ and $\delta Y = 2Y - a$. The

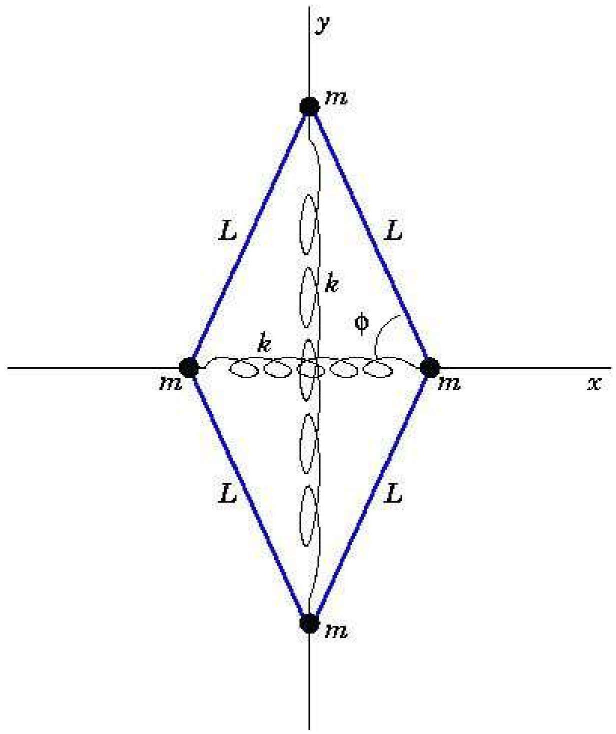

Figure 6.4: The thingy: a rhombus with opening angles 2ϕ and $\pi - 2\phi$.

kinetic and potential energies are therefore

$$T = M\big(\dot{X}^2 + \dot{Y}^2\big) = \tfrac{1}{2} M a^2 \, \dot{\phi}^2 \tag{6.82}$$

and

$$\begin{aligned} U &= \tfrac{1}{2} k \big(\delta X\big)^2 + \tfrac{1}{2} k \big(\delta Y\big)^2 \\ &= \tfrac{1}{2} k a^2 \left\{ \big(\sqrt{2}\,\cos\phi - 1\big)^2 + \big(\sqrt{2}\,\sin\phi - 1\big)^2 \right\} \\ &= \tfrac{1}{2} k a^2 \left\{ 3 - 2\sqrt{2}(\cos\phi + \sin\phi) \right\} . \end{aligned} \tag{6.83}$$

Note that minimizing $U(\phi)$ gives $\sin\phi = \cos\phi$, *i.e.* $\phi_{\rm eq} = \frac{\pi}{4}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} M a^2 \, \dot{\phi}^2 + \sqrt{2}\, k a^2 \big(\cos\phi + \sin\phi \big) + \text{const.} \tag{6.84}$$

The equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \qquad \Rightarrow \qquad M a^2 \, \ddot{\phi} = \sqrt{2}\, k a^2 \, (\cos\phi - \sin\phi) \tag{6.85}$$

It's always smart to expand about equilibrium, so let's write $\phi = \frac{\pi}{4} + \delta$, which leads to

$$\ddot{\delta} + \omega_0^2 \, \sin\delta = 0 \ , \tag{6.86}$$

with $\omega_0 = \sqrt{2k/M}$. This is the equation of a pendulum! Linearizing gives $\ddot{\delta} + \omega_0^2 \, \delta = 0$, so the small oscillation frequency is just ω_0.

6.7 Appendix : Virial Theorem

The virial theorem is a statement about the time-averaged motion of a mechanical system. Define the *virial*,

$$G(q,p) = \sum_\sigma p_\sigma \, q_\sigma \ . \tag{6.87}$$

Then

$$\begin{aligned} \frac{dG}{dt} &= \sum_\sigma \big(\dot{p}_\sigma \, q_\sigma + p_\sigma \, \dot{q}_\sigma\big) \\ &= \sum_\sigma q_\sigma \, F_\sigma + \sum_\sigma \dot{q}_\sigma \, \frac{\partial L}{\partial \dot{q}_\sigma} \ . \end{aligned} \tag{6.88}$$

Now suppose that $T = \frac{1}{2}\sum_{\sigma,\sigma'} \mathrm{T}_{\sigma\sigma'} \, \dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k=2$ in $\dot{q}$, and that U is homogeneous of degree zero in $\dot{q}$. Then

$$\sum_\sigma \dot{q}_\sigma \, \frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma \, \frac{\partial T}{\partial \dot{q}_\sigma} = 2\,T, \tag{6.89}$$

which follows from Euler's theorem on homogeneous functions.

Now consider the time average of $\dot{G}$ over a period τ:

$$\begin{aligned} \Big\langle \frac{dG}{dt} \Big\rangle &= \frac{1}{\tau} \int\limits_0^\tau \! dt \, \frac{dG}{dt} \\ &= \frac{1}{\tau} \Big[G(\tau) - G(0) \Big] \ . \end{aligned} \tag{6.90}$$

If $G(t)$ is bounded, then in the limit $\tau \to \infty$ we must have $\langle \dot{G} \rangle = 0$. Any bounded motion, such as the orbit of the earth around the Sun, will result in $\langle \dot{G} \rangle_{\tau \to \infty} = 0$. But then

$$\Big\langle \frac{dG}{dt} \Big\rangle = 2\,\langle T \rangle + \Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = 0 \ , \tag{6.91}$$

which implies

$$\begin{aligned} \langle T \rangle &= -\tfrac{1}{2} \Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = +\Big\langle \tfrac{1}{2} \sum_\sigma q_\sigma \, \frac{\partial U}{\partial q_\sigma} \Big\rangle \\ &= \Big\langle \tfrac{1}{2} \sum_i \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_{\!i} U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \Big\rangle \\ &= \tfrac{1}{2} k \, \langle U \rangle \ , \end{aligned} \tag{6.92}$$

where the last line pertains to homogeneous potentials of degree k. Finally, since $T+U=E$ is conserved, we have

$$\langle T\rangle = \frac{k\,E}{k+2} \qquad , \qquad \langle U\rangle = \frac{2\,E}{k+2} \ . \tag{6.93}$$

Chapter 7

Noether's Theorem

7.1 Continuous Symmetry Implies Conserved Charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector $\boldsymbol{r}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) - U(r)\ , \tag{7.1}$$

where we have chosen generalized coordinates (r, ϕ). The momentum conjugate to ϕ is $p_\phi = m\,r^2\,\dot{\phi}$. The generalized force F_ϕ clearly vanishes, since L does not depend on the coordinate ϕ. (One says that L is 'cyclic' in ϕ.) Thus, although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = m\,r^2\,\dot{\phi}$ is constant. This is the conserved angular momentum about the $\hat{\boldsymbol{z}}$ axis.

If instead the particle moved in a potential $U(y)$, independent of x, then writing

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(y)\ , \tag{7.2}$$

we have that the momentum $p_x = \partial L/\partial\dot{x} = m\dot{x}$ is conserved, because the generalized force $F_x = \partial L/\partial x = 0$ vanishes. This situation pertains in a uniform gravitational field, with $U(x, y) = mgy$, independent of x. The horizontal component of momentum is conserved.

In general, whenever the system exhibits a *continuous symmetry*, there is an associated *conserved charge*. (The terminology 'charge' is from field theory.) Indeed, this is a rigorous result, known as *Noether's Theorem*. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta)\ , \tag{7.3}$$

where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, *i.e.* $\tilde{q}_\sigma(q, 0) = q_\sigma$. The transformation may be nonlinear in the generalized coordinates. Suppose further that the Lagrangian L is invariant under the replacement $q \to \tilde{q}$. Then we must have

$$\begin{aligned} 0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(\tilde{q},\dot{\tilde{q}},t) &= \frac{\partial L}{\partial q_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} \\ &= \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg)\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg)_{\zeta=0} \\ &= \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg)_{\zeta=0} \ . \end{aligned} \tag{7.4}$$

Thus, there is an associated conserved charge

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} \ . \tag{7.5}$$

7.1.1 Examples of one-parameter families of transformations

Consider the Lagrangian

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U\big(\sqrt{x^2+y^2}\big) \ . \tag{7.6}$$

In two-dimensional polar coordinates, we have

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) \ , \tag{7.7}$$

and we may now define

$$\begin{aligned} \tilde{r}(\zeta) &= r \\ \tilde{\phi}(\zeta) &= \phi + \zeta \ . \end{aligned} \tag{7.8}$$

Note that $\tilde{r}(0) = r$ and $\tilde{\phi}(0) = \phi$, *i.e.* the transformation is the identity when $\zeta = 0$. We now have

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} = \frac{\partial L}{\partial \dot{r}}\,\frac{\partial \tilde{r}}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \tilde{\phi}}{\partial\zeta}\bigg|_{\zeta=0} = mr^2\dot{\phi} \ . \tag{7.9}$$

Another way to derive the same result which is somewhat instructive is to work out the transformation in Cartesian coordinates. We then have

$$\begin{aligned} \tilde{x}(\zeta) &= x\,\cos\zeta - y\,\sin\zeta \\ \tilde{y}(\zeta) &= x\,\sin\zeta + y\,\cos\zeta \ . \end{aligned} \tag{7.10}$$

Thus,

$$\frac{\partial\tilde{x}}{\partial\zeta} = -\tilde{y} \quad , \quad \frac{\partial\tilde{y}}{\partial\zeta} = \tilde{x} \tag{7.11}$$

and

$$\Lambda = \frac{\partial L}{\partial \dot{x}}\,\frac{\partial \tilde{x}}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}}\,\frac{\partial \tilde{y}}{\partial\zeta}\bigg|_{\zeta=0} = m(x\dot{y} - y\dot{x}) \ . \tag{7.12}$$

But

$$m(x\dot{y} - y\dot{x}) = m\hat{\boldsymbol{z}} \cdot \boldsymbol{r} \times \dot{\boldsymbol{r}} = mr^2\dot{\phi} \ . \tag{7.13}$$

As another example, consider the potential

$$U(\rho, \phi, z) = V(\rho, a\phi + z) \ , \tag{7.14}$$

where (ρ, ϕ, z) are cylindrical coordinates for a particle of mass m, and where a is a constant with dimensions of length. The Lagrangian is

$$\tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\big) - V(\rho, a\phi + z) \ . \tag{7.15}$$

This model possesses a helical symmetry, with a one-parameter family

$$\begin{aligned} \tilde{\rho}(\zeta) &= \rho \\ \tilde{\phi}(\zeta) &= \phi + \zeta \\ \tilde{z}(\zeta) &= z - \zeta a \ . \end{aligned} \tag{7.16}$$

Note that

$$a\tilde{\phi} + \tilde{z} = a\phi + z \ , \tag{7.17}$$

so the potential energy, and the Lagrangian as well, is invariant under this one-parameter family of transformations. The conserved charge for this symmetry is

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}}\,\frac{\partial \tilde{\rho}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{z}}\,\frac{\partial \tilde{z}}{\partial \zeta}\bigg|_{\zeta=0} = m\rho^2\dot{\phi} - ma\dot{z} \ . \tag{7.18}$$

We can check explicitly that Λ is conserved, using the equations of motion

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\phi}}\bigg) &= \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) = \frac{\partial L}{\partial \phi} = -a\frac{\partial V}{\partial z} \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{z}}\bigg) &= \frac{d}{dt}(m\dot{z}) = \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z} \ . \end{aligned} \tag{7.19}$$

Thus,

$$\dot{\Lambda} = \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) - a\,\frac{d}{dt}(m\dot{z}) = 0 \ . \tag{7.20}$$

7.1.2 Conservation of Linear and Angular Momentum

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol{n}}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol{x}}_a = \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \ , \tag{7.21}$$

and the associated conserved Noether charge is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{P} \ , \tag{7.22}$$

where $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is the *total momentum* of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\begin{aligned} \tilde{\boldsymbol{x}}_a &= R(\zeta, \hat{\boldsymbol{n}})\, \boldsymbol{x}_a \\ &= \boldsymbol{x}_a + \zeta\, \hat{\boldsymbol{n}} \times \boldsymbol{x}_a + \mathcal{O}(\zeta^2) \ , \end{aligned} \tag{7.23}$$

where we have expanded the rotation matrix $R(\zeta, \hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} \times \boldsymbol{x}_a = \hat{\boldsymbol{n}} \cdot \sum_a \boldsymbol{x}_a \times \boldsymbol{p}_a = \hat{\boldsymbol{n}} \cdot \boldsymbol{L} \ , \tag{7.24}$$

where $\boldsymbol{L}$ is the *total angular momentum* of the system.

7.1.3 Invariance of L *vs.* Invariance of S

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action S[1]. Suppose S is invariant under

$$\begin{aligned} t &\rightarrow \tilde{t}(q, t, \zeta) \\ q_\sigma(t) &\rightarrow \tilde{q}_\sigma(q, t, \zeta) \ . \end{aligned} \tag{7.25}$$

Then invariance of S means

$$S = \int\limits_{t_a}^{t_b} dt\, L(q, \dot{q}, t) = \int\limits_{\tilde{t}_a}^{\tilde{t}_b} dt\, L(\tilde{q}, \dot{\tilde{q}}, t) \ . \tag{7.26}$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde{t}$. The endpoints of the integral, however, do change under the transformation. Now consider an infinitesimal transformation, for which $\delta t = \tilde{t} - t$ and $\delta q = \tilde{q}(\tilde{t}) - q(t)$ are both small. Thus,

$$S = \int\limits_{t_a}^{t_b} dt\, L(q, \dot{q}, t) = \int\limits_{t_a + \delta t_a}^{t_b + \delta t_b} dt \left\{ L(q, \dot{q}, t) + \frac{\partial L}{\partial q_\sigma}\, \bar{\delta} q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}\, \bar{\delta} \dot{q}_\sigma + \ldots \right\} \ , \tag{7.27}$$

where

$$\begin{aligned} \bar{\delta} q_\sigma(t) &\equiv \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \tilde{q}_\sigma(\tilde{t}) - \tilde{q}_\sigma(\tilde{t}) + \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \delta q_\sigma - \dot{q}_\sigma\, \delta t + \mathcal{O}(\delta q\, \delta t) \end{aligned} \tag{7.28}$$

[1] Indeed, we should be demanding that S only change by a function of the endpoint values.

Subtracting eqn. 7.27 from eqn. 7.26, we obtain

$$\begin{aligned}0 &= L_b\,\delta t_b - L_a\,\delta t_a + \frac{\partial L}{\partial \dot q_\sigma}\bigg|_b \bar\delta q_{\sigma,b} - \frac{\partial L}{\partial \dot q_\sigma}\bigg|_a \bar\delta q_{\sigma,a} + \int\limits_{t_a+\delta t_a}^{t_b+\delta t_b} \!dt\,\bigg\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot q_\sigma}\bigg)\bigg\}\,\bar\delta q_\sigma(t) \\ &= \int\limits_{t_a}^{t_b}\! dt\,\frac{d}{dt}\bigg\{\bigg(L - \frac{\partial L}{\partial \dot q_\sigma}\,\dot q_\sigma\bigg)\,\delta t + \frac{\partial L}{\partial \dot q_\sigma}\,\delta q_\sigma\bigg\}\ ,\end{aligned} \tag{7.29}$$

where $L_{a,b}$ is $L(q,\dot q,t)$ evaluated at $t = t_{a,b}$. Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal, and

$$\begin{aligned}\delta t &= A(q,t)\,\delta\zeta \\ \delta q_\sigma &= B_\sigma(q,t)\,\delta\zeta\ ,\end{aligned} \tag{7.30}$$

then the conserved charge is

$$\begin{aligned}\Lambda &= \bigg(L - \frac{\partial L}{\partial \dot q_\sigma}\,\dot q_\sigma\bigg)\,A(q,t) + \frac{\partial L}{\partial \dot q_\sigma}\,B_\sigma(q,t) \\ &= -\,H(q,p,t)\,A(q,t) + p_\sigma\,B_\sigma(q,t)\ .\end{aligned} \tag{7.31}$$

Thus, when $A = 0$, we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance: $t \to t + \zeta$, for which $A = 1$ and $B_\sigma = 0$. Here we have written

$$H = p_\sigma\,\dot q_\sigma - L\ , \tag{7.32}$$

and expressed it in terms of the momenta p_σ, the coordinates q_σ, and time t. H is called the *Hamiltonian*.

7.2 The Hamiltonian

The Lagrangian is a function of generalized coordinates, velocities, and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma}\ . \tag{7.33}$$

The Hamiltonian is a function of coordinates, *momenta*, and time. It is defined as the Legendre transform of L:

$$H(q,p,t) = \sum_\sigma p_\sigma\,\dot q_\sigma - L\ . \tag{7.34}$$

Let's examine the differential of H:

$$\begin{aligned}dH &= \sum_\sigma \bigg(\dot q_\sigma\,dp_\sigma + p_\sigma\,d\dot q_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma - \frac{\partial L}{\partial \dot q_\sigma}\,d\dot q_\sigma\bigg) - \frac{\partial L}{\partial t}\,dt \\ &= \sum_\sigma \bigg(\dot q_\sigma\,dp_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma\bigg) - \frac{\partial L}{\partial t}\,dt\ ,\end{aligned} \tag{7.35}$$

where we have invoked the definition of p_σ to cancel the coefficients of $d\dot{q}_\sigma$. Since $\dot{p}_\sigma = \partial L/\partial q_\sigma$, we have *Hamilton's equations of motion*,

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad , \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \ . \tag{7.36}$$

Thus, we can write

$$dH = \sum_\sigma \Big(\dot{q}_\sigma \, dp_\sigma - \dot{p}_\sigma \, dq_\sigma \Big) - \frac{\partial L}{\partial t} \, dt \ . \tag{7.37}$$

Dividing by dt, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{7.38}$$

which says that the Hamiltonian is *conserved* (*i.e.* it does not change with time) whenever there is no *explicit* time dependence to L.

Example #1 : For a simple $d = 1$ system with $L = \frac{1}{2} m\dot{x}^2 - U(x)$, we have $p = m\dot{x}$ and

$$H = p\,\dot{x} - L = \tfrac{1}{2} m\dot{x}^2 + U(x) = \frac{p^2}{2m} + U(x) \ . \tag{7.39}$$

Example #2 : Consider now the mass point – wedge system analyzed above, with

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2} m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha \ , \tag{7.40}$$

The canonical momenta are

$$P = \frac{\partial L}{\partial \dot{X}} = (M+m)\,\dot{X} + m\dot{x} \tag{7.41}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{X} + m\,(1+\tan^2\!\alpha)\,\dot{x} \ . \tag{7.42}$$

The Hamiltonian is given by

$$\begin{aligned} H &= P\,\dot{X} + p\,\dot{x} - L \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2} m\,(1+\tan^2\!\alpha)\,\dot{x}^2 + mg\,x\,\tan\alpha \ . \end{aligned} \tag{7.43}$$

However, this is not quite H, since $H = H(X,x,P,p,t)$ must be expressed in terms of the coordinates and the *momenta* and not the coordinates and velocities. So we must eliminate $\dot{X}$ and $\dot{x}$ in favor of P and p. We do this by inverting the relations

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} M+m & m \\ m & m\,(1+\tan^2\!\alpha) \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} \tag{7.44}$$

to obtain

$$\begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} = \frac{1}{m\,\big(M + (M+m)\tan^2\!\alpha\big)} \begin{pmatrix} m\,(1+\tan^2\!\alpha) & -m \\ -m & M+m \end{pmatrix} \begin{pmatrix} P \\ p \end{pmatrix} \ . \tag{7.45}$$

Substituting into 7.43, we obtain

$$H = \frac{M+m}{2m} \, \frac{P^2\,\cos^2\!\alpha}{M + m\,\sin^2\!\alpha} - \frac{Pp\,\cos^2\!\alpha}{M+m\,\sin^2\!\alpha} + \frac{p^2}{2\,(M + m\,\sin^2\!\alpha)} + mg\,x\,\tan\alpha \ . \tag{7.46}$$

Notice that $\dot{P} = 0$ since $\frac{\partial L}{\partial X} = 0$. P is the total horizontal momentum of the system (wedge plus particle) and it is conserved.

7.2.1 Is $H = T + U$?

The most general form of the kinetic energy is

$$\begin{aligned} T &= T_2 + T_1 + T_0 \\ &= \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t)\ , \end{aligned}$$

where $T^{(n)}(q,\dot{q},t)$ is homogeneous of degree n in the velocities[2]. We assume a potential energy of the form

$$\begin{aligned} U &= U_1 + U_0 \\ &= U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + U^{(0)}(q,t)\ , \end{aligned} \tag{7.47}$$

which allows for velocity-dependent forces, as we have with charged particles moving in an electromagnetic field. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t) - U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma - U^{(0)}(q,t)\ . \tag{7.48}$$

The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T^{(2)}_{\sigma\sigma'}\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t) - U^{(1)}_\sigma(q,t) \tag{7.49}$$

which is inverted to give

$$\dot{q}_\sigma = {T^{(2)}_{\sigma\sigma'}}^{-1} \left(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'} \right) . \tag{7.50}$$

The Hamiltonian is then

$$\begin{aligned} H &= p_\sigma\,\dot{q}_\sigma - L \\ &= \tfrac{1}{2}\,{T^{(2)}_{\sigma\sigma'}}^{-1} \left(p_\sigma - T^{(1)}_\sigma + U^{(1)}_\sigma \right) \left(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'} \right) - T_0 + U_0 \\ &= T_2 - T_0 + U_0\ . \end{aligned} \tag{7.51}$$

If T_0, T_1, and U_1 vanish, *i.e.* if $T(q,\dot{q},t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q,t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, or the potential is velocity-dependent, then $H \neq T + U$.

7.2.2 Example: A bead on a rotating hoop

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity $\dot{\phi} = \omega$ about the $\hat{\boldsymbol{z}}$-axis, as shown in Fig. 7.1.

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[2] A homogeneous function of degree k satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$. It is then easy to prove *Euler's theorem*, $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = k f$.

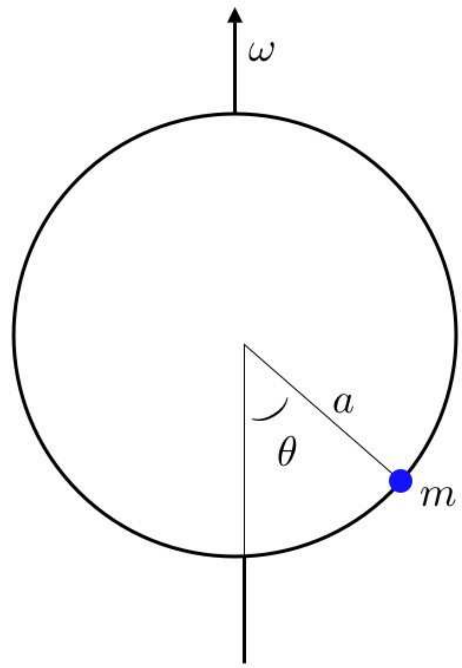

Figure 7.1: A bead of mass m on a rotating hoop of radius a.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$\begin{aligned} T &= \tfrac{1}{2}m\big(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\big) \\ &= \tfrac{1}{2}ma^2\big(\dot{\theta}^2 + \omega^2\sin^2\theta\big)\ . \end{aligned} \tag{7.52}$$

Thus, $T_2 = \frac{1}{2}ma^2\dot{\theta}^2$ and $T_0 = \frac{1}{2}ma^2\omega^2\sin^2\theta$. The potential energy is $U(\theta) = mga(1-\cos\theta)$. The momentum conjugate to θ is $p_\theta = ma^2\dot{\theta}$, and thus

$$\begin{aligned} H(\theta, p) &= T_2 - T_0 + U \\ &= \tfrac{1}{2}ma^2\dot{\theta}^2 - \tfrac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta) \\ &= \frac{p_\theta^2}{2ma^2} - \tfrac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta)\ . \end{aligned} \tag{7.53}$$

For this problem, we can define the *effective potential*

$$\begin{aligned} U_{\rm eff}(\theta) \equiv U - T_0 &= mga(1-\cos\theta) - \tfrac{1}{2}ma^2\omega^2\sin^2\theta \\ &= mga\bigg(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\bigg)\ , \end{aligned} \tag{7.54}$$

where $\omega_0^2 \equiv g/a$. The Lagrangian may then be written

$$L = \tfrac{1}{2}ma^2\dot{\theta}^2 - U_{\rm eff}(\theta)\ , \tag{7.55}$$

and thus the equations of motion are

$$ma^2\ddot{\theta} = -\frac{\partial U_{\rm eff}}{\partial\theta}\ . \tag{7.56}$$

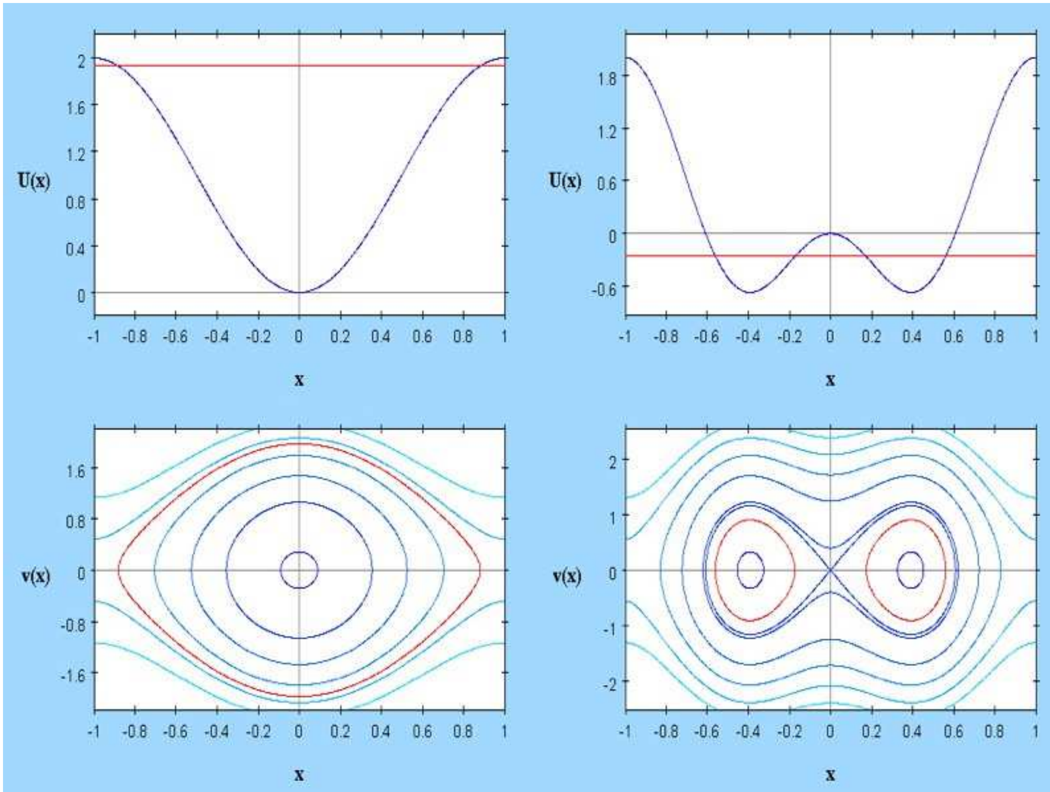

Figure 7.2: The effective potential $U_{\rm eff}(\theta) = mga\big[1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\big]$. (The dimensionless potential $\tilde U_{\rm eff}(x) = U_{\rm eff}/mga$ is shown, where $x = \theta/\pi$.) Left panels: $\omega = \frac{1}{2}\sqrt{3}\,\omega_0$. Right panels: $\omega = \sqrt{3}\,\omega_0$.

Equilibrium is achieved when $U'_{\rm eff}(\theta) = 0$, which gives

$$\frac{\partial U_{\rm eff}}{\partial\theta} = mga\sin\theta\,\bigg\{1 - \frac{\omega^2}{\omega_0^2}\cos\theta\bigg\} = 0\ , \tag{7.57}$$

i.e. $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibria is assessed by examining the sign of $U''_{\rm eff}(\theta^*)$. We have

$$U''_{\rm eff}(\theta) = mga\,\bigg\{\cos\theta - \frac{\omega^2}{\omega_0^2}\big(2\cos^2\theta - 1\big)\bigg\}\ . \tag{7.58}$$

Thus,

$$U''_{\rm eff}(\theta^*) = \begin{cases} mga\left(1 - \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = 0 \\ \\ -mga\left(1 + \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = \pi \\ \\ mga\left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & \text{at } \theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right)\ . \end{cases} \tag{7.59}$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, *i.e.* when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in Fig. 7.2.

7.2.3 Charged Particle in a Magnetic Field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r},\dot{\boldsymbol{r}}) = q\,\phi(\boldsymbol{r},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{r}}\ , \tag{7.60}$$

which *is* velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\,\dot{\boldsymbol{r}}^2$, as usual. Here $\phi(\boldsymbol{r})$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r})$ the vector potential. The electric and magnetic fields are given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}\ . \tag{7.61}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial\dot{\boldsymbol{r}}} = m\,\dot{\boldsymbol{r}} + \frac{q}{c}\,\boldsymbol{A}\ , \tag{7.62}$$

and hence the Hamiltonian is

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{p},t) &= \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L \\ &= m\dot{\boldsymbol{r}}^2 + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\,\phi \\ &= \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 + q\,\phi \\ &= \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\Big)^2 + q\,\phi(\boldsymbol{r},t)\ . \end{aligned} \tag{7.63}$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

Let's work out the equations of motion. We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial\boldsymbol{r}} \tag{7.64}$$

which gives

$$m\,\ddot{\boldsymbol{r}} + \frac{q}{c}\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{r}})\ , \tag{7.65}$$

or, in component notation,

$$m\,\ddot{x}_i + \frac{q}{c}\frac{\partial A_i}{\partial x_j}\,\dot{x}_j + \frac{q}{c}\frac{\partial A_i}{\partial t} = -q\,\frac{\partial\phi}{\partial x_i} + \frac{q}{c}\frac{\partial A_j}{\partial x_i}\,\dot{x}_j\ , \tag{7.66}$$

which is to say

$$m\,\ddot{x}_i = -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j\ . \tag{7.67}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk} \, \frac{\partial A_k}{\partial x_j} \ , \tag{7.68}$$

and using the result

$$\epsilon_{ijk} \, \epsilon_{imn} = \delta_{jm} \, \delta_{kn} - \delta_{jn} \, \delta_{km} \ , \tag{7.69}$$

we have $\epsilon_{ijk} \, B_i = \partial_j \, A_k - \partial_k \, A_j$, and

$$m \, \ddot{x}_i = -q \, \frac{\partial \phi}{\partial x_i} - \frac{q}{c} \, \frac{\partial A_i}{\partial t} + \frac{q}{c} \, \epsilon_{ijk} \, \dot{x}_j \, B_k \ , \tag{7.70}$$

or, in vector notation,

$$\begin{aligned} m \, \ddot{\boldsymbol{r}} &= -q \, \boldsymbol{\nabla} \phi - \frac{q}{c} \, \frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c} \, \dot{\boldsymbol{r}} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \\ &= q \, \boldsymbol{E} + \frac{q}{c} \, \dot{\boldsymbol{r}} \times \boldsymbol{B} \ , \end{aligned} \tag{7.71}$$

which is, of course, the Lorentz force law.

7.3 Fast Perturbations : Rapidly Oscillating Fields

Consider a free particle moving under the influence of an oscillating force,

$$m\ddot{q} = F \sin \omega t \ . \tag{7.72}$$

The motion of the system is then

$$q(t) = q_{\rm h}(t) - \frac{F \, \cos \omega t}{m\omega^2} \ , \tag{7.73}$$

where $q_{\rm h}(t) = A + Bt$ is the solution to the homogeneous (unforced) equation of motion. Note that the amplitude of the response $q - q_{\rm h}$ goes as ω^{-2} and is therefore small when ω is large.

Now consider a general $n = 1$ system, with

$$H(q,p,t) = H^0(q,p) + V(q) \cos(\omega t) \ . \tag{7.74}$$

We assume that ω is much greater than any natural oscillation frequency associated with H_0. We separate the motion $q(t)$ and $p(t)$ into slow and fast components:

$$\begin{aligned} q(t) &= Q(t) + \zeta(t) \\ p(t) &= P(t) + \pi(t) \ , \end{aligned} \tag{7.75}$$

where $\zeta(t)$ and $\pi(t)$ oscillate with the driving frequency ω. Since ζ and π will be small, we expand Hamilton's equations in these quantities:

$$\begin{aligned}\dot{Q}+\dot{\zeta}&=\frac{\partial H^0}{\partial P}+\frac{\partial^2 H^0}{\partial P^2}\,\pi+\frac{\partial^2 H^0}{\partial Q\,\partial P}\,\zeta+\frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q^2\,\partial P}\,\zeta^2+\frac{\partial^3 H^0}{\partial Q\,\partial P^2}\,\zeta\pi+\frac{1}{2}\,\frac{\partial^3 H^0}{\partial P^3}\,\pi^2+\ldots\\ \dot{P}+\dot{\pi}&=-\frac{\partial H^0}{\partial Q}-\frac{\partial^2 H^0}{\partial Q^2}\,\zeta-\frac{\partial^2 H^0}{\partial Q\,\partial P}\,\pi-\frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q^3}\,\zeta^2-\frac{\partial^3 H^0}{\partial Q^2\,\partial P}\,\zeta\pi-\frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q\,\partial P^2}\,\pi^2\\ &\qquad\qquad-\frac{\partial V}{\partial Q}\,\cos(\omega t)-\frac{\partial^2 V}{\partial Q^2}\,\zeta\cos(\omega t)-\ldots\ .\end{aligned}\tag{7.76}$$

We now average over the fast degrees of freedom to obtain an equation of motion for the slow variables Q and P, which we here carry to lowest nontrivial order in averages of fluctuating quantities:

$$\begin{aligned}\dot{Q}&=H^0_P+\tfrac{1}{2}H^0_{QQP}\,\langle\zeta^2\rangle+H^0_{QPP}\,\langle\zeta\pi\rangle+\tfrac{1}{2}H^0_{PPP}\,\langle\pi^2\rangle\\ \dot{P}&=-H^0_Q-\tfrac{1}{2}H^0_{QQQ}\,\langle\zeta^2\rangle-H^0_{QQP}\,\langle\zeta\pi\rangle-\tfrac{1}{2}H^0_{QPP}\,\langle\pi^2\rangle-V_{QQ}\,\langle\zeta\cos\omega t\rangle\ ,\end{aligned}\tag{7.77}$$

where we now adopt the shorthand notation $H^0_{QQP}=\partial^3 H^0/\partial^2 Q\,\partial P$, *etc.* The fast degrees of freedom obey

$$\begin{aligned}\dot{\zeta}&=H^0_{QP}\,\zeta+H^0_{PP}\,\pi\\ \dot{\pi}&=-H^0_{QQ}\,\zeta-H^0_{QP}\,\pi-V_Q\,\cos(\omega t)\ .\end{aligned}\tag{7.78}$$

We can solve these by replacing $V_Q\,\cos\omega t$ above with $V_Q\,e^{-i\omega t}$, and writing $\zeta(t)=\zeta_0\,e^{-i\omega t}$ and $\pi(t)=\pi_0\,e^{-i\omega t}$, resulting in

$$\begin{pmatrix}H^0_{QP}+i\omega & H^0_{PP}\\ -H^0_{QQ} & -H^0_{QP}+i\omega\end{pmatrix}\begin{pmatrix}\zeta_0\\ \pi_0\end{pmatrix}=\begin{pmatrix}0\\ V_Q\end{pmatrix}\quad .\tag{7.79}$$

We now invert the matrix to obtain ζ_0 and π_0, then take the real part, which yields

$$\begin{aligned}\zeta(t)&=\frac{H^0_{PP}\,V_Q}{\omega^2+(H^0_{QP})^2-H^0_{QQ}\,H^0_{PP}}\,\cos\omega t\\ \pi(t)&=-\frac{H^0_{QP}\,V_Q}{\omega^2+{H^0_{QP}}^2-H^0_{QQ}\,H^0_{PP}}\,\cos\omega t-\frac{\omega\,V_Q}{\omega^2+(H^0_{QP})^2-H^0_{QQ}\,H^0_{PP}}\,\sin\omega t\quad .\end{aligned}\tag{7.80}$$

Invoking $\big\langle\cos^2(\omega t)\big\rangle=\big\langle\sin^2(\omega t)\big\rangle=\frac{1}{2}$ and $\big\langle\cos(\omega t)\sin(\omega t)\big\rangle=0$, we substitute into Eqns. 7.77 to obtain

$$\dot{Q}=H^0_P+\frac{H^0_{QQP}\,(H^0_{PP})^2-2\,H^0_{QPP}\,H^0_{QP}\,H^0_{PP}+H^0_{PPP}\,(H^0_{QP})^2+\omega^2\,H^0_{PPP}}{4\left(\omega^2+(H^0_{QP})^2-H^0_{QQ}\,H^0_{PP}\right)^2}\,V_Q^2\tag{7.81}$$

and

$$\dot{P}=-H^0_Q-\frac{H^0_{QQQ}\,(H^0_{QP})^2=2\,H^0_{QQP}\,H^0_{QP}\,H^0_{PP}+H^0_{QPP}\,(H^0_{QP})^2+\omega^2\,H^0_{QPP}}{4\left(\omega^2+(H^0_{QP})^2-H^0_{QQ}\,H^0_{PP}\right)^2}\,V_Q^2\tag{7.82}$$

These equations may be written compactly as

$$\dot{Q}=\frac{\partial K}{\partial P}\quad,\quad\dot{P}=-\frac{\partial K}{\partial Q}\ ,\tag{7.83}$$

where

$$K = H^0 + \frac{\frac{1}{4}\, H^0_{PP}\, V_Q^2}{\omega^2 + (H^0_{QP})^2 - H^0_{QQ}\, H^0_{PP}} \quad . \tag{7.84}$$

We are licensed only to retain the leading order term in the denominator, hence

$$K(Q,P) = H^0(Q,P) + \frac{1}{4\omega^2}\,\frac{\partial^2 H^0}{\partial P^2}\left(\frac{\partial V}{\partial Q}\right)^2 \quad . \tag{7.85}$$

7.3.1 Example : pendulum with oscillating support

Consider a pendulum with a vertically oscillating point of support. The coordinates of the pendulum bob are

$$x = \ell\,\sin\theta \quad , \quad y = a(t) - \ell\,\cos\theta \ . \tag{7.86}$$

The Lagrangian is easily obtained:

$$\begin{aligned} L &= \tfrac{1}{2} m\ell^2\,\dot\theta^2 + m\ell\dot a\,\dot\theta\sin\theta + mg\ell\cos\theta + \tfrac{1}{2}m\dot a^2 - mga \\ &= \tfrac{1}{2} m\ell^2\,\dot\theta^2 + m(g+\ddot a)\ell\cos\theta + \overbrace{\tfrac{1}{2}m\dot a^2 - mga - \frac{d}{dt}\big(m\ell\dot a\,\cos\theta\big)}^{\text{these may be dropped}} \quad . \end{aligned} \tag{7.87}$$

Thus we may take the Lagrangian to be

$$\bar L = \tfrac{1}{2} m\ell^2\,\dot\theta^2 + m(g+\ddot a)\ell\cos\theta \ , \tag{7.88}$$

from which we derive the Hamiltonian

$$\begin{aligned} H(\theta,p_\theta,t) &= \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta - m\ell\ddot a\,\cos\theta \\ &= H_0(\theta,p_\theta,t) + V_1(\theta)\,\sin\omega t \ . \end{aligned} \tag{7.89}$$

We have assumed $a(t) = a_0 \sin\omega t$, so

$$V_1(\theta) = m\ell a_0\,\omega^2\cos\theta \ . \tag{7.90}$$

The effective Hamiltonian, per eqn. 7.85, is

$$K(\bar\theta, P_\theta) = \frac{P_\theta}{2m\ell^2} - mg\ell\cos\bar\theta + \tfrac{1}{4} m\, a_0^2\,\omega^2\sin^2\bar\theta \ . \tag{7.91}$$

Let's define the dimensionless parameter

$$\epsilon \equiv \frac{2g\ell}{\omega^2 a_0^2} \ . \tag{7.92}$$

The slow variable $\bar\theta$ executes motion in the *effective potential* $V_{\text{eff}}(\bar\theta) = mg\ell\, v(\bar\theta)$, with

$$v(\bar\theta) = -\cos\bar\theta + \frac{1}{2\epsilon}\,\sin^2\bar\theta \ . \tag{7.93}$$

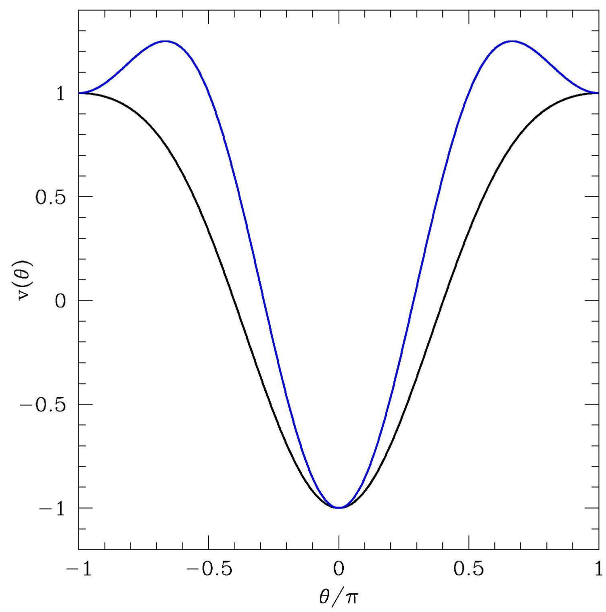

Figure 7.3: Dimensionless potential $v(\theta)$ for $\epsilon = 1.5$ (black curve) and $\epsilon = 0.5$ (blue curve).

Differentiating, and dropping the bar on θ, we find that $V_{\rm eff}(\theta)$ is stationary when

$$v'(\theta) = 0 \quad \Rightarrow \quad \sin\theta\cos\theta = -\epsilon\sin\theta \ . \tag{7.94}$$

Thus, $\theta = 0$ and $\theta = \pi$, where $\sin\theta = 0$, are equilibria. When $\epsilon < 1$ (note $\epsilon > 0$ always), there are two new solutions, given by the roots of $\cos\theta = -\epsilon$.

To assess stability of these equilibria, we compute the second derivative:

$$v''(\theta) = \cos\theta + \frac{1}{\epsilon}\cos 2\theta \ . \tag{7.95}$$

From this, we see that $\theta = 0$ is stable (*i.e.* $v''(\theta = 0) > 0$) always, but $\theta = \pi$ is stable for $\epsilon < 1$ and unstable for $\epsilon > 1$. When $\epsilon < 1$, two new solutions appear, at $\cos\theta = -\epsilon$, for which

$$v''(\cos^{-1}(-\epsilon)) = \epsilon - \frac{1}{\epsilon} \ , \tag{7.96}$$

which is always negative since $\epsilon < 1$ in order for these equilibria to exist. The situation is sketched in fig. 7.3, showing $v(\theta)$ for two representative values of the parameter ϵ. For $\epsilon > 1$, the equilibrium at $\theta = \pi$ is unstable, but as ϵ decreases, a subcritical pitchfork bifurcation is encountered at $\epsilon = 1$, and $\theta = \pi$ becomes stable, while the outlying $\theta = \cos^{-1}(-\epsilon)$ solutions are unstable.

7.4 Field Theory: Systems with Several Independent Variables

Suppose $\phi_a(\boldsymbol{x})$ depends on several independent variables: $\{x^1, x^2, \ldots, x^n\}$. Furthermore, suppose

$$S\big[\{\phi_a(\boldsymbol{x})\}\big] = \int\limits_{\Omega} d\boldsymbol{x}\, \mathcal{L}(\phi_a\, \partial_\mu \phi_a, \boldsymbol{x}) \ , \tag{7.97}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^K. Then the first variation of S is

$$\begin{aligned}\delta S &= \int\limits_\Omega \! d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} \! d\Sigma\, n^\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a + \int\limits_\Omega \! d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right\}\delta\phi_a \ ,\end{aligned} \tag{7.98}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ or $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) . \tag{7.99}$$

As an example, consider the case of a stretched string of linear mass density μ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 , \tag{7.100}$$

whence the Euler-Lagrange equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= -\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) \\ &= \tau\,\frac{\partial^2 y}{\partial x^2} - \mu\,\frac{\partial^2 y}{\partial t^2} \ ,\end{aligned} \tag{7.101}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Thus, $\mu\ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a,t) = \delta y(x_b,t) = \delta y(x,t_a) = \delta y(x,t_b) = 0$.

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \tfrac{1}{c}\,j_\mu\,A^\mu \ . \tag{7.102}$$

The equations of motion are then

$$\frac{\partial\mathcal{L}}{\partial A^\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad\Rightarrow\quad \partial_\mu\,F^{\mu\nu} = \frac{4\pi}{c}\,j^\nu \ , \tag{7.103}$$

which are Maxwell's equations.

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_\sigma}\,\frac{\partial\tilde{q}_\sigma}{\partial\zeta}\right)_{\zeta=0} = 0 \ , \tag{7.104}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t) \ , \tag{7.105}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $d\Lambda/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{7.106}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu \, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} \ . \tag{7.107}$$

We call $\Lambda = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu \, J^\mu$ over the spatial region Ω gives

$$\frac{d\Lambda}{dt} = \int\limits_\Omega d^3\!x \, \partial_0 \, J^0 = -\int\limits_\Omega d^3\!x \, \boldsymbol{\nabla} \cdot \boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma \, \hat{\boldsymbol{n}} \cdot \boldsymbol{J} = 0 \ , \tag{7.108}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[3]

$$\mathcal{L}(\psi,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2} K \, (\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \ . \tag{7.109}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i \, e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \ , \tag{7.110}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} \cdot (i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)} \cdot (-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \ . \end{aligned} \tag{7.111}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

[3]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

7.4.1 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\left(|\psi|^2 - n_0\right)^2 \ . \tag{7.112}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x}, t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned}
\delta S[\psi^*, \psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\
&= \int\!dt \int\!d^dx \, \bigg\{ i\hbar\,\psi^* \frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^* \cdot \boldsymbol{\nabla}\psi \\
&\qquad\qquad - 2g\left(|\psi|^2 - n_0\right)\left(\psi^*\delta\psi + \psi\delta\psi^*\right) \bigg\} \\
&= \int\!dt \int\!d^dx \, \bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\left(|\psi|^2 - n_0\right)\psi^* \bigg]\,\delta\psi \\
&\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\left(|\psi|^2 - n_0\right)\psi \bigg]\,\delta\psi^* \bigg\} \ ,
\end{aligned}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*, \psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\left(|\psi|^2 - n_0\right)\psi \tag{7.113}$$

as well as its complex conjugate,

$$-i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\left(|\psi|^2 - n_0\right)\psi^* \ . \tag{7.114}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\begin{aligned}
\frac{\delta S}{\delta\psi} &= \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right) \\
\frac{\delta S}{\delta\psi^*} &= \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right) \ ,
\end{aligned} \tag{7.115}$$

with $x^\mu = (t, \boldsymbol{x})$[4] Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi} = -2g\left(|\psi|^2 - n_0\right)\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{7.116}$$

and

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\,\psi - 2g\left(|\psi|^2 - n_0\right)\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi \ , \tag{7.117}$$

[4] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) \ . \tag{7.118}$$

Thus, the conserved Noether current is then

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\,\frac{\partial\tilde{\psi}}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\,\frac{\partial\tilde{\psi}^*}{\partial\zeta}\bigg|_{\zeta=0} \\ J^0 &= -\hbar\,|\psi|^2 \\ \boldsymbol{J} &= -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \ . \end{aligned} \tag{7.119}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \ , \tag{7.120}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \ . \tag{7.121}$$

are the particle density and the particle current, respectively.

7.5 Hamiltonian Mechanics

Recall that $L = L(q,\dot{q},t)$, and

$$p_\sigma = \frac{\partial L}{\partial\dot{q}_\sigma} \ . \tag{7.122}$$

The Hamiltonian, $H(q,p)$ is obtained by a Legendre transformation,

$$H(q,p) = \sum_{\sigma=1}^{n} p_\sigma\,\dot{q}_\sigma - L \ . \tag{7.123}$$

Note that

$$\begin{aligned} dH &= \sum_{\sigma=1}^{n}\left(p_\sigma\,d\dot{q}_\sigma + \dot{q}_\sigma\,dp_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma - \frac{\partial L}{\partial\dot{q}_\sigma}\,d\dot{q}_\sigma\right) - \frac{\partial L}{\partial t}\,dt \\ &= \sum_{\sigma=1}^{n}\left(\dot{q}_\sigma\,dp_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma\right) - \frac{\partial L}{\partial t}\,dt \ . \end{aligned} \tag{7.124}$$

Thus, we obtain Hamilton's equations of motion,

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma \quad , \quad \frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -\dot{p}_\sigma \tag{7.125}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \ . \tag{7.126}$$

Some remarks:

- As an example, consider a particle moving in three dimensions, described by spherical polar coordinates (r, θ, ϕ). Then

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\right) - U(r,\theta,\phi) \ . \tag{7.127}$$

  We have

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\,\dot{\theta} \quad , \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\,\sin^2\!\theta\,\dot{\phi} \quad , \tag{7.128}$$

  and thus

$$\begin{aligned} H &= p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\phi\,\dot{\phi} - L \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\!\theta} + U(r,\theta,\phi) \ . \end{aligned} \tag{7.129}$$

  Note that H is time-independent, hence $\frac{\partial H}{\partial t} = \frac{dH}{dt} = 0$, and therefore H is a constant of the motion.

- In order to obtain $H(q,p)$ we must invert the relation $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = p_\sigma(q,\dot{q})$ to obtain $\dot{q}_\sigma(q,p)$. This is possible if the Hessian,

$$\frac{\partial p_\alpha}{\partial \dot{q}_\beta} = \frac{\partial^2 L}{\partial \dot{q}_\alpha\,\partial \dot{q}_\beta} \tag{7.130}$$

  is nonsingular. This is the content of the 'inverse function theorem' of multivariable calculus.

- Define the rank $2n$ vector, ξ, by its components,

$$\xi_i = \begin{cases} q_i & \text{if } 1 \le i \le n \\ p_{i-n} & \text{if } n < i \le 2n \ . \end{cases} \tag{7.131}$$

  Then we may write Hamilton's equations compactly as

$$\dot{\xi}_i = \mathbb{J}_{ij}\,\frac{\partial H}{\partial \xi_j} \ , \tag{7.132}$$

  where

$$\mathbb{J} = \begin{pmatrix} \mathbb{O}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix} \tag{7.133}$$

  is a rank $2n$ matrix. Note that $\mathbb{J}^{\rm t} = -\mathbb{J}$, *i.e.* $\mathbb{J}$ is antisymmetric, and that $\mathbb{J}^2 = -\mathbb{I}_{2n\times 2n}$. We shall utilize this 'symplectic structure' to Hamilton's equations shortly.

7.5.1 Modified Hamilton's principle

We have that

$$\begin{aligned} 0 = \delta \int\limits_{t_a}^{t_b} \! dt \, L &= \delta \int\limits_{t_a}^{t_b} \! dt \, \big(p_\sigma \, \dot q_\sigma - H \big) \\ &= \int\limits_{t_a}^{t_b} \! dt \, \bigg\{ p_\sigma \, \delta \dot q_\sigma + \dot q_\sigma \, \delta p_\sigma - \frac{\partial H}{\partial q_\sigma} \, \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \, \delta p_\sigma \bigg\} \\ &= \int\limits_{t_a}^{t_b} \! dt \, \Bigg\{ - \bigg(\dot p_\sigma + \frac{\partial H}{\partial q_\sigma} \bigg) \, \delta q_\sigma + \bigg(\dot q_\sigma - \frac{\partial H}{\partial p_\sigma} \bigg) \, \delta p_\sigma \Bigg\} + \big(p_\sigma \, \delta q_\sigma \big) \Big|_{t_a}^{t_b} \ , \end{aligned} \tag{7.134}$$

assuming $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. Setting the coefficients of δq_σ and δp_σ to zero, we recover Hamilton's equations.

7.5.2 Phase flow is incompressible

A flow for which $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ is *incompressible* – we shall see why in a moment. Let's check that the divergence of the phase space velocity does indeed vanish:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \dot{\boldsymbol{\xi}} &= \sum_{\sigma=1}^n \bigg\{ \frac{\partial \dot q_\sigma}{\partial q_\sigma} + \frac{\partial \dot p_\sigma}{\partial p_\sigma} \bigg\} \\ &= \sum_{i=1}^{2n} \frac{\partial \dot \xi_i}{\partial \xi_i} = \sum_{i,j} \mathbb{J}_{ij} \, \frac{\partial^2 H}{\partial \xi_i \, \partial \xi_j} = 0 \ . \end{aligned} \tag{7.135}$$

Now let $\rho(\boldsymbol{\xi}, t)$ be a distribution on phase space. Continuity implies

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \, \dot{\boldsymbol{\xi}}) = 0 \ . \tag{7.136}$$

Invoking $\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\xi}} = 0$, we have that

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla} \rho = 0 \ , \tag{7.137}$$

where $D\rho/Dt$ is sometimes called the *convective derivative* – it is the total derivative of the function $\rho\big(\boldsymbol{\xi}(t), t\big)$, evaluated at a point $\xi(t)$ in phase space which moves according to the dynamics. This says that the density in the "comoving frame" is locally constant.

7.5.3 Poincaré recurrence theorem

Let g_τ be the 'τ-advance mapping' which evolves points in phase space according to Hamilton's equations

$$\dot q_\sigma = + \frac{\partial H}{\partial p_\sigma} \qquad , \qquad \dot p_\sigma = - \frac{\partial H}{\partial q_\sigma} \tag{7.138}$$

for a time interval $\Delta t = \tau$. Consider a region Ω in phase space. Define $g_\tau^n \Omega$ to be the n^{th} image of Ω under the mapping g_τ. Clearly g_τ is invertible; the inverse is obtained by integrating the equations of motion backward in time. We denote the inverse of g_τ by g_τ^{-1}. By Liouville's theorem, g_τ is volume preserving when acting on regions in phase space, since the evolution of any given point is Hamiltonian. This follows from the continuity equation for the phase space density,

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\boldsymbol{u} \varrho) = 0 \tag{7.139}$$

where $\boldsymbol{u} = \{\dot{\boldsymbol{q}}, \dot{\boldsymbol{p}}\}$ is the velocity vector in phase space, and Hamilton's equations, which say that the phase flow is incompressible, *i.e.* $\nabla \cdot \boldsymbol{u} = 0$:

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= \sum_{\sigma=1}^{n} \left\{ \frac{\partial \dot{q}_\sigma}{\partial q_\sigma} + \frac{\partial \dot{p}_\sigma}{\partial p_\sigma} \right\} \\ &= \sum_{\sigma=1}^{n} \left\{ \frac{\partial}{\partial q_\sigma} \left(\frac{\partial H}{\partial p_\sigma} \right) + \frac{\partial}{\partial p_\sigma} \left(-\frac{\partial H}{\partial q_\sigma} \right) \right\} = 0 \ . \end{aligned} \tag{7.140}$$

Thus, we have that the convective derivative vanishes, *viz.*

$$\frac{D\varrho}{Dt} \equiv \frac{\partial \varrho}{\partial t} + \boldsymbol{u} \cdot \nabla \varrho = 0 \ , \tag{7.141}$$

which guarantees that the density remains constant in a frame moving with the flow.

The proof of the recurrence theorem is simple. Assume that g_τ is invertible and volume-preserving, as is the case for Hamiltonian flow. Further assume that phase space volume is finite. Since the energy is preserved in the case of time-independent Hamiltonians, we simply ask that the volume of phase space *at fixed total energy* E be finite, *i.e.*

$$\int \! d\mu \, \delta\big(E - H(\boldsymbol{q}, \boldsymbol{p})\big) < \infty \ , \tag{7.142}$$

where $d\mu = \prod_i dq_i \, dp_i$ is the phase space uniform integration measure.

Theorem: In any finite neighborhood Ω of phase space there exists a point $\boldsymbol{\varphi}_0$ which will return to Ω after n applications of g_τ, where n is finite.

Proof: Assume the theorem fails; we will show this assumption results in a contradiction. Consider the set Υ formed from the union of all sets $g_\tau^m \, \Omega$ for all m:

$$\Upsilon = \bigcup_{m=0}^{\infty} g_\tau^m \, \Omega \tag{7.143}$$

We assume that the set $\{g_\tau^m \, \Omega \,|\, m \in \mathbb{Z} \ , m \geq 0\}$ is disjoint. The volume of a union of disjoint sets is the sum of the individual volumes. Thus,

$$\mathsf{vol}(\Upsilon) = \sum_{m=0}^{\infty} \mathsf{vol}(g_\tau^m \, \Omega) = \mathsf{vol}(\Omega) \cdot \sum_{m=1}^{\infty} 1 = \infty \ , \tag{7.144}$$

since $\mathsf{vol}(g_\tau^m\,\Omega) = \mathsf{vol}(\Omega)$ from volume preservation. But clearly Υ is a subset of the entire phase space, hence we have a contradiction, because by assumption phase space is of finite volume.

Thus, the assumption that the set $\{g_\tau^m\,\Omega\,|\,m\in\mathbb{Z}\,,m\geq 0\}$ is disjoint fails. This means that there exists some pair of integers k and l, with $k\neq l$, such that $g_\tau^k\,\Omega\cap g_\tau^l\,\Omega\neq\emptyset$. Without loss of generality we may assume $k>l$. Apply the inverse g_τ^{-1} to this relation l times to get $g_\tau^{k-l}\,\Omega\cap\Omega\neq\emptyset$. Now choose any point $\boldsymbol{\varphi}\in g_\tau^n\,\Omega\cap\Omega$, where $n=k-l$, and define $\boldsymbol{\varphi}_0 = g_\tau^{-n}\boldsymbol{\varphi}$. Then by construction both $\boldsymbol{\varphi}_0$ and $g_\tau^n\,\boldsymbol{\varphi}_0$ lie within Ω and the theorem is proven.

Each of the two central assumptions – invertibility and volume preservation – is crucial. Without either of them, the proof fails. Consider, for example, a volume-preserving map which is not invertible. An example might be a mapping $f\colon\mathbb{R}\to\mathbb{R}$ which takes any real number to its fractional part. Thus, $f(\pi) = 0.14159265\ldots$. Let us restrict our attention to intervals of width less than unity. Clearly f is then volume preserving. The action of f on the interval $[2,3)$ is to map it to the interval $[0,1)$. But $[0,1)$ remains fixed under the action of f, so no point within the interval $[2,3)$ will ever return under repeated iterations of f. Thus, f does not exhibit Poincaré recurrence.

Consider next the case of the damped harmonic oscillator. In this case, phase space volumes contract. For a one-dimensional oscillator obeying $\ddot{x}+2\beta\dot{x}+\Omega_0^2 x=0$ one has $\nabla\!\cdot\!\boldsymbol{u} = -2\beta < 0$ ($\beta>0$ for damping). Thus the convective derivative is equal to $D_t\varrho = -(\nabla\cdot\boldsymbol{u})\varrho = +2\beta\varrho$ which says that the density increases exponentially in the comoving frame, as $\varrho(t) = e^{2\beta t}\,\varrho(0)$. Thus, phase space volumes collapse, and are not preserved by the dynamics. In this case, it is possible for the set Υ to be of finite volume, even if it is the union of an infinite number of sets $g_\tau^n\,\Omega$, because the volumes of these component sets themselves decrease exponentially, as $\mathsf{vol}(g_\tau^n\,\Omega) = e^{-2n\beta\tau}\,\mathsf{vol}(\Omega)$. A damped pendulum, released from rest at some small angle θ_0, will not return arbitrarily close to these initial conditions.

7.5.4 Poisson brackets

The time evolution of any function $F(\boldsymbol{q},\boldsymbol{p})$ over phase space is given by

$$\begin{aligned}\frac{d}{dt}\,F\big(\boldsymbol{q}(t),\boldsymbol{p}(t),t\big) &= \frac{\partial F}{\partial t} + \sum_{\sigma=1}^n \left\{\frac{\partial F}{\partial q_\sigma}\,\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\,\dot{p}_\sigma\right\} \\ &\equiv \frac{\partial F}{\partial t} + \big\{F,H\big\}\ ,\end{aligned} \tag{7.145}$$

where the *Poisson bracket* $\{\cdot\,,\cdot\}$ is given by

$$\begin{aligned}\big\{A,B\big\} &\equiv \sum_{\sigma=1}^n \left(\frac{\partial A}{\partial q_\sigma}\,\frac{\partial B}{\partial p_\sigma} - \frac{\partial A}{\partial p_\sigma}\,\frac{\partial B}{\partial q_\sigma}\right) \\ &= \sum_{i,j=1}^{2n} \mathbb{J}_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j}\ .\end{aligned} \tag{7.146}$$

Properties of the Poisson bracket:

- Antisymmetry:

$$\big\{f,g\big\} = -\big\{g,f\big\}\ . \tag{7.147}$$

- Bilinearity: if λ is a constant, and f, g, and h are functions on phase space, then

$$\big\{f + \lambda\, g, h\big\} = \big\{f, h\big\} + \lambda\{g, h\big\} \ . \tag{7.148}$$

  Linearity in the second argument follows from this and the antisymmetry condition.

- Associativity:

$$\big\{fg, h\big\} = f\big\{g, h\big\} + g\big\{f, h\big\} \ . \tag{7.149}$$

- Jacobi identity:

$$\big\{f, \{g, h\}\big\} + \big\{g, \{h, f\}\big\} + \big\{h, \{f, g\}\big\} = 0 \ . \tag{7.150}$$

Some other useful properties:

- If $\{A, H\} = 0$ and $\frac{\partial A}{\partial t} = 0$, then $\frac{dA}{dt} = 0$, *i.e.* $A(q, p)$ is a constant of the motion.
- If $\{A, H\} = 0$ and $\{B, H\} = 0$, then $\big\{\{A, B\}, H\big\} = 0$. If in addition A and B have no explicit time dependence, we conclude that $\{A, B\}$ is a constant of the motion.
- It is easily established that

$$\{q_\alpha, q_\beta\} = 0 \quad , \quad \{p_\alpha, p_\beta\} = 0 \quad , \quad \{q_\alpha, p_\beta\} = \delta_{\alpha\beta} \quad . \tag{7.151}$$

7.6 Canonical Transformations

7.6.1 Point transformations in Lagrangian mechanics

In Lagrangian mechanics, we are free to redefine our generalized coordinates, *viz.*

$$Q_\sigma = Q_\sigma(q_1, \ldots, q_n, t) \ . \tag{7.152}$$

This is called a "point transformation." The transformation is invertible if

$$\mathsf{det}\left(\frac{\partial Q_\alpha}{\partial q_\beta}\right) \neq 0 \ . \tag{7.153}$$

The transformed Lagrangian, $\tilde{L}$, written as a function of the new coordinates $\boldsymbol{Q}$ and velocities $\dot{\boldsymbol{Q}}$, is

$$\tilde{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = L\big(\boldsymbol{q}(\boldsymbol{Q}, t), \dot{q}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t), t\big) + \frac{d}{dt}\, F\big(\boldsymbol{q}(\boldsymbol{Q}, t), t\big) \ , \tag{7.154}$$

where $F(\boldsymbol{q}, t)$ is a function only of the coordinates $q_\sigma(\boldsymbol{Q}, t)$ and time[5]. Finally, Hamilton's principle,

$$\delta \int\limits_{t_1}^{t_b} dt\, \tilde{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = 0 \tag{7.155}$$

[5]We must have that the relation $Q_\sigma = Q_\sigma(\boldsymbol{q}, t)$ is invertible.

with $\delta Q_\sigma(t_a) = \delta Q_\sigma(t_b) = 0$, still holds, and the form of the Euler-Lagrange equations remains unchanged:

$$\frac{\partial \tilde{L}}{\partial Q_\sigma} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = 0 \ . \tag{7.156}$$

The invariance of the equations of motion under a point transformation may be verified explicitly. We first evaluate

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma}\right) \quad , \tag{7.157}$$

where the relation $\partial\dot{q}_\alpha/\partial\dot{Q}_\sigma = \partial q_\alpha/\partial Q_\sigma$ follows from $\dot{q}_\alpha = \frac{\partial q_\alpha}{\partial Q_\sigma}\,\dot{Q}_\sigma + \frac{\partial q_\alpha}{\partial t}$. We know that adding a total time derivative of a function $\tilde{F}(\boldsymbol{Q},t) = F\big(\boldsymbol{q}(\boldsymbol{Q},t),t\big)$ to the Lagrangian does not alter the equations of motion. Hence we can set $F = 0$ and compute

$$\begin{aligned}
\frac{\partial \tilde{L}}{\partial Q_\sigma} &= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial \dot{q}_\alpha}{\partial Q_\sigma} \\
&= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial Q_{\sigma'}}\,\dot{Q}_{\sigma'} + \frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial t}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{d}{dt}\left(\frac{\partial q_\alpha}{\partial Q_\sigma}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial q_\alpha}{\partial Q_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) ,
\end{aligned} \tag{7.158}$$

where the last equality is what we obtained earlier in eqn. 7.157.

7.6.2 Canonical transformations in Hamiltonian mechanics

In Hamiltonian mechanics, we will deal with a much broader class of transformations – ones which mix all the q's and p's. The general form for a canonical transformation (CT) is

$$\begin{aligned}
q_\sigma &= q_\sigma\big(Q_1,\ldots,Q_n\,;P_1,\ldots,P_n;t\big) \\
p_\sigma &= p_\sigma\big(Q_1,\ldots,Q_n\,;P_1,\ldots,P_n;t\big) \ ,
\end{aligned} \tag{7.159}$$

with $\sigma \in \{1,\ldots,n\}$. We may also write

$$\xi_i = \xi_i\big(\Xi_1,\ldots,\Xi_{2n};t\big) \ , \tag{7.160}$$

with $i \in \{1,\ldots,2n\}$. The transformed Hamiltonian is $\tilde{H}(\boldsymbol{Q},\boldsymbol{P},t)$., where, as we shall see below, $\tilde{H}(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + \frac{\partial}{\partial t}F(\boldsymbol{q},\boldsymbol{Q},t)$.

What sorts of transformations are allowed? Well, if Hamilton's equations are to remain invariant, then

$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \quad , \quad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} \ , \tag{7.161}$$

which gives

$$\frac{\partial \dot{Q}_\sigma}{\partial Q_\sigma} + \frac{\partial \dot{P}_\sigma}{\partial P_\sigma} = 0 = \frac{\partial \dot{\Xi}_i}{\partial \Xi_i} \ . \tag{7.162}$$

I.e. the flow remains incompressible in the new (Q, P) variables. We will also require that phase space volumes are preserved by the transformation, *i.e.*

$$\mathsf{det}\left(\frac{\partial \Xi_i}{\partial \xi_j}\right) = \left|\left|\frac{\partial(\boldsymbol{Q},\boldsymbol{P})}{\partial(\boldsymbol{q},\boldsymbol{p})}\right|\right| = 1 \ . \tag{7.163}$$

Additional conditions will be discussed below.

7.6.3 Hamiltonian evolution

Hamiltonian evolution itself defines a canonical transformation. Let $\xi_i = \xi_i(t)$ and let $\xi'_i = \xi_i(t + dt)$. Then from the dynamics $\dot{\xi}_i = \mathbb{J}_{ij}\, \partial H/\partial \xi_j$, we have

$$\xi_i(t+dt) = \xi_i(t) + \mathbb{J}_{ij} \frac{\partial H}{\partial \xi_j}\, dt + \mathcal{O}\big(dt^2\big) \ . \tag{7.164}$$

Thus,

$$\begin{aligned}\frac{\partial \xi'_i}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\left(\xi_i + \mathbb{J}_{ik} \frac{\partial H}{\partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big)\right) \\ &= \delta_{ij} + \mathbb{J}_{ik} \frac{\partial^2 H}{\partial \xi_j\, \partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big) \ .\end{aligned} \tag{7.165}$$

Now, using the result $\mathsf{det}\big(1 + \epsilon M\big) = 1 + \epsilon\ \mathsf{Tr}\ M + \mathcal{O}(\epsilon^2)$, we have

$$\left|\left|\frac{\partial \xi'_i}{\partial \xi_j}\right|\right| = 1 + \mathbb{J}_{jk} \frac{\partial^2 H}{\partial \xi_j\, \partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big) = 1 + \mathcal{O}\big(dt^2\big) \ . \tag{7.166}$$

7.6.4 Symplectic structure

We have that

$$\dot{\xi}_i = \mathbb{J}_{ij} \frac{\partial H}{\partial \xi_j} \ . \tag{7.167}$$

Suppose we make a time-independent canonical transformation to new phase space coordinates, $\Xi_a = \Xi_a(\xi)$. We then have

$$\dot{\Xi}_a = \frac{\partial \Xi_a}{\partial \xi_j}\, \dot{\xi}_j = \frac{\partial \Xi_a}{\partial \xi_j}\, \mathbb{J}_{jk} \frac{\partial H}{\partial \xi_k} \ . \tag{7.168}$$

But if the transformation is canonical, then the equations of motion are preserved, and we also have

$$\dot{\Xi}_a = \mathbb{J}_{ab} \frac{\partial \tilde{H}}{\partial \Xi_b} = \mathbb{J}_{ab} \frac{\partial H}{\partial \xi_k} \frac{\partial \xi_k}{\partial \Xi_b} \ . \tag{7.169}$$

Equating these two expressions, we have

$$M_{aj}\,\mathbb{J}_{jk}\,\frac{\partial H}{\partial \xi_k} = \mathbb{J}_{ab}\,M^{-1}_{kb}\,\frac{\partial H}{\partial \xi_k}\ , \tag{7.170}$$

where $M_{aj} \equiv \partial \Xi_a / \partial \xi_j$ is the Jacobian of the transformation. Since the equality must hold for all ξ, we conclude

$$M\mathbb{J} = \mathbb{J}\big(M^{\rm t}\big)^{-1} \qquad \Longrightarrow \qquad M\mathbb{J}M^{\rm t} = \mathbb{J}\ . \tag{7.171}$$

A matrix M satisfying $MM^{\rm t} = \mathbb{I}$ is of course an *orthogonal* matrix. A matrix M satisfying $M\mathbb{J}M^{\rm t} = \mathbb{J}$ is called *symplectic*. We write $M \in \mathsf{Sp}(2n)$, *i.e.* M is an element of the group of *symplectic matrices*[6] of rank $2n$.

The symplectic property of M guarantees that the Poisson brackets are preserved under a canonical transformation:

$$\begin{aligned}\big\{A,B\big\}_\xi &= \mathbb{J}_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j} = \mathbb{J}_{ij}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial \Xi_a}{\partial \xi_i}\,\frac{\partial B}{\partial \Xi_b}\,\frac{\partial \Xi_b}{\partial \xi_j} \\ &= \big(M_{ai}\,\mathbb{J}_{ij}\,M^{\rm t}_{jb}\big)\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} = \mathbb{J}_{ab}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} = \big\{A,B\big\}_\Xi\ .\end{aligned} \tag{7.172}$$

7.6.5 Generating functions for canonical transformations

For a transformation to be canonical, we require

$$\delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{p_\sigma\,\dot q_\sigma - H(\boldsymbol{q},\boldsymbol{p},t)\Big\} = 0 = \delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{P_\sigma\,\dot Q_\sigma - \tilde H(\boldsymbol{Q},\boldsymbol{P},t)\Big\}\ . \tag{7.173}$$

This is satisfied provided

$$\Big\{p_\sigma\,\dot q_\sigma - H(\boldsymbol{q},\boldsymbol{p},t)\Big\} = \lambda\,\bigg\{P_\sigma\,\dot Q_\sigma - \tilde H(\boldsymbol{Q},\boldsymbol{P},t) + \frac{dF}{dt}\bigg\}\ , \tag{7.174}$$

where λ is a constant. For canonical transformations[7], $\lambda = 1$. Thus,

$$\begin{aligned}\tilde H(Q,P,t) = H(q,p,t) &+ P_\sigma\,\dot Q_\sigma - p_\sigma\,\dot q_\sigma + \frac{\partial F}{\partial q_\sigma}\,\dot q_\sigma + \frac{\partial F}{\partial Q_\sigma}\,\dot Q_\sigma \\ &+ \frac{\partial F}{\partial p_\sigma}\,\dot p_\sigma + \frac{\partial F}{\partial P_\sigma}\,\dot P_\sigma + \frac{\partial F}{\partial t}\ .\end{aligned} \tag{7.175}$$

Thus, we require

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial p_\sigma} = 0 \quad , \quad \frac{\partial F}{\partial P_\sigma} = 0\ , \tag{7.176}$$

[6] Note that the rank of a symplectic matrix is always even. Note also $M\mathbb{J}M^{\rm t} = \mathbb{J}$ implies $M^{\rm t}\mathbb{J}M = \mathbb{J}$.

[7] Solutions of eqn. 7.174 with $\lambda \neq 1$ are known as *extended* canonical transformations. We can always rescale coordinates and/or momenta to achieve $\lambda = 1$.

which says that $F = F(\boldsymbol{q}, \boldsymbol{Q}, t)$ is only a function of $(\boldsymbol{q}, \boldsymbol{Q}, t)$ and not a function of the momentum variables $\boldsymbol{p}$ and $\boldsymbol{P}$. The transformed Hamiltonian is then

$$\tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial F(\boldsymbol{q}, \boldsymbol{Q}, t)}{\partial t} \ . \tag{7.177}$$

There are four possibilities, corresponding to the freedom to make Legendre transformations with respect to the coordinate arguments of $F(q, Q, t)$:

$$F(\boldsymbol{q}, \boldsymbol{Q}, t) = \begin{cases} F_1(\boldsymbol{q}, \boldsymbol{Q}, t) & ; \quad p_\sigma = +\frac{\partial F_1}{\partial q_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \quad \text{(type I)} \\ F_2(\boldsymbol{q}, \boldsymbol{P}, t) - P_\sigma \, Q_\sigma & ; \quad p_\sigma = +\frac{\partial F_2}{\partial q_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_2}{\partial P_\sigma} \quad \text{(type II)} \\ F_3(\boldsymbol{p}, \boldsymbol{Q}, t) + p_\sigma \, q_\sigma & ; \quad q_\sigma = -\frac{\partial F_3}{\partial p_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_3}{\partial Q_\sigma} \quad \text{(type III)} \\ F_4(\boldsymbol{p}, \boldsymbol{P}, t) + p_\sigma \, q_\sigma - P_\sigma \, Q_\sigma & ; \quad q_\sigma = -\frac{\partial F_4}{\partial p_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_4}{\partial P_\sigma} \quad \text{(type IV)} \end{cases}$$

In each case $(\gamma = 1, 2, 3, 4)$, we have

$$\tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial F_\gamma}{\partial t} \ . \tag{7.178}$$

Let's work out some examples:

- Consider the type-II transformation generated by

$$F_2(\boldsymbol{q}, \boldsymbol{P}) = A_\sigma(\boldsymbol{q}) \, P_\sigma \ , \tag{7.179}$$

where $A_\sigma(\boldsymbol{q})$ is an arbitrary function of the $\{q_\sigma\}$. We then have

$$Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} = A_\sigma(\boldsymbol{q}) \qquad , \qquad p_\sigma = \frac{\partial F_2}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} \, P_\alpha \ . \tag{7.180}$$

Thus,

$$Q_\sigma = A_\sigma(\boldsymbol{q}) \qquad , \qquad P_\sigma = \frac{\partial q_\alpha}{\partial Q_\sigma} \, p_\alpha \ . \tag{7.181}$$

This is a general point transformation of the kind discussed in eqn. 7.152. For a general linear point transformation, $Q_\alpha = M_{\alpha\beta} \, q_\beta$, we have $P_\alpha = p_\beta \, M^{-1}_{\beta\alpha}$, *i.e.* $\boldsymbol{Q} = M\boldsymbol{q}$, $\boldsymbol{P} = \boldsymbol{p} \, M^{-1}$. If $M_{\alpha\beta} = \delta_{\alpha\beta}$, this is the identity transformation. $F_2 = q_1 P_3 + q_3 P_1$ interchanges labels 1 and 3, *etc.*

- Consider the type-I transformation generated by

$$F_1(\boldsymbol{q}, \boldsymbol{Q}) = A_\sigma(\boldsymbol{q}) \, Q_\sigma \ . \tag{7.182}$$

We then have

$$\begin{aligned} p_\sigma &= \frac{\partial F_1}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} \, Q_\alpha \\ P_\sigma &= -\frac{\partial F_1}{\partial Q_\sigma} = -A_\sigma(\boldsymbol{q}) \ . \end{aligned} \tag{7.183}$$

Note that $A_\sigma(\boldsymbol{q}) = q_\sigma$ generates the transformation

$$\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} \longrightarrow \begin{pmatrix} -\boldsymbol{P} \\ +\boldsymbol{Q} \end{pmatrix} . \tag{7.184}$$

- A mixed transformation is also permitted. For example,

$$F(\boldsymbol{q},\boldsymbol{Q}) = q_1\, Q_1 + (q_3 - Q_2)\, P_2 + (q_2 - Q_3)\, P_3 \tag{7.185}$$

is of type-I with respect to index $\sigma = 1$ and type-II with respect to indices $\sigma = 2, 3$. The transformation effected is

$$Q_1 = p_1 \quad , \quad Q_2 = q_3 \quad , \quad Q_3 = q_2 \quad , \quad P_1 = -q_1 \quad , \quad P_2 = p_3 \quad , \quad P_3 = p_2 \ . \tag{7.186}$$

- Consider the $n = 1$ harmonic oscillator,

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 \ . \tag{7.187}$$

If we could find a time-independent canonical transformation such that

$$p = \sqrt{2mf(P)}\,\cos Q \qquad , \qquad q = \sqrt{\frac{2\,f(P)}{k}}\,\sin Q \ , \tag{7.188}$$

where $f(P)$ is some function of P, then we'd have $\tilde{H}(Q,P) = f(P)$, which is cyclic in Q. To find this transformation, we take the ratio of p and q to obtain

$$p = \sqrt{mk}\,q\ \mathrm{ctn}\,Q \ , \tag{7.189}$$

which suggests the type-I transformation

$$F_1(q,Q) = \tfrac{1}{2}\sqrt{mk}\,q^2\ \mathrm{ctn}\,Q \ . \tag{7.190}$$

This leads to

$$p = \frac{\partial F_1}{\partial q} = \sqrt{mk}\,q\ \mathrm{ctn}\,Q \quad , \quad P = -\frac{\partial F_1}{\partial Q} = \frac{\sqrt{mk}\,q^2}{2\,\sin^2\!Q} \ . \tag{7.191}$$

Thus,

$$q = \frac{\sqrt{2P}}{\sqrt[4]{mk}}\,\sin Q \quad \Longrightarrow \quad f(P) = \sqrt{\frac{k}{m}}\,P = \omega P \ , \tag{7.192}$$

where $\omega = \sqrt{k/m}$ is the oscillation frequency. We therefore have $\tilde{H}(Q,P) = \omega P$, whence $P = E/\omega$. The equations of motion are

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \qquad , \qquad \dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \omega \ , \tag{7.193}$$

which yields

$$Q(t) = \omega t + \varphi_0 \qquad , \qquad q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin\big(\omega t + \varphi_0\big) \ . \tag{7.194}$$

Chapter 8

Constraints

A mechanical system of N point particles in d dimensions possesses $n = dN$ degrees of freedom[1]. To specify these degrees of freedom, we can choose any independent set of generalized coordinates $\{q_1, \ldots, q_K\}$. Oftentimes, however, not all n coordinates are independent.

Consider, for example, the situation in Fig. 8.1, where a cylinder of radius a rolls over a half-cylinder of radius R. If there is no slippage, then the angles θ_1 and θ_2 are not independent, and they obey the *equation of constraint*,

$$R\,\theta_1 = a\,(\theta_2 - \theta_1) \ . \tag{8.1}$$

In this case, we can easily solve the constraint equation and substitute $\theta_2 = \left(1 + \frac{R}{a}\right)\theta_1$. In other cases, though, the equation of constraint might not be so easily solved (*e.g.* it may be nonlinear). How then do we proceed?

8.1 Constraints and Variational Calculus

Before addressing the subject of constrained dynamical systems, let's consider the issue of constraints in the broader context of variational calculus. Suppose we have a functional

$$F[y(x)] = \int\limits_{x_a}^{x_b} dx\, L(y, y', x) \ , \tag{8.2}$$

which we want to extremize subject to some constraints. Here y may stand for a set of functions $\{y_\sigma(x)\}$. There are two classes of constraints we will consider:

[1]For N rigid bodies, the number of degrees of freedom is $n' = \frac{1}{2}d(d+1)N$, corresponding to d center-of-mass coordinates and $\frac{1}{2}d(d-1)$ angles of orientation for each particle. The dimension of the group of rotations in d dimensions is $\frac{1}{2}d(d-1)$, corresponding to the number of parameters in a general rank-d orthogonal matrix (*i.e.* an element of the group $O(d)$).

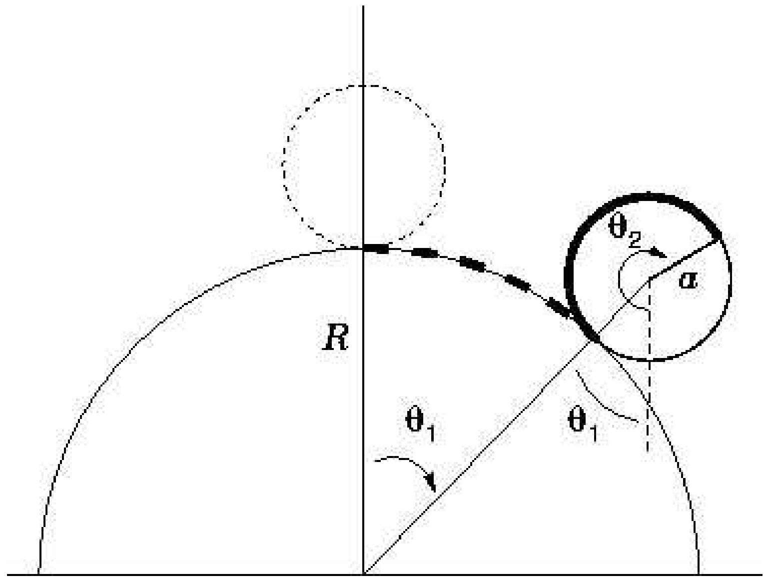

Figure 8.1: A cylinder of radius a rolls along a half-cylinder of radius R. When there is no slippage, the angles θ_1 and θ_2 obey the constraint equation $R\theta_1 = a(\theta_2 - \theta_1)$.

1. *Integral constraints:* These are of the form

$$\int\limits_{x_a}^{x_b} dx\, N_j(y, y', x) = C_j \ , \tag{8.3}$$

   where j labels the constraint.

2. *Holonomic constraints:* These are of the form

$$G_j(y, x) = 0 \ . \tag{8.4}$$

The cylinders system in Fig. 8.1 provides an example of a holonomic constraint. There, $G(\theta, t) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0$. As an example of a problem with an integral constraint, suppose we want to know the shape of a hanging rope of fixed length C. This means we minimize the rope's potential energy,

$$U[y(x)] = \lambda g \int\limits_{x_a}^{x_b} ds\, y(x) = \lambda g \int\limits_{x_a}^{x_b} dx\, y \sqrt{1 + y'^2} \ , \tag{8.5}$$

where λ is the linear mass density of the rope, subject to the fixed-length constraint

$$C = \int\limits_{x_a}^{x_b} ds = \int\limits_{x_a}^{x_b} dx\, \sqrt{1 + y'^2} \ . \tag{8.6}$$

Note $ds = \sqrt{dx^2 + dy^2}$ is the differential element of arc length along the rope. To solve problems like these, we turn to Lagrange's method of *undetermined multipliers*.

8.2 Constrained Extremization of Functions

Given $F(x_1, \ldots, x_n)$ to be extremized subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ where $j = 1, \ldots, k$, construct

$$F^*\big(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \, G_j(x_1, \ldots, x_n) \tag{8.7}$$

which is a function of the $(n+k)$ variables $\big\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big\}$. Now freely extremize the extended function F^*:

$$\begin{aligned} dF^* &= \sum_{\sigma=1}^{n} \frac{\partial F^*}{\partial x_\sigma} \, dx_\sigma + \sum_{j=1}^{k} \frac{\partial F^*}{\partial \lambda_j} \, d\lambda_j \\ &= \sum_{\sigma=1}^{n} \Bigg(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} \Bigg) \, dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j = 0 \end{aligned} \tag{8.8}$$

This results in the $(n+k)$ equations

$$\begin{aligned} \frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} &= 0 \qquad (\sigma = 1, \ldots, n) \\ G_j &= 0 \qquad (j = 1, \ldots, k) \ . \end{aligned} \tag{8.9}$$

The interpretation of all this is as follows. The n equations in 8.9 can be written in vector form as

$$\boldsymbol{\nabla} F + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{\nabla} G_j = 0 \ . \tag{8.10}$$

This says that the (n-component) vector $\boldsymbol{\nabla} F$ is linearly dependent upon the k vectors $\boldsymbol{\nabla} G_j$. Thus, any movement in the direction of $\boldsymbol{\nabla} F$ must necessarily entail movement along one or more of the directions $\boldsymbol{\nabla} G_j$. This would require violating the constraints, since movement along $\boldsymbol{\nabla} G_j$ takes us off the level set $G_j = 0$. Were $\boldsymbol{\nabla} F$ linearly *independent* of the set $\{\boldsymbol{\nabla} G_j\}$, this would mean that we could find a differential displacement $d\boldsymbol{x}$ which has finite overlap with $\boldsymbol{\nabla} F$ but zero overlap with each $\boldsymbol{\nabla} G_j$. Thus $\boldsymbol{x} + d\boldsymbol{x}$ would still satisfy $G_j(\boldsymbol{x} + d\boldsymbol{x}) = 0$, but F would change by the finite amount $dF = \boldsymbol{\nabla} F(\boldsymbol{x}) \cdot d\boldsymbol{x}$.

8.3 Extremization of Functionals : Integral Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int_{x_a}^{x_b} \! dx \, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{8.11}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$\int\limits_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l=1,\ldots,k)\ , \tag{8.12}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j\}\big] \equiv \int\limits_{x_a}^{x_b} dx \left\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{l=1}^{k} \lambda_l\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big)\right\} - \sum_{l=1}^{k} \lambda_l\, C_l \tag{8.13}$$

and freely extremize over $\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\}$. This results in $(n+k)$ equations

$$\begin{aligned} \frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{l=1}^{k} \lambda_l \left\{ \frac{\partial N_l}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial N_l}{\partial y'_\sigma}\bigg)\right\} = 0 \qquad (\sigma=1,\ldots,n) \\ \int\limits_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l=1,\ldots,k)\ . \end{aligned} \tag{8.14}$$

8.4 Extremization of Functionals : Holonomic Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int\limits_{x_a}^{x_b} dx\, L\big(\{y_\sigma\},\{y'_\sigma\},x\big) \qquad (\sigma=1,\ldots,n) \tag{8.15}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$G_j\big(\{y_\sigma(x)\},x\big) = 0 \qquad (j=1,\ldots,k)\ , \tag{8.16}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j(x)\}\big] \equiv \int\limits_{x_a}^{x_b} dx \left\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{j=1}^{k} \lambda_j\, G_j\big(\{y_\sigma\}\big)\right\} \tag{8.17}$$

and freely extremize over $\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\}$:

$$\delta F^* = \int\limits_{x_a}^{x_b} dx \left\{ \sum_{\sigma=1}^{n} \bigg(\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma}\bigg)\delta y_\sigma + \sum_{j=1}^{k} G_j\, \delta\lambda_j \right\} = 0\ , \tag{8.18}$$

resulting in the $(n+k)$ equations

$$\begin{aligned} \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma} \qquad (\sigma=1,\ldots,n) \\ G_j\big(\{y_\sigma\},x\big) = 0 \qquad (j=1,\ldots,k)\ . \end{aligned} \tag{8.19}$$

8.4.1 Examples of extremization with constraints

Volume of a cylinder : As a warm-up problem, let's maximize the volume $V = \pi a^2 h$ of a cylinder of radius a and height h, subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \tag{8.20}$$

We therefore define

$$V^*(a,h,\lambda) \equiv V(a,h) + \lambda\, G(a,h) \ , \tag{8.21}$$

and set

$$\begin{aligned} \frac{\partial V^*}{\partial a} &= 2\pi a h + 2\pi\lambda = 0 \\ \frac{\partial V^*}{\partial h} &= \pi a^2 + 2\lambda\, \frac{h}{b} = 0 \\ \frac{\partial V^*}{\partial \lambda} &= 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \end{aligned} \tag{8.22}$$

Solving these three equations simultaneously gives

$$a = \frac{2\ell}{5\pi} \quad , \quad h = \sqrt{\frac{b\ell}{5}} \quad , \quad \lambda = \frac{2\pi}{5^{3/2}}\, b^{1/2}\, \ell^{3/2} \quad , \quad V = \frac{4}{5^{5/2}\,\pi}\, \ell^{5/2}\, b^{1/2} \ . \tag{8.23}$$

Hanging rope : We minimize the energy functional

$$E\big[y(x)\big] = \mu g \int\limits_{x_1}^{x_2} \! dx\, y \sqrt{1 + y'^2} \ , \tag{8.24}$$

where μ is the linear mass density, subject to the constraint of fixed total length,

$$C\big[y(x)\big] = \int\limits_{x_1}^{x_2} \! dx\, \sqrt{1 + y'^2} \ . \tag{8.25}$$

Thus,

$$E^*\big[y(x),\lambda\big] = E\big[y(x)\big] + \lambda C\big[y(x)\big] = \int\limits_{x_1}^{x_2} \! dx\, L^*(y,y',x) \ , \tag{8.26}$$

with

$$L^*(y,y',x) = (\mu g y + \lambda)\, \sqrt{1 + y'^2} \ . \tag{8.27}$$

Since $\frac{\partial L^*}{\partial x} = 0$ we have that

$$\mathcal{J} = y'\, \frac{\partial L^*}{\partial y'} - L^* = -\frac{\mu g y + \lambda}{\sqrt{1 + y'^2}} \tag{8.28}$$

is constant. Thus,

$$\frac{dy}{dx} = \pm\mathcal{J}^{-1}\sqrt{(\mu g y + \lambda)^2 - \mathcal{J}^2} \ , \tag{8.29}$$

with solution

$$y(x) = -\frac{\lambda}{\mu g} + \frac{\mathcal{J}}{\mu g}\cosh\Big(\frac{\mu g}{\mathcal{J}}\,(x-a)\Big) \ . \tag{8.30}$$

Here, $\mathcal{J}$, a, and λ are constants to be determined by demanding $y(x_i) = y_i$ $(i = 1, 2)$, and that the total length of the rope is C.

Geodesic on a curved surface : Consider next the problem of a geodesic on a curved surface. Let the equation for the surface be

$$G(x, y, z) = 0 \ . \tag{8.31}$$

We wish to extremize the distance,

$$D = \int\limits_a^b ds = \int\limits_a^b \sqrt{dx^2 + dy^2 + dz^2} \ . \tag{8.32}$$

We introduce a parameter t defined on the unit interval: $t \in [0, 1]$, such that $x(0) = x_a$, $x(1) = x_b$, *etc.* Then D may be regarded as a functional, *viz.*

$$D\big[x(t), y(t), z(t)\big] = \int\limits_0^1 dt\,\sqrt{\dot x^2 + \dot y^2 + \dot z^2} \ . \tag{8.33}$$

We impose the constraint by forming the extended functional, D^*:

$$D^*\big[x(t), y(t), z(t), \lambda(t)\big] \equiv \int\limits_0^1 dt\,\bigg\{\sqrt{\dot x^2 + \dot y^2 + \dot z^2} + \lambda\, G(x, y, z)\bigg\} \ , \tag{8.34}$$

and we demand that the first functional derivatives of D^* vanish:

$$\begin{aligned}
\frac{\delta D^*}{\delta x(t)} &= -\frac{d}{dt}\bigg(\frac{\dot x}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\bigg) + \lambda\,\frac{\partial G}{\partial x} = 0 \\
\frac{\delta D^*}{\delta y(t)} &= -\frac{d}{dt}\bigg(\frac{\dot y}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\bigg) + \lambda\,\frac{\partial G}{\partial y} = 0 \\
\frac{\delta D^*}{\delta z(t)} &= -\frac{d}{dt}\bigg(\frac{\dot z}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\bigg) + \lambda\,\frac{\partial G}{\partial z} = 0 \\
\frac{\delta D^*}{\delta \lambda(t)} &= G(x, y, z) = 0 \ .
\end{aligned} \tag{8.35}$$

Thus,

$$\lambda(t) = \frac{v\ddot x - \dot x \dot v}{v^2\,\partial_x G} = \frac{v\ddot y - \dot y \dot v}{v^2\,\partial_y G} = \frac{v\ddot z - \dot z \dot v}{v^2\,\partial_z G} \ , \tag{8.36}$$

with $v = \sqrt{\dot x^2 + \dot y^2 + \dot z^2}$ and $\partial_x \equiv \frac{\partial}{\partial x}$, *etc.* These three equations are supplemented by $G(x, y, z) = 0$, which is the fourth.

8.5 Application to Mechanics

Let us write our system of constraints in the differential form

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, dq_\sigma + h_j(q,t)dt = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.37}$$

If the partial derivatives satisfy

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma} \ , \tag{8.38}$$

then the differential can be integrated to give $dG(q,t) = 0$, where

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad , \quad h_j = \frac{\partial G_j}{\partial t} \ . \tag{8.39}$$

The action functional is

$$S[\{q_\sigma(t)\}] = \int\limits_{t_a}^{t_b} dt\, L\big(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\big) \qquad (\sigma = 1, \ldots, n) \ , \tag{8.40}$$

subject to boundary conditions $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. The first variation of S is given by

$$\delta S = \int\limits_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \right\} \delta q_\sigma \ . \tag{8.41}$$

Since the $\{q_\sigma(t)\}$ are no longer independent, we cannot infer that the term in brackets vanishes for each σ. What are the constraints on the variations $\delta q_\sigma(t)$? The constraints are expressed in terms of *virtual displacements* which take no time: $\delta t = 0$. Thus,

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \delta q_\sigma(t) = 0 \ , \tag{8.42}$$

where $j = 1, \ldots, k$ is the constraint index. We may now relax the constraint by introducing k undetermined functions $\lambda_j(t)$, by adding integrals of the above equations with undetermined coefficient functions to δS:

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0 \ . \tag{8.43}$$

Now we can demand that the term in brackets vanish for all σ. Thus, we obtain a set of $(n+k)$ equations,

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \equiv Q_\sigma \\ &g_{j\sigma}(q,t)\, \dot{q}_\sigma + h_j(q,t) = 0 \ , \end{aligned} \tag{8.44}$$

in $(n+k)$ unknowns $\{q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k\}$. Here, Q_σ is the *force of constraint conjugate to the generalized coordinate* q_σ. Thus, with

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad F_\sigma = \frac{\partial L}{\partial q_\sigma} \quad , \quad Q_\sigma = \sum_{j=1}^{k} \lambda_j \, g_{j\sigma} \ , \tag{8.45}$$

we write Newton's second law as

$$\dot{p}_\sigma = F_\sigma + Q_\sigma \ . \tag{8.46}$$

Note that we can write

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\bigg) \tag{8.47}$$

and that the *instantaneous* constraints may be written

$$\boldsymbol{g}_j \cdot \delta \boldsymbol{q} = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.48}$$

Thus, by demanding

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{g}_j = 0 \tag{8.49}$$

we require that the functional derivative be linearly dependent on the k vectors $\boldsymbol{g}_j$.

8.5.1 Constraints and conservation laws

We have seen how invariance of the Lagrangian with respect to a one-parameter family of coordinate transformations results in an associated conserved quantity Λ, and how a lack of explicit time dependence in L results in the conservation of the Hamiltonian H. In deriving both these results, however, we used the equations of motion $\dot{p}_\sigma = F_\sigma$. What happens when we have constraints, in which case $\dot{p}_\sigma = F_\sigma + Q_\sigma$?

Let's begin with the Hamiltonian. We have $H = \dot{q}_\sigma \, p_\sigma - L$, hence

$$\begin{aligned} \frac{dH}{dt} &= \bigg(p_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\bigg)\ddot{q}_\sigma + \bigg(\dot{p}_\sigma - \frac{\partial L}{\partial q_\sigma}\bigg)\dot{q}_\sigma - \frac{\partial L}{\partial t} \\ &= Q_\sigma \, \dot{q}_\sigma - \frac{\partial L}{\partial t} \ . \end{aligned} \tag{8.50}$$

We now use

$$Q_\sigma \, \dot{q}_\sigma = \lambda_j \, g_{j\sigma} \, \dot{q}_\sigma = -\lambda_j \, h_j \tag{8.51}$$

to obtain

$$\frac{dH}{dt} = -\lambda_j \, h_j - \frac{\partial L}{\partial t} \ . \tag{8.52}$$

We therefore conclude that *in a system with constraints of the form* $g_{j\sigma} \, \dot{q}_\sigma + h_j = 0$*, the Hamiltonian is conserved if each* $h_j = 0$ *and if* L *is not explicitly dependent on time.* In the case of holonomic constraints, $h_j = \frac{\partial G_j}{\partial t}$, so H is conserved if neither L nor any of the constraints G_j is explicitly time-dependent.

Next, let us rederive Noether's theorem when constraints are present. We assume a one-parameter family of transformations $q_\sigma \to \tilde{q}_\sigma(\zeta)$ leaves L invariant. Then

$$\begin{aligned} 0 = \frac{dL}{d\zeta} &= \frac{\partial L}{\partial \tilde{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \frac{\partial L}{\partial \dot{\tilde{q}}_\sigma} \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta} \\ &= \big(\dot{\tilde{p}}_\sigma - \tilde{Q}_\sigma\big) \frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \tilde{p}_\sigma \frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) \\ &= \frac{d}{dt}\bigg(\tilde{p}_\sigma \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) - \lambda_j \, \tilde{g}_{j\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta} \ . \end{aligned} \tag{8.53}$$

Now let us write the constraints in differential form as

$$\tilde{g}_{j\sigma} \, d\tilde{q}_\sigma + \tilde{h}_j \, dt + \tilde{k}_j \, d\zeta = 0 \ . \tag{8.54}$$

We now have

$$\frac{d\Lambda}{dt} = \lambda_j \, \tilde{k}_j \ , \tag{8.55}$$

which says that *if the constraints are independent of ζ then Λ is conserved.* For holonomic constraints, this means that

$$G_j\big(\tilde{q}(\zeta), t\big) = 0 \quad \Rightarrow \quad \tilde{k}_j = \frac{\partial G_j}{\partial \zeta} = 0 \ , \tag{8.56}$$

i.e. $G_j(\tilde{q}, t)$ has no explicit ζ dependence.

8.6 Worked Examples

Here we consider several example problems of constrained dynamics, and work each out in full detail.

8.6.1 One cylinder rolling off another

As an example of the constraint formalism, consider the system in Fig. 8.1, where a cylinder of radius a rolls atop a cylinder of radius R. We have two constraints:

$$\begin{aligned} G_1(r, \theta_1, \theta_2) &= r - R - a = 0 \qquad \text{(cylinders in contact)} \\ G_2(r, \theta_1, \theta_2) &= R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0 \qquad \text{(no slipping)} \ , \end{aligned} \tag{8.57}$$

from which we obtain the $g_{j\sigma}$:

$$g_{j\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix} \ , \tag{8.58}$$

which is to say

$$\frac{\partial G_1}{\partial r} = 1 \quad , \quad \frac{\partial G_1}{\partial \theta_1} = 0 \quad , \quad \frac{\partial G_1}{\partial \theta_2} = 0 \quad , \quad \frac{\partial G_2}{\partial r} = 0 \quad , \quad \frac{\partial G_2}{\partial \theta_1} = R + a \quad , \quad \frac{\partial G_2}{\partial \theta_2} = -a \ . \tag{8.59}$$

The Lagrangian is

$$L = T - U = \tfrac{1}{2}M\big(\dot{r}^2 + r^2\,\dot{\theta}_1^2\big) + \tfrac{1}{2}I\,\dot{\theta}_2^2 - Mgr\,\cos\theta_1 \ , \tag{8.60}$$

where M and I are the mass and rotational inertia of the rolling cylinder, respectively. Note that the kinetic energy is a sum of center-of-mass translation $T_{\rm tr} = \frac{1}{2}M\big(\dot{r}^2 + r^2\,\dot{\theta}_1^2\big)$ and rotation about the center-of-mass, $T_{\rm rot} = \frac{1}{2}I\,\dot{\theta}_2^2$. The equations of motion are

$$\begin{aligned}
\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{r}}\bigg) - \frac{\partial L}{\partial r} &= M\ddot{r} - Mr\,\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1 \equiv Q_r \\
\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\theta}_1}\bigg) - \frac{\partial L}{\partial \theta_1} &= Mr^2\ddot{\theta}_1 + 2Mr\dot{r}\,\dot{\theta}_1 - Mgr\sin\theta_1 = (R+a)\,\lambda_2 \equiv Q_{\theta_1} \\
\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\theta}_2}\bigg) - \frac{\partial L}{\partial \theta_2} &= I\ddot{\theta}_2 = -a\,\lambda_2 \equiv Q_{\theta_2} \ .
\end{aligned} \tag{8.61}$$

To these three equations we add the two constraints, resulting in five equations in the five unknowns $\big\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\big\}$.

We solve by first implementing the constraints, which give $r = (R+a)$ a constant (*i.e.* $\dot{r} = 0$), and $\dot{\theta}_2 = \big(1 + \frac{R}{a}\big)\,\dot{\theta}_1$. Substituting these into the above equations gives

$$\begin{gathered}
-M(R+a)\,\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1 \\
M(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = (R+a)\,\lambda_2 \\
I\bigg(\frac{R+a}{a}\bigg)\ddot{\theta}_1 = -a\lambda_2 \ .
\end{gathered} \tag{8.62}$$

From the last of eqns. 8.62 we obtain

$$\lambda_2 = -\frac{I}{a}\,\ddot{\theta}_2 = -\frac{R+a}{a^2}\,I\,\ddot{\theta}_1 \ , \tag{8.63}$$

which we substitute into the second of eqns. 8.62 to obtain

$$\bigg(M + \frac{I}{a^2}\bigg)(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = 0 \ . \tag{8.64}$$

Multiplying by $\dot{\theta}_1$, we obtain an exact differential, which may be integrated to yield

$$\tfrac{1}{2}M\bigg(1 + \frac{I}{Ma^2}\bigg)\dot{\theta}_1^2 + \frac{Mg}{R+a}\,\cos\theta_1 = \frac{Mg}{R+a}\,\cos\theta_1^\circ \ . \tag{8.65}$$

Here, we have assumed that $\dot{\theta}_1 = 0$ when $\theta_1 = \theta_1^\circ$, *i.e.* the rolling cylinder is released from rest at $\theta_1 = \theta_1^\circ$. Finally, inserting this result into the first of eqns. 8.62, we obtain the radial force of constraint,

$$Q_r = \frac{Mg}{1+\alpha}\Big\{(3+\alpha)\,\cos\theta_1 - 2\,\cos\theta_1^\circ\Big\} \ , \tag{8.66}$$

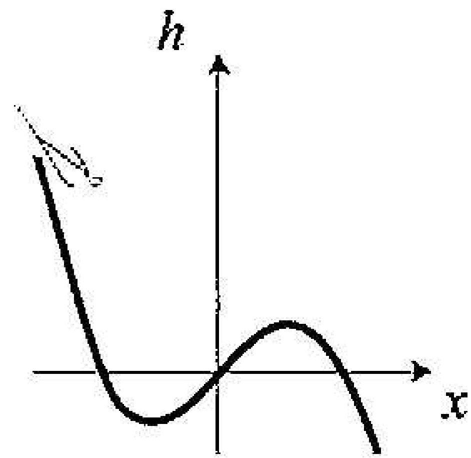

Figure 8.2: Frictionless motion under gravity along a curved surface. The skier flies off the surface when the normal force vanishes.

where $\alpha = I/Ma^2$ is a dimensionless parameter ($0 \leq \alpha \leq 1$). This is the radial component of the normal force between the two cylinders. When Q_r vanishes, the cylinders lose contact – the rolling cylinder flies off. Clearly this occurs at an angle $\theta_1 = \theta_1^*$, where

$$\theta_1^* = \cos^{-1}\left(\frac{2\cos\theta_1^\circ}{3+\alpha}\right) . \tag{8.67}$$

The detachment angle θ_1^* is an increasing function of α, which means that larger I delays detachment. This makes good sense, since when I is larger the gain in kinetic energy is split between translational and rotational motion of the rolling cylinder.

8.6.2 Frictionless motion along a curve

Consider the situation in Fig. 8.2 where a skier moves frictionlessly under the influence of gravity along a general curve $y = h(x)$. The Lagrangian for this problem is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{8.68}$$

and the (holonomic) constraint is

$$G(x, y) = y - h(x) = 0 . \tag{8.69}$$

Accordingly, the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \lambda\,\frac{\partial G}{\partial q_\sigma} , \tag{8.70}$$

where $q_1 = x$ and $q_2 = y$. Thus, we obtain

$$\begin{aligned} m\ddot{x} &= -\lambda\, h'(x) = Q_x \\ m\ddot{y} + mg &= \lambda = Q_y . \end{aligned} \tag{8.71}$$

We eliminate y in favor of x by invoking the constraint. Since we need $\ddot{y}$, we must differentiate the constraint, which gives

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.72}$$

Using the second Euler-Lagrange equation, we then obtain

$$\frac{\lambda}{m} = g + h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.73}$$

Finally, we substitute this into the first E-L equation to obtain an equation for x alone:

$$\Big(1 + \big[h'(x)\big]^2\Big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 + g\,h'(x) = 0 \ . \tag{8.74}$$

Had we started by eliminating $y = h(x)$ at the outset, writing

$$L(x, \dot{x}) = \tfrac{1}{2}m\Big(1 + \big[h'(x)\big]^2\Big)\,\dot{x}^2 - mg\,h(x) \ , \tag{8.75}$$

we would also have obtained this equation of motion.

The skier flies off the curve when the vertical force of constraint $Q_y = \lambda$ starts to become negative, because the curve can only supply a positive normal force. Suppose the skier starts from rest at a height y_0. We may then determine the point x at which the skier detaches from the curve by setting $\lambda(x) = 0$. To do so, we must eliminate $\dot{x}$ and $\ddot{x}$ in terms of x. For $\ddot{x}$, we may use the equation of motion to write

$$\ddot{x} = -\bigg(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\bigg) \ , \tag{8.76}$$

which allows us to write

$$\lambda = m\bigg(\frac{g + h''\,\dot{x}^2}{1 + h'^2}\bigg) \ . \tag{8.77}$$

To eliminate $\dot{x}$, we use conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot{x}^2 + mgh \ , \tag{8.78}$$

which fixes

$$\dot{x}^2 = 2g\bigg(\frac{y_0 - h}{1 + h'^2}\bigg) \ . \tag{8.79}$$

Putting it all together, we have

$$\lambda(x) = \frac{mg}{\big(1 + h'^2\big)^2}\,\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\} \ . \tag{8.80}$$

The skier detaches from the curve when $\lambda(x) = 0$, *i.e.* when

$$1 + h'^2 + 2(y_0 - h)\,h'' = 0 \ . \tag{8.81}$$

There is a somewhat easier way of arriving at the same answer. This is to note that the skier must fly off when the local centripetal force equals the gravitational force normal to the curve, *i.e.*

$$\frac{m\,v^2(x)}{R(x)} = mg\,\cos\theta(x) \ , \tag{8.82}$$

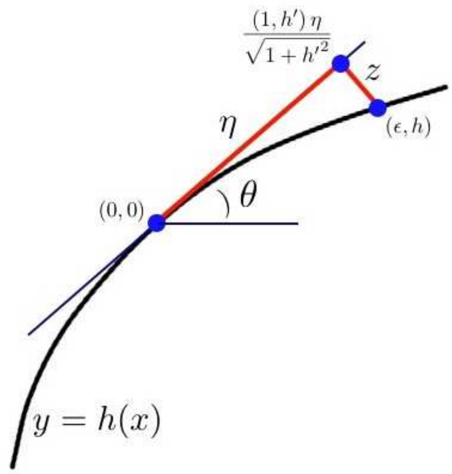

Figure 8.3: Finding the local radius of curvature: $z = \eta^2/2R$.

where $R(x)$ is the local radius of curvature. Now $\tan\theta = h'$, so $\cos\theta = \big(1 + h'^2\big)^{-1/2}$. The square of the velocity is $v^2 = \dot{x}^2 + \dot{y}^2 = \big(1 + h'^2\big)\,\dot{x}^2$. What is the local radius of curvature $R(x)$? This can be determined from the following argument, and from the sketch in Fig. 8.3. Writing $x = x^* + \epsilon$, we have

$$y = h(x^*) + h'(x^*)\,\epsilon + \tfrac{1}{2}h''(x^*)\,\epsilon^2 + \ldots\ . \tag{8.83}$$

We now drop a perpendicular segment of length z from the point (x, y) to the line which is tangent to the curve at $\big(x^*, h(x^*)\big)$. According to Fig. 8.3, this means

$$\begin{pmatrix} \epsilon \\ y \end{pmatrix} = \eta \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} 1 \\ h' \end{pmatrix} - z \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} -h' \\ 1 \end{pmatrix} \ . \tag{8.84}$$

Thus, we have

$$\begin{aligned} y &= h'\,\epsilon + \tfrac{1}{2}h''\,\epsilon^2 \\ &= h'\left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}}\right) + \tfrac{1}{2}h''\left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}}\right)^2 \\ &= \frac{\eta\,h' + z\,h'^2}{\sqrt{1+h'^2}} + \frac{h''\,\eta^2}{2\big(1+h'^2\big)} + \mathcal{O}(\eta z) \\ &= \frac{\eta\,h' - z}{\sqrt{1+h'^2}} \ , \end{aligned} \tag{8.85}$$

from which we obtain

$$z = -\frac{h''\,\eta^2}{2\big(1+h'^2\big)^{3/2}} + \mathcal{O}(\eta^3) \tag{8.86}$$

and therefore

$$R(x) = -\frac{1}{h''(x)} \cdot \Big(1 + \big[h'(x)\big]^2\Big)^{3/2} \ . \tag{8.87}$$

Thus, the detachment condition,

$$\frac{mv^2}{R} = -\frac{m\,h''\,\dot{x}^2}{\sqrt{1+h'^2}} = \frac{mg}{\sqrt{1+h'^2}} = mg\,\cos\theta \tag{8.88}$$

reproduces the result from eqn. 8.77.

8.6.3 Disk rolling down an inclined plane

A hoop of mass m and radius R rolls without slipping down an inclined plane. The inclined plane has opening angle α and mass M, and itself slides frictionlessly along a horizontal surface. Find the motion of the system.

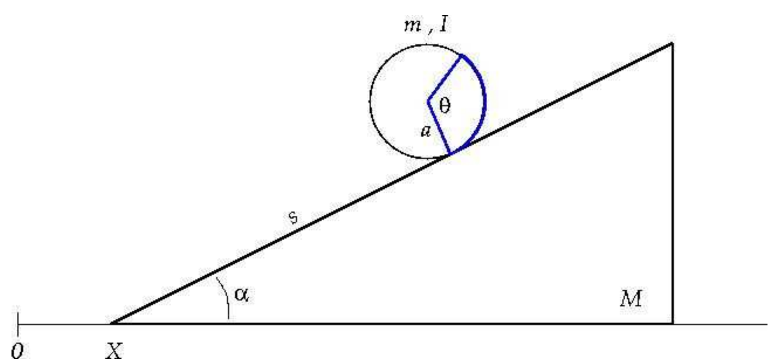

Figure 8.4: A hoop rolling down an inclined plane lying on a frictionless surface.

Solution : Referring to the sketch in Fig. 8.4, the center of the hoop is located at

$$\begin{aligned} x &= X + s\,\cos\alpha - a\,\sin\alpha \\ y &= s\,\sin\alpha + a\,\cos\alpha\ , \end{aligned}$$

where X is the location of the lower left corner of the wedge, and s is the distance along the wedge to the bottom of the hoop. If the hoop rotates through an angle θ, the no-slip condition is $a\,\dot{\theta} + \dot{s} = 0$. Thus,

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \\ &= \tfrac{1}{2}\bigg(m + \frac{I}{a^2}\bigg)\dot{s}^2 + \tfrac{1}{2}(M+m)\dot{X}^2 + m\cos\alpha\,\dot{X}\,\dot{s} - mgs\,\sin\alpha - mga\,\cos\alpha\ . \end{aligned}$$

Since X is cyclic in L, the momentum

$$P_X = (M+m)\dot{X} + m\cos\alpha\,\dot{s}\ ,$$

is preserved: $\dot{P}_X = 0$. The second equation of motion, corresponding to the generalized coordinate s, is

$$\bigg(1 + \frac{I}{ma^2}\bigg)\ddot{s} + \cos\alpha\,\ddot{X} = -g\,\sin\alpha\ .$$

Using conservation of P_X, we eliminate $\ddot{s}$ in favor of $\ddot{X}$, and immediately obtain

$$\ddot{X} = \frac{g \sin\alpha \cos\alpha}{\left(1 + \frac{M}{m}\right)\left(1 + \frac{I}{ma^2}\right) - \cos^2\alpha} \equiv a_X \ .$$

The result

$$\ddot{s} = -\frac{g\left(1 + \frac{M}{m}\right)\sin\alpha}{\left(1 + \frac{M}{m}\right)\left(1 + \frac{I}{ma^2}\right) - \cos^2\alpha} \equiv a_s$$

follows immediately. Thus,

$$\begin{aligned} X(t) &= X(0) + \dot{X}(0)\, t + \tfrac{1}{2} a_X t^2 \\ s(t) &= s(0) + \dot{s}(0)\, t + \tfrac{1}{2} a_s t^2 \ . \end{aligned}$$

Note that $a_s < 0$ while $a_X > 0$, *i.e.* the hoop rolls down and to the left as the wedge slides to the right. Note that $I = ma^2$ for a hoop; we've computed the answer here for general I.

8.6.4 Pendulum with nonrigid support

A particle of mass m is suspended from a flexible string of length ℓ in a uniform gravitational field. While hanging motionless in equilibrium, it is struck a horizontal blow resulting in an initial angular velocity ω_0. Treating the system as one with *two* degrees of freedom and a constraint, answer the following:

(a) Compute the Lagrangian, the equation of constraint, and the equations of motion.

Solution : The Lagrangian is

$$L = \tfrac{1}{2} m \left(\dot{r}^2 + r^2\, \dot{\theta}^2\right) + mgr\, \cos\theta \ .$$

The constraint is $r = \ell$. The equations of motion are

$$\begin{aligned} m\ddot{r} - mr\,\dot{\theta}^2 - mg\, \cos\theta &= \lambda \\ mr^2\,\ddot{\theta} + 2mr\,\dot{r}\,\dot{\theta} - mg\sin\theta &= 0 \ . \end{aligned}$$

(b) Compute the tension in the string as a function of angle θ.

Solution : Energy is conserved, hence

$$\tfrac{1}{2} m\ell^2\, \dot{\theta}^2 - mg\ell\cos\theta = \tfrac{1}{2} m\ell^2\, \dot{\theta}_0^2 - mg\ell\cos\theta_0 \ .$$

We take $\theta_0 = 0$ and $\dot{\theta}_0 = \omega_0$. Thus,

$$\dot{\theta}^2 = \omega_0^2 - 2\,\Omega^2\left(1 - \cos\theta\right) \ ,$$

with $\Omega = \sqrt{g/\ell}$. Substituting this into the equation for λ, we obtain

$$\lambda = mg\left\{2 - 3\cos\theta - \frac{\omega_0^2}{\Omega^2}\right\} \ .$$

(c) Show that if $\omega_0^2 < 2g/\ell$ then the particle's motion is confined below the horizontal and that the tension in the string is always positive (defined such that positive means exerting a pulling force and negative means exerting a pushing force). Note that the difference between a string and a rigid rod is that the string can only pull but the rod can pull or push. Thus, *the string tension must always be positive or else the string goes "slack"*.

Solution : Since $\dot\theta^2 \geq 0$, we must have

$$\frac{\omega_0^2}{2\Omega^2} \geq 1 - \cos\theta \ .$$

The condition for slackness is $\lambda = 0$, or

$$\frac{\omega_0^2}{2\Omega^2} = 1 - \tfrac{3}{2}\cos\theta \ .$$

Thus, if $\omega_0^2 < 2\Omega^2$, we have

$$1 > \frac{\omega_0^2}{2\Omega^2} > 1 - \cos\theta > 1 - \tfrac{3}{2}\cos\theta \ ,$$

and the string never goes slack. Note the last equality follows from $\cos\theta > 0$. The string rises to a maximum angle

$$\theta_{\rm max} = \cos^{-1}\left(1 - \frac{\omega_0^2}{2\Omega^2}\right) \ .$$

(d) Show that if $2g/\ell < \omega_0^2 < 5g/\ell$ the particle rises above the horizontal and the string becomes slack (the tension vanishes) at an angle θ^*. Compute θ^*.

Solution : When $\omega^2 > 2\Omega^2$, the string rises above the horizontal and goes slack at an angle

$$\theta^* = \cos^{-1}\left(\tfrac{2}{3} - \frac{\omega_0^2}{3\Omega^2}\right) \ .$$

This solution craps out when the string is still taut at $\theta = \pi$, which means $\omega_0^2 = 5\Omega^2$.

(e) Show that if $\omega_0^2 > 5g/\ell$ the tension is always positive and the particle executes circular motion.

Solution : For $\omega_0^2 > 5\Omega^2$, the string never goes slack. Furthermore, $\dot\theta$ never vanishes. Therefore, the pendulum undergoes circular motion, albeit not with constant angular velocity.

8.6.5 Falling ladder

A uniform ladder of length ℓ and mass m has one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest and makes an angle θ_0 with respect to the horizontal.

(a) Make a convenient choice of generalized coordinates and find the Lagrangian.

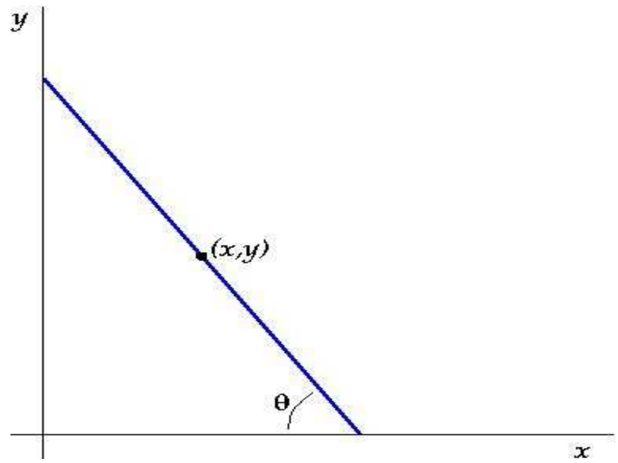

Figure 8.5: A ladder sliding down a wall and across a floor.

Solution : I choose as generalized coordinates the Cartesian coordinates (x, y) of the ladder's center of mass, and the angle θ it makes with respect to the floor. The Lagrangian is then

$$L = \tfrac{1}{2}\, m\,(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\, I\,\dot{\theta}^2 + mgy \ .$$

There are two constraints: one enforcing contact along the wall, and the other enforcing contact along the floor. These are written

$$\begin{aligned} G_1(x, y, \theta) &= x - \tfrac{1}{2}\,\ell\,\cos\theta = 0 \\ G_2(x, y, \theta) &= y - \tfrac{1}{2}\,\ell\,\sin\theta = 0 \ . \end{aligned}$$

(b) Prove that the ladder leaves the wall when its upper end has fallen to a height $\frac{2}{3}L\sin\theta_0$. The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_j \lambda_j\,\frac{\partial G_j}{\partial q_\sigma} \ .$$

Thus, we have

$$\begin{aligned} m\,\ddot{x} &= \lambda_1 = Q_x \\ m\,\ddot{y} + mg &= \lambda_2 = Q_y \\ I\,\ddot{\theta} &= \tfrac{1}{2}\ell\big(\lambda_1\sin\theta - \lambda_2\cos\theta\big) = Q_\theta \ . \end{aligned}$$

We now implement the constraints to eliminate x and y in terms of θ. We have

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta} & \qquad \ddot{x} &= -\tfrac{1}{2}\,\ell\,\cos\theta\,\dot{\theta}^2 - \tfrac{1}{2}\,\ell\,\sin\theta\,\ddot{\theta} \\ \dot{y} &= \tfrac{1}{2}\,\ell\,\cos\theta\,\dot{\theta} & \qquad \ddot{y} &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta}^2 + \tfrac{1}{2}\,\ell\,\cos\theta\,\ddot{\theta} \ . \end{aligned}$$

We can now obtain the forces of constraint in terms of the function $\theta(t)$:

$$\begin{aligned} \lambda_1 &= -\tfrac{1}{2}m\ell\,\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \\ \lambda_2 &= +\tfrac{1}{2}m\ell\,\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big) + mg \ . \end{aligned}$$

We substitute these into the last equation of motion to obtain the result

$$I\,\ddot{\theta} = -I_0\,\ddot{\theta} - \tfrac{1}{2} mg\ell\,\cos\theta \ ,$$

or

$$(1+\alpha)\,\ddot{\theta} = -2\omega_0^2\,\cos\theta \ ,$$

with $I_0 = \frac{1}{4}m\ell^2$, $\alpha \equiv I/I_0$ and $\omega_0 = \sqrt{g/\ell}$. This may be integrated once (multiply by $\dot{\theta}$ to convert to a total derivative) to yield

$$\tfrac{1}{2}(1+\alpha)\,\dot{\theta}^2 + 2\,\omega_0^2\,\sin\theta = 2\,\omega_0^2\,\sin\theta_0 \ ,$$

which is of course a statement of energy conservation. This,

$$\dot{\theta}^2 = \frac{4\,\omega_0^2\,(\sin\theta_0 - \sin\theta)}{1+\alpha}$$
$$\ddot{\theta} = -\frac{2\,\omega_0^2\,\cos\theta}{1+\alpha} \ .$$

We may now obtain $\lambda_1(\theta)$ and $\lambda_2(\theta)$:

$$\lambda_1(\theta) = -\frac{mg}{1+\alpha}\Big(3\sin\theta - 2\sin\theta_0\Big)\,\cos\theta$$
$$\lambda_2(\theta) = \frac{mg}{1+\alpha}\Big\{\Big(3\sin\theta - 2\sin\theta_0\Big)\,\sin\theta + \alpha\Big\} \ .$$

Demanding $\lambda_1(\theta) = 0$ gives the detachment angle $\theta = \theta_{\rm d}$, where

$$\sin\theta_{\rm d} = \tfrac{2}{3}\sin\theta_0 \ .$$

Note that $\lambda_2(\theta_{\rm d}) = mg\alpha/(1+\alpha) > 0$, so the normal force from the floor is always positive for $\theta > \theta_{\rm d}$. The time to detachment is

$$T_1(\theta_0) = \int\frac{d\theta}{\dot{\theta}} = \frac{\sqrt{1+\alpha}}{2\,\omega_0}\int\limits_{\theta_{\rm d}}^{\theta_0}\frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \ .$$

(c) Show that the subsequent motion can be reduced to quadratures (*i.e.* explicit integrals).

Solution : After the detachment, there is no longer a constraint G_1. The equations of motion are

$$m\,\ddot{x} = 0 \qquad \text{(conservation of } x\text{-momentum)}$$
$$m\,\ddot{y} + m\,g = \lambda$$
$$I\,\ddot{\theta} = -\tfrac{1}{2}\,\ell\,\lambda\,\cos\theta \ ,$$

along with the constraint $y = \frac{1}{2}\,\ell\,\sin\theta$. Eliminating y in favor of θ using the constraint, the second equation yields

$$\lambda = mg - \tfrac{1}{2} m\ell\,\sin\theta\,\dot\theta^2 + \tfrac{1}{2} m\ell\,\cos\theta\,\ddot\theta\ .$$

Plugging this into the third equation of motion, we find

$$I\,\ddot\theta = -2\,I_0\,\omega_0^2\,\cos\theta + I_0\,\sin\theta\,\cos\theta\,\dot\theta^2 - I_0\,\cos^2\theta\,\ddot\theta\ .$$

Multiplying by $\dot\theta$ one again obtains a total time derivative, which is equivalent to rediscovering energy conservation:

$$E = \tfrac{1}{2}\big(I + I_0\,\cos^2\theta\big)\,\dot\theta^2 + 2\,I_0\,\omega_0^2\,\sin\theta\ .$$

```
In[37]:= T[x_] := NIntegrate[√((4 / 3) / (x - Sin[y])), {y, ArcSin[2 x / 3], ArcSin[x] - 10^-9}] / 2

In[38]:= S[x_] := NIntegrate[
           √((1 + (4 / 3) (Cos[y]) ^ 2) / ((1 - (x / 3) ^ 2) x - Sin[y])), {y, 0, ArcSin[2 x / 3]}] / 2

In[39]:= Q[x_] := T[x] + S[x]

In[43]:= Plot[Q[x], {x, 0, 1}]
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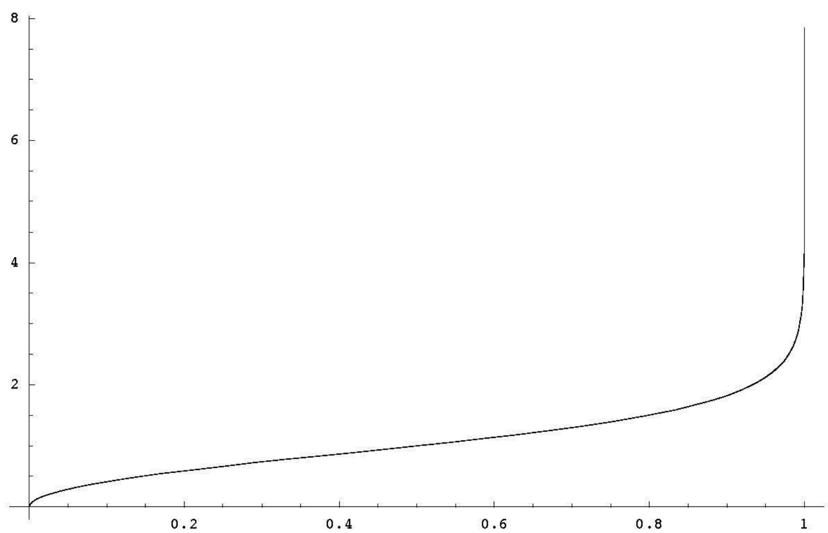


Figure 8.6: Plot of time to fall for the slipping ladder. Here $x = \sin\theta_0$.

By continuity with the first phase of the motion, we obtain the initial conditions for this second phase:

$$\theta = \sin^{-1}\left(\tfrac{2}{3}\sin\theta_0\right)$$
$$\dot\theta = -2\,\omega_0\,\sqrt{\frac{\sin\theta_0}{3\,(1+\alpha)}}\ .$$

Thus,

$$\begin{aligned} E &= \tfrac{1}{2}\big(I + I_0 - \tfrac{4}{9}\,I_0\,\sin^2\theta_0\big)\cdot\frac{4\,\omega_0^2\,\sin\theta_0}{3\,(1+\alpha)} + \tfrac{1}{3}\,mg\ell\,\sin\theta_0 \\ &= 2\,I_0\,\omega_0^2\cdot\left\{1 + \tfrac{4}{27}\,\frac{\sin^2\theta_0}{1+\alpha}\right\}\sin\theta_0\ . \end{aligned}$$

(d) Find an expression for the time $T(\theta_0)$ it takes the ladder to smack against the floor. Note that, expressed in units of the time scale $\sqrt{L/g}$, T is a dimensionless function of θ_0. Numerically integrate this expression and plot T *versus* θ_0.

Solution : The time from detachment to smack is

$$T_2(\theta_0) = \int \frac{d\theta}{\dot\theta} = \frac{1}{2\,\omega_0} \int\limits_0^{\theta_{\rm d}} d\theta \sqrt{\frac{1+\alpha\,\cos^2\theta}{\left(1-\frac{4}{27}\frac{\sin^2\theta_0}{1+\alpha}\right)\sin\theta_0 - \sin\theta}} \ .$$

The total time is then $T(\theta_0) = T_1(\theta_0) + T_2(\theta_0)$. For a uniformly dense ladder, $I = \frac{1}{12}\,m\ell^2 = \frac{1}{3}\,I_0$, so $\alpha = \frac{1}{3}$.

(e) What is the horizontal velocity of the ladder at long times?

Solution : From the moment of detachment, and thereafter,

$$\dot x = -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot\theta = \sqrt{\frac{4\,g\,\ell}{27\,(1+\alpha)}}\ \sin^{3/2}\!\theta_0 \ .$$

(f) Describe in words the motion of the ladder subsequent to it slapping against the floor.

Solution : Only a fraction of the ladder's initial potential energy is converted into kinetic energy of horizontal motion. The rest is converted into kinetic energy of vertical motion and of rotation. The slapping of the ladder against the floor is an elastic collision. After the collision, the ladder must rise again, and continue to rise and fall *ad infinitum*, as it slides along with constant horizontal velocity.

8.6.6 Point mass inside rolling hoop

Consider the point mass m inside the hoop of radius R, depicted in Fig. 8.7. We choose as generalized coordinates the Cartesian coordinates (X, Y) of the center of the hoop, the Cartesian coordinates (x, y) for the point mass, the angle ϕ through which the hoop turns, and the angle θ which the point mass makes with respect to the vertical. These six coordinates are not all independent. Indeed, there are only two independent coordinates for this system, which can be taken to be θ and ϕ. Thus, there are *four* constraints:

$$\begin{aligned} X - R\phi &\equiv G_1 = 0 \\ Y - R &\equiv G_2 = 0 \\ x - X - R\sin\theta &\equiv G_3 = 0 \\ y - Y + R\cos\theta &\equiv G_4 = 0 \ . \end{aligned} \tag{8.89}$$

The kinetic and potential energies are easily expressed in terms of the Cartesian coordinates, aside from the energy of rotation of the hoop about its CM, which is expressed in terms of $\dot\phi$:

$$\begin{aligned} T &= \tfrac{1}{2}M(\dot X^2 + \dot Y^2) + \tfrac{1}{2}m(\dot x^2 + \dot y^2) + \tfrac{1}{2}I\,\dot\phi^2 \\ U &= MgY + mgy \ . \end{aligned} \tag{8.90}$$

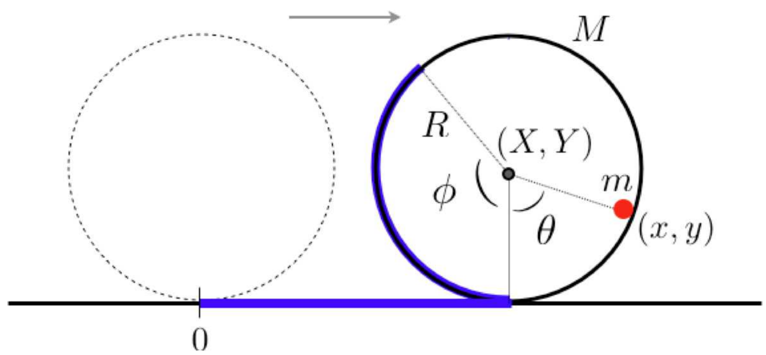


Figure 8.7: A point mass m inside a hoop of mass M, radius R, and moment of inertia I.

The moment of inertia of the hoop about its CM is $I = MR^2$, but we could imagine a situation in which I were different. For example, we could instead place the point mass inside a very short cylinder with two solid end caps, in which case $I = \frac{1}{2}MR^2$. The Lagrangian is then

$$L = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 - MgY - mgy \ . \tag{8.91}$$

Note that L as written is completely independent of θ and $\dot{\theta}$!

Continuous symmetry

Note that there is an continuous symmetry to L which is satisfied by all the constraints, under

$$\begin{aligned} \tilde{X}(\zeta) &= X + \zeta \quad , \quad & \tilde{Y}(\zeta) &= Y \\ \tilde{x}(\zeta) &= x + \zeta \quad , \quad & \tilde{y}(\zeta) &= y \\ \tilde{\phi}(\zeta) &= \phi + \frac{\zeta}{R} \quad , \quad & \tilde{\theta}(\zeta) &= \theta \ . \end{aligned} \tag{8.92}$$

Thus, according to Noether's theorem, there is a conserved quantity

$$\begin{aligned} \Lambda &= \frac{\partial L}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{x}} + \frac{1}{R}\,\frac{\partial L}{\partial \dot{\phi}} \\ &= M\dot{X} + m\dot{x} + \frac{I}{R}\,\dot{\phi} \ . \end{aligned} \tag{8.93}$$

This means $\dot{\Lambda} = 0$. This reflects the overall conservation of momentum in the x-direction.

Energy conservation

Since neither L nor any of the constraints are explicitly time-dependent, the Hamiltonian is conserved. And since T is homogeneous of degree two in the generalized velocities, we have $H = E = T + U$:

$$E = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 + MgY + mgy \ . \tag{8.94}$$

Equations of motion

We have $n = 6$ generalized coordinates and $k = 4$ constraints. Thus, there are four undetermined multipliers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ used to impose the constraints. This makes for ten unknowns:

$$X \ , \ Y \ , \ x \ , \ y \ , \ \phi \ , \ \theta \ , \ \lambda_1 \ , \ \lambda_2 \ , \ \lambda_3 \ , \ \lambda_4 \ . \tag{8.95}$$

Accordingly, we have ten equations: six equations of motion plus the four equations of constraint. The equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \ . \tag{8.96}$$

Taking each generalized coordinate in turn, the equations of motion are thus

$$\begin{aligned} M\ddot{X} &= \lambda_1 - \lambda_3 \\ M\ddot{Y} &= -Mg + \lambda_2 - \lambda_4 \\ m\ddot{x} &= \lambda_3 \\ m\ddot{y} &= -mg + \lambda_4 \\ I\,\ddot{\phi} &= -R\,\lambda_1 \\ 0 &= -R\cos\theta\,\lambda_3 - R\sin\theta\,\lambda_4 \ . \end{aligned} \tag{8.97}$$

Along with the four constraint equations, these determine the motion of the system. Note that the last of the equations of motion, for the generalized coordinate $q_\sigma = \theta$, says that $Q_\theta = 0$, which means that the force of constraint on the point mass is radial. Were the point mass replaced by a rolling object, there would be an angular component to this constraint in order that there be no slippage.

Implementation of constraints

We now use the constraint equations to eliminate X, Y, x, and y in terms of θ and ϕ:

$$X = R\phi \quad , \quad Y = R \quad , \quad x = R\phi + R\sin\theta \quad , \quad y = R(1 - \cos\theta) \ . \tag{8.98}$$

We also need the derivatives:

$$\dot{x} = R\,\dot{\phi} + R\cos\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\,\ddot{\phi} + R\cos\theta\,\ddot{\theta} - R\sin\theta\,\dot{\theta}^2 \ , \tag{8.99}$$

and

$$\dot{y} = R\sin\theta\,\dot{\theta} \quad , \quad \ddot{y} = R\sin\theta\,\ddot{\theta} + R\cos\theta\,\dot{\theta}^2 \ , \tag{8.100}$$

as well as

$$\dot{X} = R\,\dot{\phi} \quad , \quad \ddot{X} = R\,\ddot{\phi} \quad , \quad \dot{Y} = 0 \quad , \quad \ddot{Y} = 0 \ . \tag{8.101}$$

We now may write the conserved charge as

$$\Lambda = \frac{1}{R}(I + MR^2 + mR^2)\,\dot{\phi} + mR\cos\theta\,\dot{\theta} \ . \tag{8.102}$$

This, in turn, allows us to eliminate $\dot{\phi}$ in terms of $\dot{\theta}$ and the constant Λ:

$$\dot{\phi} = \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR} - \dot{\theta}\cos\theta\right) , \tag{8.103}$$

where

$$\gamma = \frac{mR^2}{I + MR^2} . \tag{8.104}$$

The energy is then

$$\begin{aligned} E &= \tfrac{1}{2}(I + MR^2)\,\dot{\phi}^2 + \tfrac{1}{2}m\big(R^2\,\dot{\phi}^2 + R^2\,\dot{\theta}^2 + 2R^2\cos\theta\,\dot{\phi}\,\dot{\theta}\big) + MgR + mgR(1-\cos\theta) \\ &= \tfrac{1}{2}mR^2\left\{\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) + \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR}\right)^2 + \frac{2Mg}{mR}\right\} . \end{aligned} \tag{8.105}$$

The last two terms inside the big bracket are constant, so we can write this as

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) = \frac{4gk}{R} . \tag{8.106}$$

Here, k is a dimensionless measure of the energy of the system, after subtracting the aforementioned constants. If $k > 1$, then $\dot{\theta}^2 > 0$ for all θ, which would result in 'loop-the-loop' motion of the point mass inside the hoop – provided, that is, the normal force of the hoop doesn't vanish and the point mass doesn't detach from the hoop's surface.

Equation motion for $\theta(t)$

The equation of motion for θ obtained by eliminating all other variables from the original set of ten equations is the same as $\dot{E} = 0$, and may be written

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\ddot{\theta} + \left(\frac{\gamma\sin\theta\cos\theta}{1+\gamma}\right)\dot{\theta}^2 = -\frac{g}{R} . \tag{8.107}$$

We can use this to write $\ddot{\theta}$ in terms of $\dot{\theta}^2$, and, after invoking eqn. 8.106, in terms of θ itself. We find

$$\begin{aligned} \dot{\theta}^2 &= \frac{4g}{R}\cdot\left(\frac{1+\gamma}{1+\gamma\sin^2\!\theta}\right)\big(k - \sin^2\!\tfrac{1}{2}\theta\big) \\ \ddot{\theta} &= -\frac{g}{R}\cdot\frac{(1+\gamma)\sin\theta}{\big(1+\gamma\sin^2\!\theta\big)^2}\Big[4\gamma\,\big(k - \sin^2\!\tfrac{1}{2}\theta\big)\cos\theta + 1 + \gamma\sin^2\!\theta\Big] . \end{aligned} \tag{8.108}$$

Forces of constraint

We can solve for the λ_j, and thus obtain the forces of constraint $Q_\sigma = \sum_j \lambda_j\,\frac{\partial G_j}{\partial q_\sigma}$.

$$\begin{aligned}
\lambda_3 &= m\ddot{x} = mR\,\ddot{\phi} + mR\cos\theta\,\ddot{\theta} - mR\sin\theta\,\dot{\theta}^2 \\
&= \frac{mR}{1+\gamma}\left[\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta\right] \\
\lambda_4 &= m\ddot{y} + mg = mg + mR\sin\theta\,\ddot{\theta} + mR\cos\theta\,\dot{\theta}^2 \\
&= mR\Big[\ddot{\theta}\,\sin\theta + \dot{\theta}^2\sin\theta + \frac{g}{R}\Big] \\
\lambda_1 &= -\frac{I}{R}\,\ddot{\phi} = \frac{(1+\gamma)I}{mR^2}\,\lambda_3 \\
\lambda_2 &= (M+m)g + m\ddot{y} = \lambda_4 + Mg \ .
\end{aligned} \tag{8.109}$$

One can check that $\lambda_3\cos\theta + \lambda_4\sin\theta = 0$.

The condition that the normal force of the hoop on the point mass vanish is $\lambda_3 = 0$, which entails $\lambda_4 = 0$. This gives

$$-\left(1+\gamma\sin^2\!\theta\right)\,\cos\theta = 4(1+\gamma)\left(k - \sin^2\!\tfrac{1}{2}\theta\right)\ . \tag{8.110}$$

Note that this requires $\cos\theta < 0$, *i.e.* the point of detachment lies above the horizontal diameter of the hoop. Clearly if k is sufficiently large, the equality cannot be satisfied, and the point mass executes a periodic 'loop-the-loop' motion. In particular, setting $\theta = \pi$, we find that

$$k_{\rm c} = 1 + \frac{1}{4(1+\gamma)}\ . \tag{8.111}$$

If $k > k_{\rm c}$, then there is periodic 'loop-the-loop' motion. If $k < k_{\rm c}$, then the point mass may detach at a critical angle θ^*, but only if the motion allows for $\cos\theta < 0$. From the energy conservation equation, we have that the maximum value of θ achieved occurs when $\dot{\theta} = 0$, which means

$$\cos\theta_{\rm max} = 1 - 2k\ . \tag{8.112}$$

If $\frac{1}{2} < k < k_{\rm c}$, then, we have the possibility of detachment. This means the energy must be large enough but not too large.

Chapter 9

Central Forces and Orbital Mechanics

9.1 Reduction to a one-body problem

Consider two particles interacting via a potential $U(\boldsymbol{r}_1, \boldsymbol{r}_2) = U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big)$. Such a potential, which depends only on the relative distance between the particles, is called a *central* potential. The Lagrangian of this system is then

$$L = T - U = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \ . \tag{9.1}$$

9.1.1 Center-of-mass (CM) and relative coordinates

The two-body central force problem may always be reduced to two independent one-body problems, by transforming to center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates (see Fig. 9.1), *viz.*

$$\begin{aligned} \boldsymbol{R} &= \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \quad , \qquad & \boldsymbol{r}_1 &= \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\, \boldsymbol{r} \\ \boldsymbol{r} &= \boldsymbol{r}_1 - \boldsymbol{r}_2 \quad , \qquad & \boldsymbol{r}_2 &= \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\, \boldsymbol{r} \end{aligned} \tag{9.2}$$

We then have

$$\begin{aligned} L &= \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1{}^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2{}^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \\ &= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r) \ . \end{aligned} \tag{9.3}$$

where

$$\begin{aligned} M &= m_1 + m_2 \qquad & &\text{(total mass)} \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \qquad & &\text{(reduced mass)} \ . \end{aligned} \tag{9.4}$$

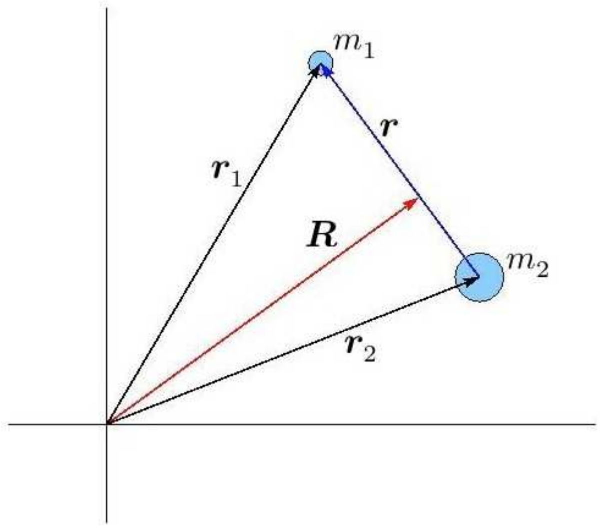


Figure 9.1: Center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates.

9.1.2 Solution to the CM problem

We have $\partial L/\partial \boldsymbol{R} = 0$, which gives $\dot{\boldsymbol{R}} = 0$ and hence

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)\, t \ . \tag{9.5}$$

Thus, the CM problem is trivial. The center-of-mass moves at constant velocity.

9.1.3 Solution to the relative coordinate problem

`Angular momentum conservation:` We have that $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant of the motion. This means that the motion $\boldsymbol{r}(t)$ is confined to a plane perpendicular to $\boldsymbol{\ell}$. It is convenient to adopt two-dimensional polar coordinates (r, ϕ). The magnitude of $\boldsymbol{\ell}$ is

$$\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}} \tag{9.6}$$

where $d\mathcal{A} = \frac{1}{2} r^2 d\phi$ is the differential element of area subtended relative to the force center. *The relative coordinate vector for a central force problem subtends equal areas in equal times.* This is known as *Kepler's Second Law.*

`Energy conservation:` The equation of motion for the relative coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \quad \Rightarrow \quad \mu \ddot{\boldsymbol{r}} = -\frac{\partial U}{\partial \boldsymbol{r}} \ . \tag{9.7}$$

Taking the dot product with $\dot{\boldsymbol{r}}$, we have

$$\begin{aligned} 0 &= \mu \ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\partial U}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \\ &= \frac{d}{dt}\Big\{ \tfrac{1}{2}\mu \dot{\boldsymbol{r}}^2 + U(r) \Big\} = \frac{dE}{dt} \ . \end{aligned} \tag{9.8}$$

Thus, the relative coordinate contribution to the total energy is itself conserved. The total energy is of course $E_{\rm tot} = E + \frac{1}{2}M\dot{\boldsymbol{R}}^2$.

Since $\boldsymbol{\ell}$ is conserved, and since $\boldsymbol{r} \cdot \boldsymbol{\ell} = 0$, all motion is confined to a plane perpendicular to $\boldsymbol{\ell}$. Choosing coordinates such that $\hat{\boldsymbol{z}} = \hat{\boldsymbol{\ell}}$, we have

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ U_{\rm eff}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \ . \end{aligned} \tag{9.9}$$

`Integration of the Equations of Motion, Step I:` The second order equation for $r(t)$ is

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{dU(r)}{dr} = -\frac{dU_{\rm eff}(r)}{dr} \ . \tag{9.10}$$

However, conservation of energy reduces this to a first order equation, via

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\Big(E - U_{\rm eff}(r)\Big)} \quad \Rightarrow \quad dt = \pm\frac{\sqrt{\frac{\mu}{2}}\,dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}} \ . \tag{9.11}$$

This gives $t(r)$, which must be inverted to obtain $r(t)$. In principle this is possible. Note that a constant of integration also appears at this stage – call it $r_0 = r(t=0)$.

`Integration of the Equations of Motion, Step II:` After finding $r(t)$ one can integrate to find $\phi(t)$ using the conservation of ℓ:

$$\dot{\phi} = \frac{\ell}{\mu r^2} \quad \Rightarrow \quad d\phi = \frac{\ell}{\mu r^2(t)}\,dt \ . \tag{9.12}$$

This gives $\phi(t)$, and introduces another constant of integration – call it $\phi_0 = \phi(t=0)$.

`Pause to Reflect on the Number of Constants:` Confined to the plane perpendicular to $\boldsymbol{\ell}$, the relative coordinate vector has two degrees of freedom. The equations of motion are second order in time, leading to *four* constants of integration. Our four constants are E, ℓ, r_0, and ϕ_0.

The original problem involves two particles, hence six positions and six velocities, making for 12 initial conditions. Six constants are associated with the CM system: $\boldsymbol{R}(0)$ and $\dot{\boldsymbol{R}}(0)$. The six remaining constants associated with the relative coordinate system are $\boldsymbol{\ell}$ (three components), E, r_0, and ϕ_0.

`Geometric Equation of the Orbit:` From $\ell = \mu r^2\dot{\phi}$, we have

$$\frac{d}{dt} = \frac{\ell}{\mu r^2}\,\frac{d}{d\phi} \ , \tag{9.13}$$

leading to

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2}\,F(r) + r \tag{9.14}$$

where $F(r) = -dU(r)/dr$ is the magnitude of the central force. This second order equation may be reduced to a first order one using energy conservation:

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\rm eff}(r)\ . \end{aligned} \tag{9.15}$$

Thus,

$$d\phi = \pm\frac{\ell}{\sqrt{2\mu}}\cdot\frac{dr}{r^2\sqrt{E - U_{\rm eff}(r)}}\ , \tag{9.16}$$

which can be integrated to yield $\phi(r)$, and then inverted to yield $r(\phi)$. Note that only one integration need be performed to obtain the geometric shape of the orbit, while two integrations – one for $r(t)$ and one for $\phi(t)$ – must be performed to obtain the full motion of the system.

It is sometimes convenient to rewrite this equation in terms of the variable $s = 1/r$:

$$\frac{d^2s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\,F\big(s^{-1}\big)\ . \tag{9.17}$$

As an example, suppose the geometric orbit is $r(\phi) = k\,e^{\alpha\phi}$, known as a logarithmic spiral. What is the force? We invoke (9.14), with $s''(\phi) = \alpha^2\,s$, yielding

$$F\big(s^{-1}\big) = -(1+\alpha^2)\,\frac{\ell^2}{\mu}\,s^3 \quad\Rightarrow\quad F(r) = -\frac{C}{r^3} \tag{9.18}$$

with

$$\alpha^2 = \frac{\mu C}{\ell^2} - 1\ . \tag{9.19}$$

The general solution for $s(\phi)$ for this force law is

$$s(\phi) = \begin{cases} A\cosh(\alpha\phi) + B\sinh(-\alpha\phi) & \text{if } \ell^2 > \mu C \\ \\ A'\,\cos\big(|\alpha|\phi\big) + B'\,\sin\big(|\alpha|\phi\big) & \text{if } \ell^2 < \mu C\ . \end{cases} \tag{9.20}$$

The logarithmic spiral shape is a special case of the first kind of orbit.

9.2 Almost Circular Orbits

A circular orbit with $r(t) = r_0$ satisfies $\ddot{r} = 0$, which means that $U'_{\rm eff}(r_0) = 0$, which says that $F(r_0) = -\ell^2/\mu r_0^3$. This is negative, indicating that a circular orbit is possible only if the force is attractive over

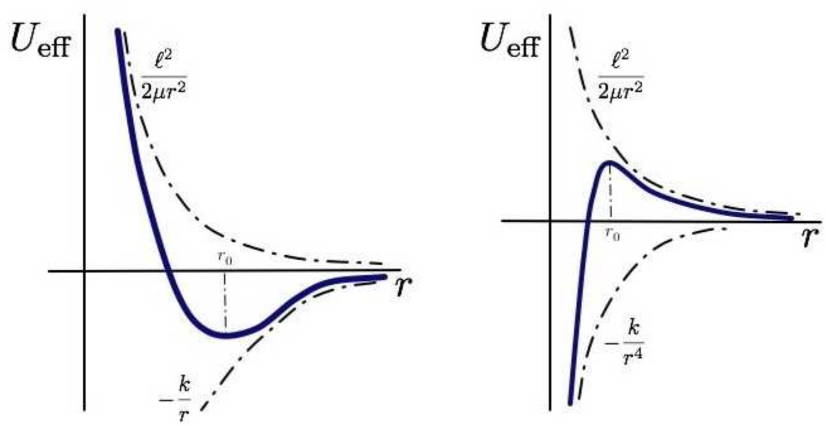


Figure 9.2: Stable and unstable circular orbits. Left panel: $U(r) = -k/r$ produces a stable circular orbit. Right panel: $U(r) = -k/r^4$ produces an unstable circular orbit.

some range of distances. Since $\dot{r} = 0$ as well, we must also have $E = U_{\rm eff}(r_0)$. An almost circular orbit has $r(t) = r_0 + \eta(t)$, where $|\eta/r_0| \ll 1$. To lowest order in η, one derives the equations

$$\frac{d^2\eta}{dt^2} = -\omega^2\,\eta \qquad , \qquad \omega^2 = \frac{1}{\mu}\,U''_{\rm eff}(r_0)\ . \tag{9.21}$$

If $\omega^2 > 0$, the circular orbit is *stable* and the perturbation oscillates harmonically. If $\omega^2 < 0$, the circular orbit is *unstable* and the perturbation grows exponentially. For the geometric shape of the perturbed orbit, we write $r = r_0 + \eta$, and from (9.14) we obtain

$$\frac{d^2\eta}{d\phi^2} = \bigg(\frac{\mu r_0^4}{\ell^2}\,F'(r_0) - 3\bigg)\,\eta = -\beta^2\,\eta\ , \tag{9.22}$$

with

$$\beta^2 = 3 + \frac{d\ln F(r)}{d\ln r}\bigg|_{r_0}\ . \tag{9.23}$$

The solution here is

$$\eta(\phi) = \eta_0\,\cos\beta(\phi - \delta_0)\ , \tag{9.24}$$

where η_0 and δ_0 are initial conditions. Setting $\eta = \eta_0$, we obtain the sequence of ϕ values

$$\phi_n = \delta_0 + \frac{2\pi n}{\beta}\ , \tag{9.25}$$

at which $\eta(\phi)$ is a local maximum, *i.e.* at *apoapsis*, where $r = r_0 + \eta_0$. Setting $r = r_0 - \eta_0$ is the condition for closest approach, *i.e. periapsis*. This yields the identical set if angles, just shifted by π. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi\big(\beta^{-1} - 1\big)\ , \tag{9.26}$$

is the amount by which the apsides (*i.e.* periapsis and apoapsis) *precess* during each cycle. If $\beta > 1$, the apsides advance, *i.e.* it takes less than a complete revolution $\Delta\phi = 2\pi$ between successive periapses. If

$\beta < 1$, the apsides retreat, and it takes longer than a complete revolution between successive periapses. The situation is depicted in Fig. 9.3 for the case $\beta = 1.1$. Below, we will exhibit a soluble model in which the precessing orbit may be determined exactly. Finally, note that if $\beta = p/q$ is a rational number, then the orbit is *closed*, *i.e.* it eventually retraces itself, after every q revolutions.

As an example, let $F(r) = -kr^{-\alpha}$. Solving for a circular orbit, we write

$$U'_{\rm eff}(r) = \frac{k}{r^\alpha} - \frac{\ell^2}{\mu r^3} = 0 \ , \tag{9.27}$$

which has a solution only for $k > 0$, corresponding to an attractive potential. We then find

$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/(3-\alpha)} \ , \tag{9.28}$$

and $\beta^2 = 3 - \alpha$. The shape of the perturbed orbits follows from $\eta'' = -\beta^2\,\eta$. Thus, while circular orbits exist whenever $k > 0$, small perturbations about these orbits are stable only for $\beta^2 > 0$, *i.e.* for $\alpha < 3$. One then has $\eta(\phi) = A\cos\beta(\phi - \phi_0)$. The perturbed orbits are closed, at least to lowest order in η, for $\alpha = 3 - (p/q)^2$, *i.e.* for $\beta = p/q$. The situation is depicted in Fig. 9.2, for the potentials $U(r) = -k/r$ ($\alpha = 2$) and $U(r) = -k/r^4$ ($\alpha = 5$).

9.3 Precession in a Soluble Model

Let's start with the answer and work backwards. Consider the geometrical orbit,

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos\beta\phi} \ . \tag{9.29}$$

Our interest is in bound orbits, for which $0 \le \epsilon < 1$ (see Fig. 9.3). What sort of potential gives rise to this orbit? Writing $s = 1/r$ as before, we have

$$s(\phi) = s_0\,(1 - \varepsilon\cos\beta\phi) \ . \tag{9.30}$$

Substituting into (9.17), we have

$$\begin{aligned} -\frac{\mu}{\ell^2 s^2}\,F\big(s^{-1}\big) &= \frac{d^2 s}{d\phi^2} + s \\ &= \beta^2\,s_0\,\epsilon\cos\beta\phi + s \\ &= (1-\beta^2)\,s + \beta^2\,s_0 \ , \end{aligned} \tag{9.31}$$

from which we conclude

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \ , \tag{9.32}$$

with

$$k = \beta^2 s_0\,\frac{\ell^2}{\mu} \quad , \quad C = (\beta^2 - 1)\,\frac{\ell^2}{\mu} \ . \tag{9.33}$$

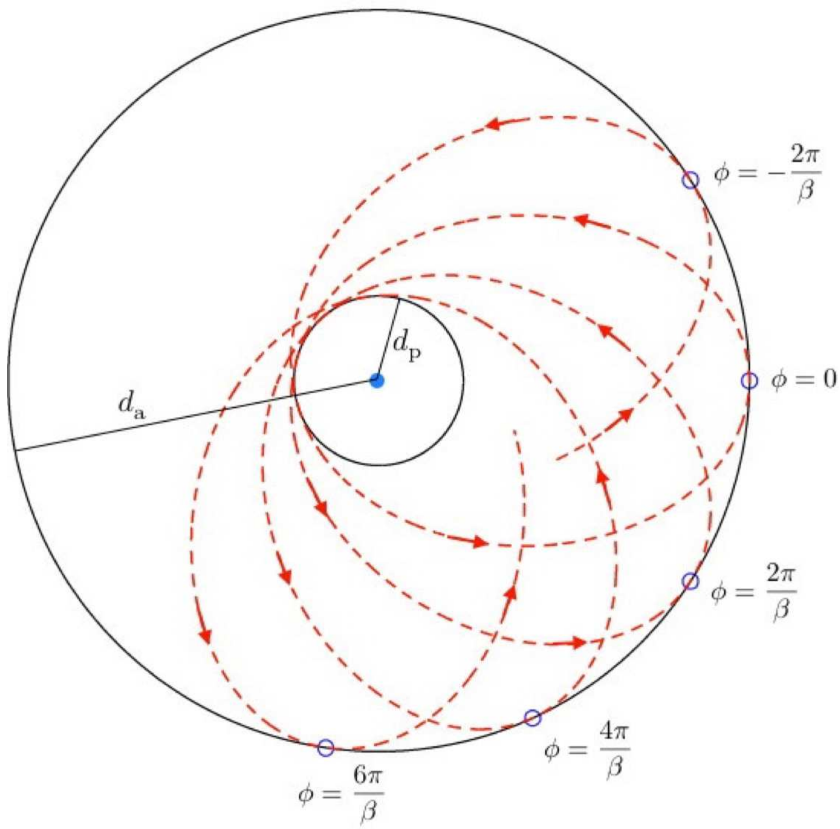


Figure 9.3: Precession in a soluble model, with geometric orbit $r(\phi) = r_0/(1 - \varepsilon \cos \beta\phi)$, shown here with $\beta = 1.1$. Periapsis and apoapsis advance by $\Delta\phi = 2\pi(1 - \beta^{-1})$ per cycle.

The corresponding potential is

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} + U_\infty \ , \tag{9.34}$$

where U_∞ is an arbitrary constant, conveniently set to zero. If μ and C are given, we have

$$r_0 = \frac{\ell^2}{\mu k} + \frac{C}{k} \quad , \quad \beta = \sqrt{1 + \frac{\mu C}{\ell^2}} \ . \tag{9.35}$$

When $C = 0$, these expressions recapitulate those from the Kepler problem. Note that when $\ell^2 + \mu C < 0$ that the effective potential is monotonically increasing as a function of r. In this case, the angular momentum barrier is overwhelmed by the (attractive, $C < 0$) inverse square part of the potential, and $U_{\text{eff}}(r)$ is monotonically increasing. The orbit then passes through the force center. It is a useful exercise to derive the total energy for the orbit,

$$E = (\varepsilon^2 - 1)\,\frac{\mu k^2}{2(\ell^2 + \mu C)} \quad \Longleftrightarrow \quad \varepsilon^2 = 1 + \frac{2E(\ell^2 + \mu C)}{\mu k^2} \ . \tag{9.36}$$

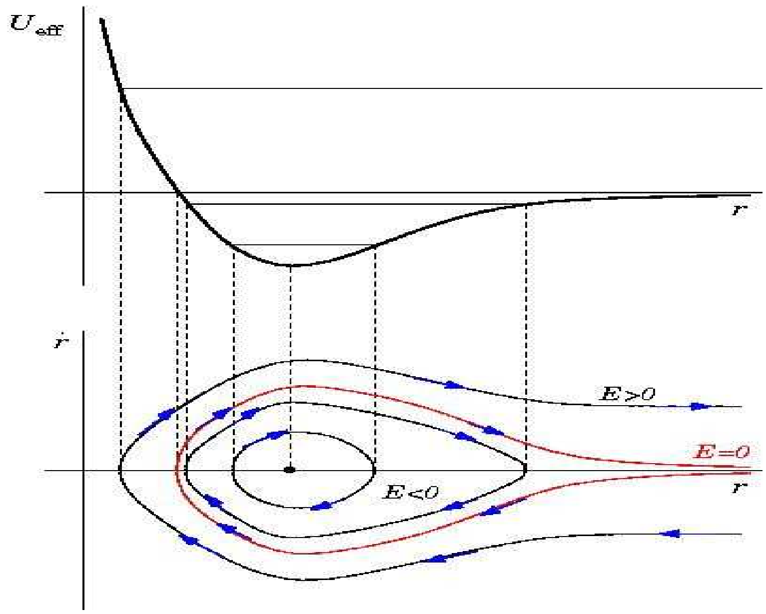


Figure 9.4: The effective potential for the Kepler problem, and associated phase curves. The orbits are geometrically described as conic sections: hyperbolae ($E > 0$), parabolae ($E = 0$), ellipses ($E_{\min} < E < 0$), and circles ($E = E_{\min}$).

9.4 The Kepler Problem: $U(r) = -k\,r^{-1}$

9.4.1 Geometric shape of orbits

The force is $F(r) = -kr^{-2}$, hence the equation for the geometric shape of the orbit is

$$\frac{d^2s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\,F(s^{-1}) = \frac{\mu k}{\ell^2}\ , \tag{9.37}$$

with $s = 1/r$. Thus, the most general solution is

$$s(\phi) = s_0 - C\cos(\phi - \phi_0)\ , \tag{9.38}$$

where C and ϕ_0 are constants. Thus,

$$r(\phi) = \frac{r_0}{1 - \varepsilon\cos(\phi - \phi_0)}\ , \tag{9.39}$$

where $r_0 = \ell^2/\mu k$ and where we have defined a new constant $\varepsilon \equiv C r_0$.

9.4.2 Laplace-Runge-Lenz vector

Consider the vector

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{\ell} - \mu k\,\hat{\boldsymbol{r}}\ , \tag{9.40}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector pointing in the direction of $\boldsymbol{r}$. We may now show that $\boldsymbol{A}$ is conserved:

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} &= \frac{d}{dt}\Big\{\boldsymbol{p}\times\boldsymbol{\ell} - \mu k\frac{\boldsymbol{r}}{r}\Big\} \\ &= \dot{\boldsymbol{p}}\times\boldsymbol{\ell} + \boldsymbol{p}\times\dot{\boldsymbol{\ell}} - \mu k\,\frac{r\dot{\boldsymbol{r}} - \boldsymbol{r}\dot{r}}{r^2} \\ &= -\frac{k\boldsymbol{r}}{r^3}\times(\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2} \\ &= -\mu k\,\frac{\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})}{r^3} + \mu k\,\frac{\dot{\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{r})}{r^3} - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2} = 0\ .\end{aligned} \tag{9.41}$$

So $\boldsymbol{A}$ is a conserved vector which clearly lies in the plane of the motion. $\boldsymbol{A}$ points toward periapsis, *i.e.* toward the point of closest approach to the force center.

Let's assume *apo*apsis occurs at $\phi = \phi_0$. Then

$$\boldsymbol{A}\cdot\boldsymbol{r} = -Ar\cos(\phi-\phi_0) = \ell^2 - \mu k r \tag{9.42}$$

giving

$$r(\phi) = \frac{\ell^2}{\mu k - A\cos(\phi-\phi_0)} = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos(\phi-\phi_0)}\ , \tag{9.43}$$

where

$$\varepsilon = \frac{A}{\mu k} \qquad , \qquad a(1-\varepsilon^2) = \frac{\ell^2}{\mu k}\ . \tag{9.44}$$

The orbit is a *conic section* with eccentricity ε. Squaring $\boldsymbol{A}$, one finds

$$\begin{aligned}\boldsymbol{A}^2 &= (\boldsymbol{p}\times\boldsymbol{\ell})^2 - 2\mu k\,\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\times\boldsymbol{\ell} + \mu^2k^2 \\ &= p^2\ell^2 - 2\mu\ell^2\,\frac{k}{r} + \mu^2k^2 \\ &= 2\mu\ell^2\bigg(\frac{p^2}{2\mu} - \frac{k}{r} + \frac{\mu k^2}{2\ell^2}\bigg) = 2\mu\ell^2\bigg(E + \frac{\mu k^2}{2\ell^2}\bigg)\end{aligned} \tag{9.45}$$

and thus

$$a = -\frac{k}{2E} \quad , \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}\ . \tag{9.46}$$

9.4.3 Kepler orbits are conic sections

There are four classes of conic sections:

- *Circle:* $\varepsilon = 0$, $E = -\mu k^2/2\ell^2$, radius $a = \ell^2/\mu k$. The force center lies at the center of circle.
- *Ellipse:* $0 < \varepsilon < 1$, $-\mu k^2/2\ell^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1-\varepsilon^2}$. The force center is at one of the foci.
- *Parabola:* $\varepsilon = 1$, $E = 0$, force center is the focus.

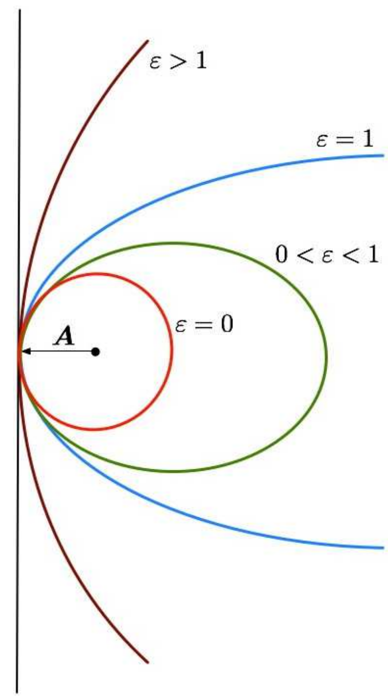


Figure 9.5: Keplerian orbits are conic sections, classified according to eccentricity: hyperbola ($\epsilon > 1$), parabola ($\epsilon = 1$), ellipse ($0 < \epsilon < 1$), and circle ($\epsilon = 0$). The Laplace-Runge-Lenz vector, $\boldsymbol{A}$, points toward periapsis.

- *Hyperbola:* $\varepsilon > 1$, $E > 0$, force center is closest focus (attractive) or farthest focus (repulsive).

To see that the Keplerian orbits are indeed conic sections, consider the ellipse of Fig. 9.6. The law of cosines gives

$$\rho^2 = r^2 + 4f^2 - 4rf\cos\phi \ , \tag{9.47}$$

where $f = \varepsilon a$ is the focal distance. Now for any point on an ellipse, the sum of the distances to the left and right foci is a constant, and taking $\phi = 0$ we see that this constant is $2a$. Thus, $\rho = 2a - r$, and we have

$$\begin{aligned} (2a-r)^2 = 4a^2 - 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(1-\varepsilon\cos\phi) &= a(1-\varepsilon^2) \ . \end{aligned} \tag{9.48}$$

Thus, we obtain

$$r(\phi) = \frac{a\,(1-\varepsilon^2)}{1-\varepsilon\cos\phi} \ , \tag{9.49}$$

and we therefore conclude that

$$r_0 = \frac{\ell^2}{\mu k} = a\,(1-\varepsilon^2) \ . \tag{9.50}$$

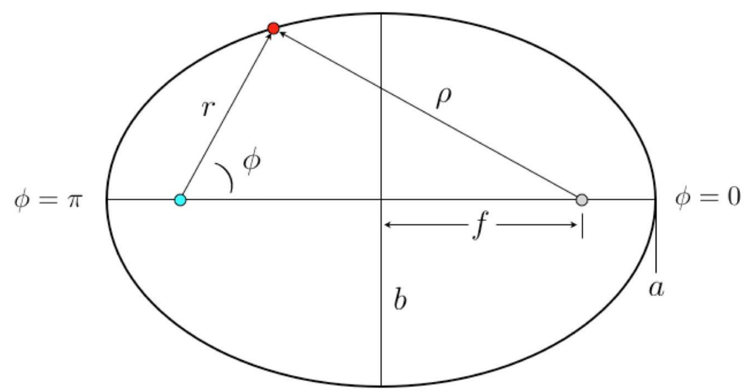


Figure 9.6: The Keplerian ellipse, with the force center at the left focus. The focal distance is $f = \varepsilon a$, where a is the semimajor axis length. The length of the semiminor axis is $b = \sqrt{1-\varepsilon^2}\,a$.

Next let us examine the energy,

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \tfrac{1}{2}\mu\bigg(\frac{\ell}{\mu r^2}\,\frac{dr}{d\phi}\bigg)^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \\ &= \frac{\ell^2}{2\mu}\left(\frac{ds}{d\phi}\right)^2 + \frac{\ell^2}{2\mu}\,s^2 - ks\ , \end{aligned} \tag{9.51}$$

with

$$s = \frac{1}{r} = \frac{\mu k}{\ell^2}\Big(1 - \varepsilon\cos\phi\Big)\ . \tag{9.52}$$

Thus,

$$\frac{ds}{d\phi} = \frac{\mu k}{\ell^2}\,\varepsilon\,\sin\phi\ , \tag{9.53}$$

and

$$\begin{aligned} \left(\frac{ds}{d\phi}\right)^2 &= \frac{\mu^2 k^2}{\ell^4}\,\varepsilon^2\,\sin^2\!\phi \\ &= \frac{\mu^2 k^2\varepsilon^2}{\ell^4} - \left(\frac{\mu k}{\ell^2} - s\right)^2 \\ &= -s^2 + \frac{2\mu k}{\ell^2}\,s + \frac{\mu^2 k^2}{\ell^4}\,\big(\varepsilon^2 - 1\big)\ . \end{aligned} \tag{9.54}$$

Substituting this into eqn. 9.51, we obtain

$$E = \frac{\mu k^2}{2\ell^2}\,\big(\varepsilon^2 - 1\big)\ . \tag{9.55}$$

For the hyperbolic orbit, depicted in Fig. 9.7, we have $r - \rho = \mp 2a$, depending on whether we are on the attractive or repulsive branch, respectively. We then have

$$\begin{aligned} (r \pm 2a)^2 = 4a^2 \pm 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow\quad r(\pm 1 + \varepsilon\cos\phi) &= a(\varepsilon^2 - 1)\ . \end{aligned} \tag{9.56}$$

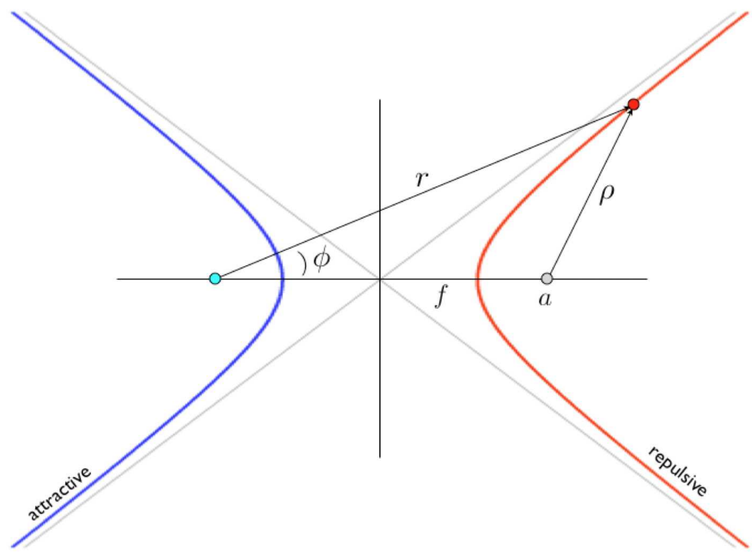


Figure 9.7: The Keplerian hyperbolae, with the force center at the left focus. The left (blue) branch corresponds to an attractive potential, while the right (red) branch corresponds to a repulsive potential. The equations of these branches are $r = \rho = \mp 2a$, where the top sign corresponds to the left branch and the bottom sign to the right branch.

This yields

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon \cos\phi} \ . \tag{9.57}$$

9.4.4 Period of bound Kepler orbits

From $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}}$, the period is $\tau = 2\mu \mathcal{A}/\ell$, where $\mathcal{A} = \pi a^2 \sqrt{1-\varepsilon^2}$ is the area enclosed by the orbit. This gives

$$\tau = 2\pi \left(\frac{\mu a^3}{k}\right)^{1/2} = 2\pi \left(\frac{a^3}{GM}\right)^{1/2} \tag{9.58}$$

as well as

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} \ , \tag{9.59}$$

where $k = Gm_1 m_2$ and $M = m_1 + m_2$ is the total mass. For planetary orbits, $m_1 = M_\odot$ is the solar mass and $m_2 = m_\mathrm{p}$ is the planetary mass. We then have

$$\frac{a^3}{\tau^2} = \left(1 + \frac{m_\mathrm{p}}{M_\odot}\right) \frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2} \ , \tag{9.60}$$

which is to an excellent approximation independent of the planetary mass. (Note that $m_\mathrm{p}/M_\odot \approx 10^{-3}$ even for Jupiter.) This analysis also holds, *mutatis mutandis,* for the case of satellites orbiting the earth, and indeed in any case where the masses are grossly disproportionate in magnitude.

9.4.5 Escape velocity

The threshold for escape from a gravitational potential occurs at $E = 0$. Since $E = T + U$ is conserved, we determine the *escape velocity* for a body a distance r from the force center by setting

$$E = 0 = \tfrac{1}{2}\mu v_{\rm esc}^2(t) - \frac{GMm}{r} \qquad \Rightarrow \qquad v_{\rm esc}(r) = \sqrt{\frac{2G(M+m)}{r}} \ . \tag{9.61}$$

When $M \gg m$, $v_{\rm esc}(r) = \sqrt{2GM/r}$. Thus, for an object at the surface of the earth, $v_{\rm esc} = \sqrt{2gR_{\rm E}} = 11.2\,{\rm km/s}$.

9.4.6 Satellites and spacecraft

A satellite in a circular orbit a distance h above the earth's surface has an orbital period

$$\tau = \frac{2\pi}{\sqrt{GM_{\rm E}}}\,(R_{\rm E} + h)^{3/2} \ , \tag{9.62}$$

where we take $m_{\rm satellite} \ll M_{\rm E}$. For low earth orbit (LEO), $h \ll R_{\rm E} = 6.37 \times 10^6\,{\rm m}$, in which case $\tau_{\rm LEO} = 2\pi\sqrt{R_{\rm E}/g} = 1.4\,{\rm hr}$.

Consider a weather satellite in an elliptical orbit whose closest approach to the earth (perigee) is 200 km above the earth's surface and whose farthest distance (apogee) is 7200 km above the earth's surface. What is the satellite's orbital period? From Fig. 9.6, we see that

$$\begin{aligned} d_{\rm apogee} &= R_{\rm E} + 7200\,{\rm km} = 13571\,{\rm km} \\ d_{\rm perigee} &= R_{\rm E} + 200\,{\rm km} = 6971\,{\rm km} \\ a &= \tfrac{1}{2}(d_{\rm apogee} + d_{\rm perigee}) = 10071\,{\rm km} \ . \end{aligned} \tag{9.63}$$

We then have

$$\tau = \left(\frac{a}{R_{\rm E}}\right)^{3/2} \cdot \tau_{\rm LEO} \approx 2.65\,{\rm hr} \ . \tag{9.64}$$

What happens if a spacecraft in orbit about the earth fires its rockets? Clearly the energy and angular momentum of the orbit will change, and this means the shape will change. If the rockets are fired (in the direction of motion) at perigee, then perigee itself is unchanged, because $\boldsymbol{v}\cdot\boldsymbol{r} = 0$ is left unchanged at this point. However, E is increased, hence the eccentricity $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$ increases. This is the most efficient way of boosting a satellite into an orbit with higher eccentricity. Conversely, and somewhat paradoxically, when a satellite in LEO loses energy due to frictional drag of the atmosphere, the energy E decreases. Initially, because the drag is weak and the atmosphere is isotropic, the orbit remains circular. Since E decreases, $\langle T\rangle = -E$ must *increase*, which means that the frictional forces cause the satellite to speed up!

9.4.7 Two examples of orbital mechanics

- Problem #1: At perigee of an elliptical Keplerian orbit, a satellite receives an impulse $\Delta\boldsymbol{p} = p_0\hat{\boldsymbol{r}}$. Describe the resulting orbit.

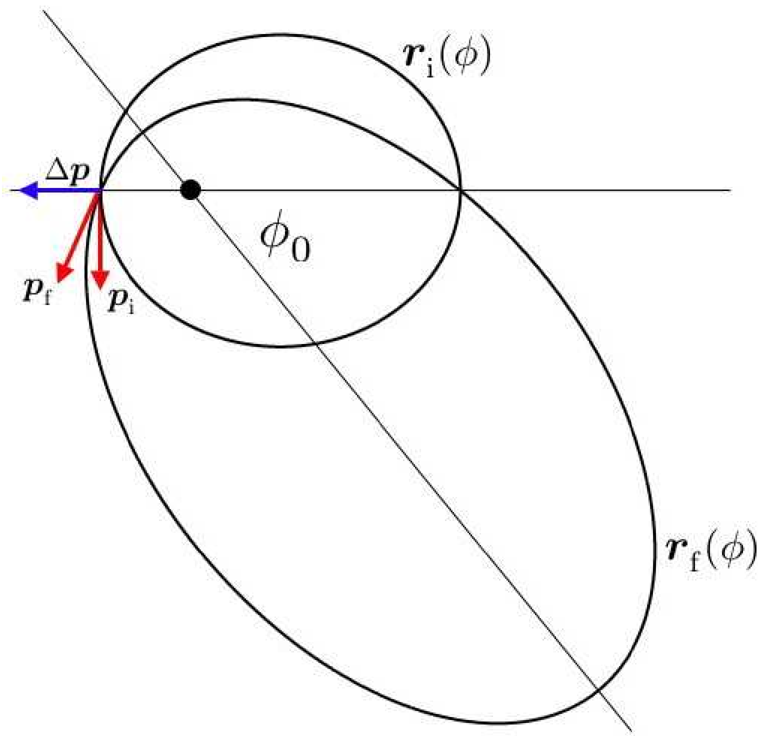


Figure 9.8: At perigee of an elliptical orbit $r_{\rm i}(\phi)$, a radial impulse $\Delta\boldsymbol{p}$ is applied. The shape of the resulting orbit $r_{\rm f}(\phi)$ is shown.

- Solution #1: Since the impulse is radial, the angular momentum $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$ is unchanged. The energy, however, does change, with $\Delta E = p_0^2/2\mu$. Thus,

$$\varepsilon_{\rm f}^2 = 1 + \frac{2E_{\rm f}\ell^2}{\mu k^2} = \varepsilon_{\rm i}^2 + \left(\frac{\ell p_0}{\mu k}\right)^2 . \tag{9.65}$$

The new semimajor axis length is

$$\begin{aligned} a_{\rm f} &= \frac{\ell^2/\mu k}{1-\varepsilon_{\rm f}^2} = a_{\rm i} \cdot \frac{1-\varepsilon_{\rm i}^2}{1-\varepsilon_{\rm f}^2} \\ &= \frac{a_{\rm i}}{1-(a_{\rm i} p_0^2/\mu k)} . \end{aligned} \tag{9.66}$$

The shape of the final orbit must also be a Keplerian ellipse, described by

$$r_{\rm f}(\phi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1-\varepsilon_{\rm f}\cos(\phi+\delta)} , \tag{9.67}$$

where the phase shift δ is determined by setting

$$r_{\rm i}(\pi) = r_{\rm f}(\pi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1+\varepsilon_{\rm i}} . \tag{9.68}$$

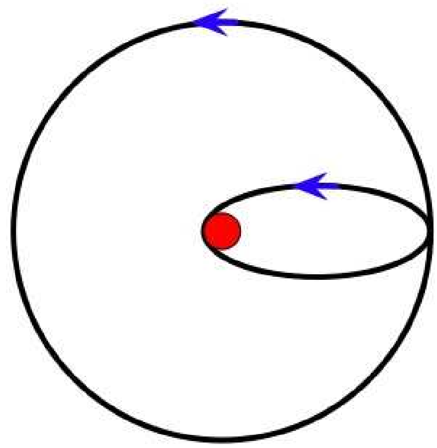


Figure 9.9: The larger circular orbit represents the orbit of the earth. The elliptical orbit represents that for an object orbiting the Sun with distance at perihelion equal to the Sun's radius.

Solving for δ, we obtain

$$\delta = \cos^{-1}\big(\varepsilon_{\rm i}/\varepsilon_{\rm f}\big) \ . \tag{9.69}$$

The situation is depicted in Fig. 9.8.

- Problem #2: Which is more energy efficient – to send nuclear waste outside the solar system, or to send it into the Sun?

- ∘ Solution #2: Escape velocity for the solar system is $v_{\rm esc,\odot}(r) = \sqrt{GM_\odot/r}$. At a distance $a_{\rm E}$, we then have $v_{\rm esc,\odot}(a_{\rm E}) = \sqrt{2}\,v_{\rm E}$, where $v_{\rm E} = \sqrt{GM_\odot/a_{\rm E}} = 2\pi a_{\rm E}/\tau_{\rm E} = 29.9\,{\rm km/s}$ is the velocity of the earth in its orbit. The satellite is launched from earth, and clearly the most energy efficient launch will be one in the direction of the earth's motion, in which case the velocity after escape from earth must be $u = \big(\sqrt{2}-1\big)v_{\rm E} = 12.4\,{\rm km/s}$. The speed just above the earth's atmosphere must then be $\tilde{u}$, where

$$\tfrac{1}{2}m\tilde{u}^2 - \frac{GM_{\rm E}m}{R_{\rm E}} = \tfrac{1}{2}mu^2 \ , \tag{9.70}$$

or, in other words,

$$\tilde{u}^2 = u^2 + v_{\rm esc,E}^2 \ . \tag{9.71}$$

We compute $\tilde{u} = 16.7\,{\rm km/s}$.

The second method is to place the trash ship in an elliptical orbit whose perihelion is the Sun's radius, $R_\odot = 6.98\times10^8$ m, and whose aphelion is $a_{\rm E}$. Using the general equation $r(\phi) = (\ell^2/\mu k)/(1-\varepsilon\cos\phi)$ for a Keplerian ellipse, we therefore solve the two equations

$$\begin{aligned} r(\phi=\pi) = R_\odot &= \frac{1}{1+\varepsilon}\cdot\frac{\ell^2}{\mu k} \\ r(\phi=0) = a_{\rm E} &= \frac{1}{1-\varepsilon}\cdot\frac{\ell^2}{\mu k} \ . \end{aligned} \tag{9.72}$$

We thereby obtain

$$\varepsilon = \frac{a_{\rm E} - R_\odot}{a_{\rm E} + R_\odot} = 0.991 \ , \tag{9.73}$$

which is a very eccentric ellipse, and

$$\begin{aligned} \frac{\ell^2}{\mu k} &= \frac{a_{\rm E}^2 \, v^2}{G(M_\odot + m)} \approx a_{\rm E} \cdot \frac{v^2}{v_{\rm E}^2} \\ &= (1-\varepsilon)\, a_{\rm E} = \frac{2a_{\rm E} R_\odot}{a_{\rm E} + R_\odot} \ . \end{aligned} \tag{9.74}$$

Hence,

$$v^2 = \frac{2R_\odot}{a_{\rm E} + R_\odot} \, v_{\rm E}^2 \ , \tag{9.75}$$

and the necessary velocity relative to earth is

$$u = \left(\sqrt{\frac{2R_\odot}{a_{\rm E} + R_\odot}} - 1 \right) v_{\rm E} \approx -0.904 \, v_{\rm E} \ , \tag{9.76}$$

i.e. $u = -27.0\,{\rm km/s}$. Launch is in the opposite direction from the earth's orbital motion, and from $\tilde u^2 = u^2 + v_{\rm esc,E}^2$ we find $\tilde u = -29.2\,{\rm km/s}$, which is larger (in magnitude) than in the first scenario. Thus, it is cheaper to ship the trash out of the solar system than to send it crashing into the Sun, by a factor $\tilde u_{\rm I}^2 / \tilde u_{\rm II}^2 = 0.327$.

9.5 Appendix I : Mission to Neptune

Four earth-launched spacecraft have escaped the solar system: *Pioneer 10* (launch 3/3/72), *Pioneer 11* (launch 4/6/73), *Voyager 1* (launch 9/5/77), and *Voyager 2* (launch 8/20/77).[1] The latter two are still functioning, and each are moving away from the Sun at a velocity of roughly 3.5 AU/yr.

As the first objects of earthly origin to leave our solar system, both *Pioneer* spacecraft featured a graphic message in the form of a 6" x 9" gold anodized plaque affixed to the spacecrafts' frame. This plaque was designed in part by the late astronomer and popular science writer Carl Sagan. The humorist Dave Barry, in an essay entitled *Bring Back Carl's Plaque*, remarks,

> But the really bad part is what they put on the plaque. I mean, if we're going to have a plaque, it ought to at least show the aliens what we're really like, right? Maybe a picture of people eating cheeseburgers and watching "The Dukes of Hazzard." Then if aliens found it, they'd say, "Ah. Just plain folks."
>
> But no. Carl came up with this incredible science-fair-wimp plaque that features drawings of – you are not going to believe this – a hydrogen atom and naked people. To represent the entire Earth! This is crazy! Walk the streets of any town on this planet, and the two things you will almost never see are hydrogen atoms and naked people.

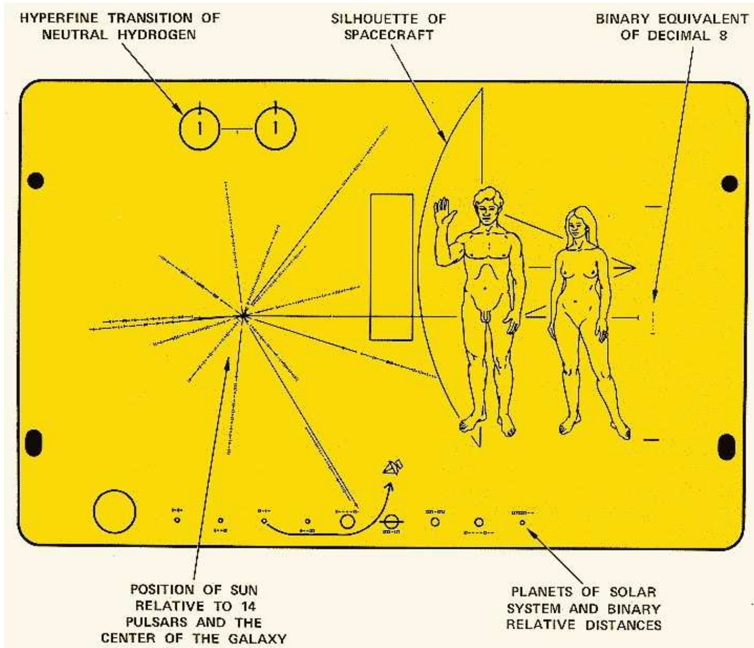


Figure 9.10: The unforgivably dorky *Pioneer 10* and *Pioneer 11* plaque.

During August, 1989, *Voyager 2* investigated the planet Neptune. A direct trip to Neptune along a Keplerian ellipse with $r_{\rm p} = a_{\rm E} = 1\,{\rm AU}$ and $r_{\rm a} = a_{\rm N} = 30.06\,{\rm AU}$ would take 30.6 years. To see this, note that $r_{\rm p} = a\,(1-\varepsilon)$ and $r_{\rm a} = a\,(1+\varepsilon)$ yield

$$a = \tfrac{1}{2}\big(a_{\rm E} + a_{\rm N}\big) = 15.53\,{\rm AU} \quad , \quad \varepsilon = \frac{a_{\rm N} - a_{\rm E}}{a_{\rm N} + a_{\rm E}} = 0.9356 \ . \tag{9.77}$$

Thus,

$$\tau = \tfrac{1}{2}\,\tau_{\rm E} \cdot \bigg(\frac{a}{a_{\rm E}}\bigg)^{3/2} = 30.6\,{\rm yr} \ . \tag{9.78}$$

The energy cost per kilogram of such a mission is computed as follows. Let the speed of the probe after its escape from earth be $v_{\rm p} = \lambda v_{\rm E}$, and the speed just above the atmosphere (*i.e.* neglecting atmospheric friction) is v_0. For the most efficient launch possible, the probe is shot in the direction of earth's instantaneous motion about the Sun. Then we must have

$$\tfrac{1}{2} m\, v_0^2 - \frac{G M_{\rm E} m}{R_{\rm E}} = \tfrac{1}{2} m\,(\lambda-1)^2\, v_{\rm E}^2 \ , \tag{9.79}$$

since the speed of the probe in the frame of the earth is $v_{\rm p} - v_{\rm E} = (\lambda - 1)\, v_{\rm E}$. Thus,

$$\begin{aligned} \frac{E}{m} &= \tfrac{1}{2} v_0^2 = \Big[\tfrac{1}{2}(\lambda-1)^2 + h\Big]\, v_{\rm E}^2 \\ v_{\rm E}^2 &= \frac{G M_\odot}{a_{\rm E}} = 6.24 \times 10^7\, R_{\rm J}/{\rm kg} \ , \end{aligned} \tag{9.80}$$

[1]There is a very nice discussion in the Barger and Olsson book on 'Grand Tours of the Outer Planets'. Here I reconstruct and extend their discussion.

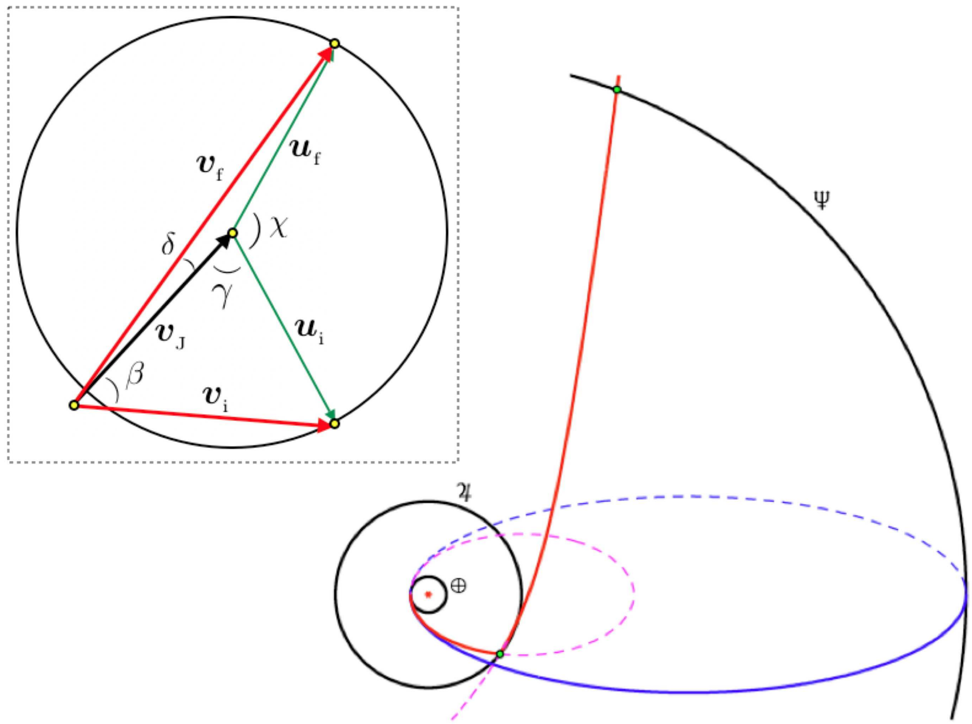


Figure 9.11: Mission to Neptune. The figure at the lower right shows the orbits of Earth, Jupiter, and Neptune in black. The cheapest (in terms of energy) direct flight to Neptune, shown in blue, would take 30.6 years. By swinging past the planet Jupiter, the satellite can pick up great speed and with even less energy the mission time can be cut to 8.5 years (red curve). The inset in the upper left shows the scattering event with Jupiter.

where

$$h \equiv \frac{M_{\rm E}}{M_\odot} \cdot \frac{a_{\rm E}}{R_{\rm E}} = 7.050 \times 10^{-2} \ . \tag{9.81}$$

Therefore, a convenient dimensionless measure of the energy is

$$\eta \equiv \frac{2E}{mv_{\rm E}^2} = \frac{v_0^2}{v_{\rm E}^2} = (\lambda - 1)^2 + 2h \ . \tag{9.82}$$

As we shall derive below, a direct mission to Neptune requires

$$\lambda \geq \sqrt{\frac{2a_{\rm N}}{a_{\rm N} + a_{\rm E}}} = 1.3913 \ , \tag{9.83}$$

which is close to the criterion for escape from the solar system, $\lambda_{\rm esc} = \sqrt{2}$. Note that about 52% of the energy is expended after the probe escapes the Earth's pull, and 48% is expended in liberating the probe from Earth itself.

This mission can be done much more economically by taking advantage of a Jupiter flyby, as shown in Fig. 9.11. The idea of a flyby is to steal some of Jupiter's momentum and then fly away very fast before Jupiter realizes and gets angry. The CM frame of the probe-Jupiter system is of course the rest frame of Jupiter, and in this frame conservation of energy means that the final velocity $\boldsymbol{u}_{\rm f}$ is of the same magnitude as the initial velocity $\boldsymbol{u}_{\rm i}$. However, in the frame of the Sun, the initial and final velocities are $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm i}$ and $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm f}$, respectively, where $\boldsymbol{v}_{\rm J}$ is the velocity of Jupiter in the rest frame of the Sun. If, as shown in the inset to Fig. 9.11, $\boldsymbol{u}_{\rm f}$ is roughly parallel to $\boldsymbol{v}_{\rm J}$, the probe's velocity in the Sun's frame will be enhanced. Thus, the motion of the probe is broken up into three segments:

I : Earth to Jupiter
II : Scatter off Jupiter's gravitational pull
III : Jupiter to Neptune

We now analyze each of these segments in detail. In so doing, it is useful to recall that the general form of a Keplerian orbit is

$$r(\phi) = \frac{d}{1 - \varepsilon\cos\phi} \quad , \quad d = \frac{\ell^2}{\mu k} = \big|\varepsilon^2 - 1\big|\, a \ . \tag{9.84}$$

The energy is

$$E = (\varepsilon^2 - 1)\,\frac{\mu k^2}{2\ell^2} \ , \tag{9.85}$$

with $k = GMm$, where M is the mass of either the Sun or a planet. In either case, M dominates, and $\mu = Mm/(M+m) \simeq m$ to extremely high accuracy. The time for the trajectory to pass from $\phi = \phi_1$ to $\phi = \phi_2$ is

$$T = \int\! dt = \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\dot\phi} = \frac{\mu}{\ell}\int\limits_{\phi_1}^{\phi_2} d\phi\, r^2(\phi) = \frac{\ell^3}{\mu k^2}\int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\big[1 - \varepsilon\cos\phi\big]^2} \ . \tag{9.86}$$

For reference,

$$a_{\rm E} = 1\,{\rm AU} \qquad a_{\rm J} = 5.20\,{\rm AU} \qquad a_{\rm N} = 30.06\,{\rm AU}$$
$$M_{\rm E} = 5.972\times 10^{24}\,{\rm kg} \qquad M_{\rm J} = 1.900\times 10^{27}\,{\rm kg} \qquad M_\odot = 1.989\times 10^{30}\,{\rm kg}$$

with $1\,{\rm AU} = 1.496\times 10^8$ km. Here $a_{\rm E,J,N}$ and $M_{\rm E,J,\odot}$ are the orbital radii and masses of Earth, Jupiter, and Neptune, and the Sun. The last thing we need to know is the radius of Jupiter,

$$R_{\rm J} = 9.558\times 10^{-4}\,{\rm AU} \ .$$

We need $R_{\rm J}$ because the distance of closest approach to Jupiter, or *perijove*, must be $R_{\rm J}$ or greater, or else the probe crashes into Jupiter!

9.5.1 I. Earth to Jupiter

The probe's velocity at perihelion is $v_{\rm p} = \lambda v_{\rm E}$. The angular momentum is $\ell = \mu a_{\rm E}\cdot\lambda v_{\rm E}$, whence

$$d = \frac{(a_{\rm E}\lambda v_{\rm E})^2}{GM_\odot} = \lambda^2\, a_{\rm E} \ . \tag{9.87}$$

From $r(\pi) = a_{\rm E}$, we obtain

$$\varepsilon_{\rm I} = \lambda^2 - 1 \ . \tag{9.88}$$

This orbit will intersect the orbit of Jupiter if $r_{\rm a} \geq a_{\rm J}$, which means

$$\frac{d}{1-\varepsilon_{\rm I}} \geq a_{\rm J} \quad \Rightarrow \quad \lambda \geq \sqrt{\frac{2a_{\rm J}}{a_{\rm J}+a_{\rm E}}} = 1.2952 \ . \tag{9.89}$$

If this inequality holds, then intersection of Jupiter's orbit will occur for

$$\phi_{\rm J} = 2\pi - \cos^{-1}\left(\frac{a_{\rm J} - \lambda^2 a_{\rm E}}{(\lambda^2-1)\, a_{\rm J}}\right) \ . \tag{9.90}$$

Finally, the time for this portion of the trajectory is

$$\tau_{\rm EJ} = \tau_{\rm E} \cdot \lambda^3 \int\limits_{\pi}^{\phi_{\rm J}} \frac{d\phi}{2\pi}\, \frac{1}{\big[1-(\lambda^2-1)\cos\phi\big]^2} \ . \tag{9.91}$$

9.5.2 II. Encounter with Jupiter

We are interested in the final speed $\boldsymbol{v}_{\rm f}$ of the probe after its encounter with Jupiter. We will determine the speed $v_{\rm f}$ and the angle δ which the probe makes with respect to Jupiter after its encounter. According to the geometry of Fig. 9.11,

$$\begin{aligned} v_{\rm f}^2 &= v_{\rm J}^2 + u^2 - 2uv_{\rm J}\cos(\chi+\gamma) \\ \cos\delta &= \frac{v_{\rm J}^2 + v_{\rm f}^2 - u^2}{2v_{\rm f} v_{\rm J}} \end{aligned} \tag{9.92}$$

Note that

$$v_{\rm J}^2 = \frac{GM_\odot}{a_{\rm J}} = \frac{a_{\rm E}}{a_{\rm J}} \cdot v_{\rm E}^2 \ . \tag{9.93}$$

But what are u, χ, and γ?

To determine u, we invoke

$$u^2 = v_{\rm J}^2 + v_{\rm i}^2 - 2v_{\rm J} v_{\rm i}\cos\beta \ . \tag{9.94}$$

The initial velocity (in the frame of the Sun) when the probe crosses Jupiter's orbit is given by energy conservation:

$$\tfrac{1}{2}m(\lambda v_{\rm E})^2 - \frac{GM_\odot m}{a_{\rm E}} = \tfrac{1}{2}mv_{\rm i}^2 - \frac{GM_\odot m}{a_{\rm J}} \ , \tag{9.95}$$

which yields

$$v_i^2 = \left(\lambda^2 - 2 + \frac{2a_{\rm E}}{a_{\rm J}}\right) v_{\rm E}^2 \ . \tag{9.96}$$

As for β, we invoke conservation of angular momentum:

$$\mu(v_{\rm i}\cos\beta)a_{\rm J} = \mu(\lambda v_{\rm E})a_{\rm E} \quad \Rightarrow \quad v_{\rm i}\cos\beta = \lambda\,\frac{a_{\rm E}}{a_{\rm J}}\, v_{\rm E} \ . \tag{9.97}$$

The angle γ is determined from

$$v_{\rm J} = v_{\rm i} \cos\beta + u\cos\gamma \ . \tag{9.98}$$

Putting all this together, we obtain

$$\begin{aligned} v_{\rm i} &= v_{\rm E}\sqrt{\lambda^2 - 2 + 2x} \\ u &= v_{\rm E}\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}} \\ \cos\gamma &= \frac{\sqrt{x} - \lambda x}{\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}}} \ , \end{aligned} \tag{9.99}$$

where

$$x \equiv \frac{a_{\rm E}}{a_{\rm J}} = 0.1923 \ . \tag{9.100}$$

We next consider the scattering of the probe by the planet Jupiter. In the Jovian frame, we may write

$$r(\phi) = \frac{\kappa R_{\rm J}\,(1+\varepsilon_{\rm J})}{1 + \varepsilon_{\rm J}\cos\phi} \ , \tag{9.101}$$

where perijove occurs at

$$r(0) = \kappa R_{\rm J} \ . \tag{9.102}$$

Here, κ is a dimensionless quantity, which is simply perijove in units of the Jovian radius. Clearly we require $\kappa > 1$ or else the probe crashes into Jupiter! The probe's energy in this frame is simply $E = \frac{1}{2}mu^2$, which means the probe enters into a hyperbolic orbit about Jupiter. Next, from

$$\begin{aligned} E &= \frac{k}{2}\,\frac{\varepsilon^2 - 1}{\ell^2/\mu k} \\ \frac{\ell^2}{\mu k} &= (1+\varepsilon)\,\kappa R_{\rm J} \end{aligned} \tag{9.103}$$

we find

$$\varepsilon_{\rm J} = 1 + \kappa\bigg(\frac{R_{\rm J}}{a_{\rm E}}\bigg)\bigg(\frac{M_\odot}{M_{\rm J}}\bigg)\bigg(\frac{u}{v_{\rm E}}\bigg)^2 \ . \tag{9.104}$$

The opening angle of the Keplerian hyperbola is then $\phi_{\rm c} = \cos^{-1}\big(\varepsilon_{\rm J}^{-1}\big)$, and the angle χ is related to $\phi_{\rm c}$ through

$$\chi = \pi - 2\phi_{\rm c} = \pi - 2\cos^{-1}\bigg(\frac{1}{\varepsilon_{\rm J}}\bigg) \ . \tag{9.105}$$

Therefore, we may finally write

$$\begin{aligned} v_{\rm f} &= \sqrt{x\,v_{\rm E}^2 + u^2 + 2\,u\,v_{\rm E}\sqrt{x}\cos(2\phi_{\rm c} - \gamma)} \\ \cos\delta &= \frac{x\,v_{\rm E}^2 + v_{\rm f}^2 - u^2}{2\,v_{\rm f}\,v_{\rm E}\sqrt{x}} \ . \end{aligned} \tag{9.106}$$

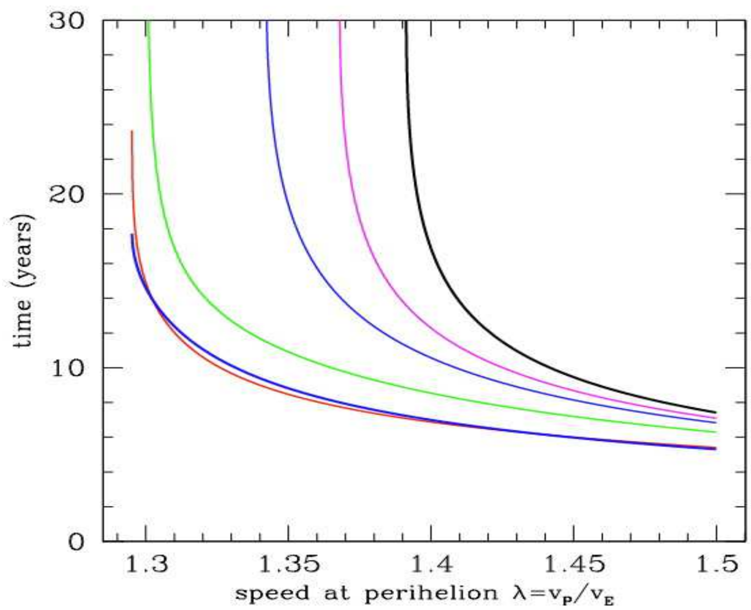


Figure 9.12: Total time for Earth-Neptune mission as a function of dimensionless velocity at perihelion, $\lambda = v_{\rm p}/v_{\rm E}$. Six different values of κ, the value of perijove in units of the Jovian radius, are shown: $\kappa = 1.0$ (thick blue), $\kappa = 5.0$ (red), $\kappa = 20$ (green), $\kappa = 50$ (blue), $\kappa = 100$ (magenta), and $\kappa = \infty$ (thick black).

9.5.3 III. Jupiter to Neptune

Immediately after undergoing gravitational scattering off Jupiter, the energy and angular momentum of the probe are

$$E = \tfrac{1}{2} m v_{\rm f}^2 - \frac{GM_\odot m}{a_{\rm J}} \tag{9.107}$$

and

$$\ell = \mu\, v_{\rm f}\, a_{\rm J} \cos\delta\ . \tag{9.108}$$

We write the geometric equation for the probe's orbit as

$$r(\phi) = \frac{d}{1 + \varepsilon\cos(\phi - \phi_{\rm J} - \alpha)}\ , \tag{9.109}$$

where

$$d = \frac{\ell^2}{\mu k} = \bigg(\frac{v_{\rm f}\, a_{\rm J}\cos\delta}{v_{\rm E}\, a_{\rm E}}\bigg)^2 a_{\rm E}\ . \tag{9.110}$$

Setting $E = (\mu k^2/2\ell^2)(\varepsilon^2 - 1)$, we obtain the eccentricity

$$\varepsilon = \sqrt{1 + \bigg(\frac{v_{\rm f}^2}{v_{\rm E}^2} - \frac{2a_{\rm E}}{a_{\rm J}}\bigg)\frac{d}{a_{\rm E}}}\ . \tag{9.111}$$

Note that the orbit is hyperbolic – the probe will escape the Sun – if $v_{\rm f} > v_{\rm E} \cdot \sqrt{2x}$. The condition that this orbit intersect Jupiter at $\phi = \phi_{\rm J}$ yields

$$\cos\alpha = \frac{1}{\varepsilon}\left(\frac{d}{a_{\rm J}} - 1\right) , \tag{9.112}$$

which determines the angle α. Interception of Neptune occurs at

$$\frac{d}{1+\varepsilon\cos(\phi_{\rm N} - \phi_{\rm J} - \alpha)} = a_{\rm N} \qquad \Rightarrow \qquad \phi_{\rm N} = \phi_{\rm J} + \alpha + \cos^{-1}\frac{1}{\varepsilon}\left(\frac{d}{a_{\rm N}} - 1\right) . \tag{9.113}$$

We then have

$$\tau_{\rm JN} = \tau_{\rm E} \cdot \left(\frac{d}{a_{\rm E}}\right)^3 \int\limits_{\phi_{\rm J}}^{\phi_{\rm N}} \frac{d\phi}{2\pi}\, \frac{1}{\big[1+\varepsilon\cos(\phi - \phi_{\rm J} - \alpha)\big]^2} . \tag{9.114}$$

The total time to Neptune is then the sum,

$$\tau_{\rm EN} = \tau_{\rm EJ} + \tau_{\rm JN} . \tag{9.115}$$

In Fig. 9.12, we plot the mission time $\tau_{\rm EN}$ *versus* the velocity at perihelion, $v_{\rm p} = \lambda v_{\rm E}$, for various values of κ. The value $\kappa = \infty$ corresponds to the case of no Jovian encounter at all.

9.6 Appendix II : Restricted Three-Body Problem

Problem : Consider the 'restricted three body problem' in which a light object of mass m (*e.g.* a satellite) moves in the presence of two celestial bodies of masses m_1 and m_2 (*e.g.* the sun and the earth, or the earth and the moon). Suppose m_1 and m_2 execute stable circular motion about their common center of mass. You may assume $m \ll m_2 \le m_1$.

(a) Show that the angular frequency for the motion of masses 1 and 2 is related to their (constant) relative separation, by

$$\omega_0^2 = \frac{GM}{r_0^3} , \tag{9.116}$$

where $M = m_1 + m_2$ is the total mass.

Solution : For a Kepler potential $U = -k/r$, the circular orbit lies at $r_0 = \ell^2/\mu k$, where $\ell = \mu r^2 \dot\phi$ is the angular momentum and $k = G m_1 m_2$. This gives

$$\omega_0^2 = \frac{\ell^2}{\mu^2\, r_0^4} = \frac{k}{\mu r_0^3} = \frac{GM}{r_0^3} , \tag{9.117}$$

with $\omega_0 = \dot\phi$.

(b) The satellite moves in the combined gravitational field of the two large bodies; the satellite itself is of course much too small to affect their motion. In deriving the motion for the satellilte, it is convenient

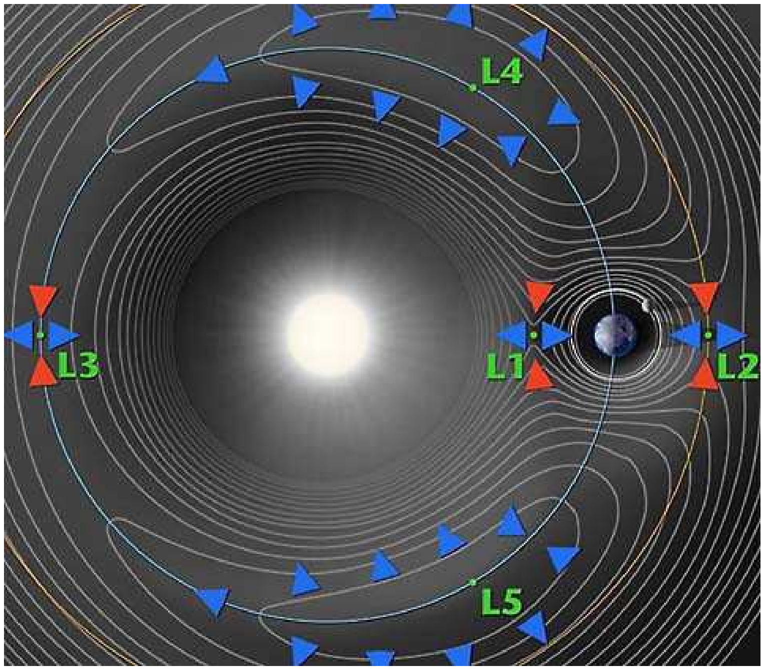


Figure 9.13: The Lagrange points for the earth-sun system. *Credit: WMAP project.*

to choose a reference frame whose origin is the CM and which rotates with angular velocity ω_0. In the rotating frame the masses m_1 and m_2 lie, respectively, at $x_1 = -\alpha r_0$ and $x_2 = \beta r_0$, with

$$\alpha = \frac{m_2}{M} \qquad , \qquad \beta = \frac{m_1}{M} \tag{9.118}$$

and with $y_1 = y_2 = 0$. Note $\alpha + \beta = 1$.

Show that the Lagrangian for the satellite in this rotating frame may be written

$$L = \tfrac{1}{2} m \big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2} m \big(\dot{y} + \omega_0\, x\big)^2 + \frac{G m_1 m}{\sqrt{(x + \alpha r_0)^2 + y^2}} + \frac{G m_2 m}{\sqrt{(x - \beta r_0)^2 + y^2}} \ . \tag{9.119}$$

Solution : Let the original (inertial) coordinates be (x_0, y_0). Then let us define the rotated coordinates (x, y) as

$$\begin{aligned} x &= \cos(\omega_0 t)\, x_0 + \sin(\omega_0 t)\, y_0 \\ y &= -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 \ . \end{aligned} \tag{9.120}$$

Therefore,

$$\begin{aligned} \dot{x} &= \cos(\omega_0 t)\, \dot{x}_0 + \sin(\omega_0 t)\, \dot{y}_0 + \omega_0\, y \\ \dot{y} &= -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 - \omega_0\, x \ . \end{aligned} \tag{9.121}$$

Therefore

$$(\dot{x} - \omega_0\, y)^2 + (\dot{y} + \omega_0\, x)^2 = \dot{x}_0^2 + \dot{y}_0^2 \ , \tag{9.122}$$

The Lagrangian is then

$$L = \tfrac{1}{2}m\big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2}m\big(\dot{y} + \omega_0\, x\big)^2 + \frac{G\,m_1\,m}{\sqrt{(x-x_1)^2+y^2}} + \frac{G\,m_2\,m}{\sqrt{(x-x_2)^2+y^2}} \ , \tag{9.123}$$

which, with $x_1 \equiv -\alpha r_0$ and $x_2 \equiv \beta r_0$, agrees with eqn. 9.119

(c) Lagrange discovered that there are five special points where the satellite remains fixed in the rotating frame. These are called the *Lagrange points* {L1, L2, L3, L4, L5}. A sketch of the Lagrange points for the earth-sun system is provided in Fig. 9.13. *Observation: In working out the rest of this problem, I found it convenient to measure all distances in units of* r_0 *and times in units of* ω_0^{-1}*, and to eliminate* G *by writing* $Gm_1 = \beta\,\omega_0^2\,r_0^3$ *and* $Gm_2 = \alpha\,\omega_0^2\,r_0^3$.

Assuming the satellite is stationary in the rotating frame, derive the equations for the positions of the Lagrange points.

Solution : At this stage it is convenient to measure all distances in units of r_0 and times in units of ω_0^{-1} to factor out a term $m\,r_0^2\,\omega_0^2$ from L, writing the dimensionless Lagrangian $\widetilde{L} \equiv L/(m\,r_0^2\,\omega_0^2)$. Using as well the definition of ω_0^2 to eliminate G, we have

$$\widetilde{L} = \tfrac{1}{2}\,(\dot{\xi} - \eta)^2 + \tfrac{1}{2}(\dot{\eta} + \xi)^2 + \frac{\beta}{\sqrt{(\xi+\alpha)^2+\eta^2}} + \frac{\alpha}{\sqrt{(\xi-\beta)^2+\eta^2}} \ , \tag{9.124}$$

with

$$\xi \equiv \frac{x}{r_0} \quad , \quad \eta \equiv \frac{y}{r_0} \quad , \quad \dot{\xi} \equiv \frac{1}{\omega_0\, r_0}\frac{dx}{dt} \quad , \quad \dot{\eta} \equiv \frac{1}{\omega_0\, r_0}\frac{dy}{dt} \ . \tag{9.125}$$

The equations of motion are then

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi - \frac{\beta(\xi+\alpha)}{d_1^3} - \frac{\alpha(\xi-\beta)}{d_2^3} \\ \ddot{\eta} + 2\dot{\xi} &= \eta - \frac{\beta\eta}{d_1^3} - \frac{\alpha\eta}{d_2^3} \ , \end{aligned} \tag{9.126}$$

where

$$d_1 = \sqrt{(\xi+\alpha)^2+\eta^2} \qquad , \qquad d_2 = \sqrt{(\xi-\beta)^2+\eta^2} \ . \tag{9.127}$$

Here, $\xi \equiv x/r_0$, $\xi = y/r_0$, *etc.* Recall that $\alpha + \beta = 1$. Setting the time derivatives to zero yields the static equations for the Lagrange points:

$$\begin{aligned} \xi &= \frac{\beta(\xi+\alpha)}{d_1^3} + \frac{\alpha(\xi-\beta)}{d_2^3} \\ \eta &= \frac{\beta\eta}{d_1^3} + \frac{\alpha\eta}{d_2^3} \ , \end{aligned} \tag{9.128}$$

(d) Show that the Lagrange points with $y = 0$ are determined by a single nonlinear equation. Show graphically that this equation always has three solutions, one with $x < x_1$, a second with $x_1 < x < x_2$, and a third with $x > x_2$. These solutions correspond to the points L3, L1, and L2, respectively.

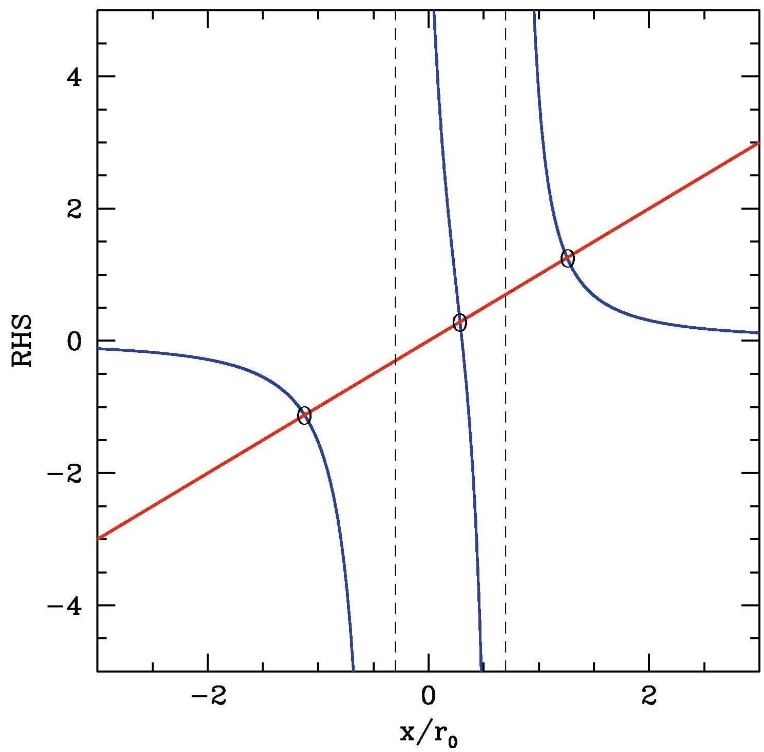


Figure 9.14: Graphical solution for the Lagrange points L1, L2, and L3.

Solution : If $\eta = 0$ the second equation is automatically satisfied. The first equation then gives

$$\xi = \beta \cdot \frac{\xi + \alpha}{\left|\xi + \alpha\right|^3} + \alpha \cdot \frac{\xi - \beta}{\left|\xi - \beta\right|^3} \ . \tag{9.129}$$

The RHS of the above equation diverges to $+\infty$ for $\xi = -\alpha + 0^+$ and $\xi = \beta + 0^+$, and diverges to $-\infty$ for $\xi = -\alpha - 0^+$ and $\xi = \beta - 0^+$, where 0^+ is a positive infinitesimal. The situation is depicted in Fig. 9.14. Clearly there are three solutions, one with $\xi < -\alpha$, one with $-\alpha < \xi < \beta$, and one with $\xi > \beta$.

(e) Show that the remaining two Lagrange points, L4 and L5, lie along equilateral triangles with the two masses at the other vertices.

Solution : If $\eta \neq 0$, then dividing the second equation by η yields

$$1 = \frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3} \ . \tag{9.130}$$

Substituting this into the first equation,

$$\xi = \left(\frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3}\right)\xi + \left(\frac{1}{d_1^3} - \frac{1}{d_2^3}\right)\alpha\beta \ , \tag{9.131}$$

gives

$$d_1 = d_2 \ . \tag{9.132}$$

Reinserting this into the previous equation then gives the remarkable result,

$$d_1 = d_2 = 1 \ , \tag{9.133}$$

which says that each of L4 and L5 lies on an equilateral triangle whose two other vertices are the masses m_1 and m_2. The side length of this equilateral triangle is r_0. Thus, the dimensionless coordinates of L4 and L5 are

$$(\xi_{\rm L4},\ \eta_{\rm L4}) = \Big(\tfrac{1}{2} - \alpha,\ \tfrac{\sqrt{3}}{2}\Big) \qquad , \qquad (\xi_{\rm L5},\ \eta_{\rm L5}) = \Big(\tfrac{1}{2} - \alpha,\ -\tfrac{\sqrt{3}}{2}\Big) \ . \tag{9.134}$$

It turns out that L1, L2, and L3 are always unstable. Satellites placed in these positions must undergo periodic course corrections in order to remain approximately fixed. The SOlar and Heliopheric Observation satellite, *SOHO*, is located at L1, which affords a continuous unobstructed view of the Sun.

(f) Show that the Lagrange points L4 and L5 are stable (obviously you need only consider one of them) provided that the mass ratio m_1/m_2 is sufficiently large. Determine this critical ratio. Also find the frequency of small oscillations for motion in the vicinity of L4 and L5.

Solution : Now we write

$$\xi = \xi_{\rm L4} + \delta\xi \qquad , \qquad \eta = \eta_{\rm L4} + \delta\eta \ , \tag{9.135}$$

and derive the linearized dynamics. Expanding the equations of motion to lowest order in $\delta\xi$ and $\delta\eta$, we have

$$\begin{aligned} \delta\ddot{\xi} - 2\delta\dot{\eta} &= \bigg(1 - \beta + \tfrac{3}{2}\beta\, \frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} - \alpha - \tfrac{3}{2}\alpha\, \frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3}{2}\beta\, \frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} - \tfrac{3}{2}\alpha\, \frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3}{4}\,\delta\xi + \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\eta \end{aligned} \tag{9.136}$$

and

$$\begin{aligned} \delta\ddot{\eta} + 2\delta\dot{\xi} &= \bigg(\tfrac{3\sqrt{3}}{2}\beta\, \frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\, \frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3\sqrt{3}}{2}\beta\, \frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\, \frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\xi + \tfrac{9}{4}\,\delta\eta \ , \end{aligned} \tag{9.137}$$

where we have defined

$$\varepsilon \equiv \beta - \alpha = \frac{m_1 - m_2}{m_1 + m_2} \ . \tag{9.138}$$

As defined, $\varepsilon \in [0,1]$.

Fourier transforming the differential equation, we replace each time derivative by $(-i\nu)$, and thereby obtain

$$\begin{pmatrix} \nu^2 + \frac{3}{4} & -2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon \\ 2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon & \nu^2 + \frac{9}{4} \end{pmatrix} \begin{pmatrix} \delta\hat{\xi} \\ \delta\hat{\eta} \end{pmatrix} = 0 \ . \tag{9.139}$$

Nontrivial solutions exist only when the determinant D vanishes. One easily finds

$$D(\nu^2) = \nu^4 - \nu^2 + \tfrac{27}{16}\left(1 - \varepsilon^2\right) \ , \tag{9.140}$$

which yields a quadratic equation in ν^2, with roots

$$\nu^2 = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{27\,\varepsilon^2 - 23} \ . \tag{9.141}$$

These frequencies are dimensionless. To convert to dimensionful units, we simply multiply the solutions for ν by ω_0, since we have rescaled time by ω_0^{-1}.

Note that the L4 and L5 points are stable only if $\varepsilon^2 > \frac{23}{27}$. If we define the mass ratio $\gamma \equiv m_1/m_2$, the stability condition is equivalent to

$$\gamma = \frac{m_1}{m_2} > \frac{\sqrt{27}+\sqrt{23}}{\sqrt{27}-\sqrt{23}} = 24.960 \ , \tag{9.142}$$

which is satisfied for both the Sun-Jupiter system ($\gamma = 1047$) – and hence for the Sun and any planet – and also for the Earth-Moon system ($\gamma = 81.2$).

Objects found at the L4 and L5 points are called *Trojans*, after the three large asteroids Agamemnon, Achilles, and Hector found orbiting in the L4 and L5 points of the Sun-Jupiter system. No large asteroids have been found in the L4 and L5 points of the Sun-Earth system.

Personal aside : David T. Wilkinson

The image in fig. 9.13 comes from the education and outreach program of the Wilkinson Microwave Anisotropy Probe (WMAP) project, a NASA mission, launched in 2001, which has produced some of the most important recent data in cosmology. The project is named in honor of David T. Wilkinson, who was a leading cosmologist at Princeton, and a founder of the Cosmic Background Explorer (COBE) satellite (launched in 1989). WMAP was sent to the L2 Lagrange point, on the night side of the earth, where it can constantly scan the cosmos with an ultra-sensitive microwave detector, shielded by the earth from interfering solar electromagnetic radiation. The L2 point is of course unstable, with a time scale of about 23 days. Satellites located at such points must undergo regular course and attitude corrections to remain situated.

During the summer of 1981, as an undergraduate at Princeton, I was a member of Wilkinson's "gravity group," working under Jeff Kuhn and Ken Libbrecht. It was a pretty big group and Dave – everyone would call him Dave – used to throw wonderful parties at his home, where we'd always play volleyball. I was very fortunate to get to know David Wilkinson a bit – after working in his group that summer I took a class from him the following year. He was a wonderful person, a superb teacher, and a world class physicist.

Chapter 10

Small Oscillations

10.1 Coupled Coordinates

We assume, for a set of n generalized coordinates $\{q_1, \ldots, q_n\}$, that the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} \ , \tag{10.1}$$

where the sum on σ and σ' from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix T_{ij} is

$$T_{\sigma\sigma'} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\!\theta \end{pmatrix} \qquad \Longrightarrow \qquad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2\big) \ . \tag{10.2}$$

The potential $U(q_1, \ldots, q_n)$ is assumed to be a function of the generalized coordinates alone: $U = U(q)$. A more general formulation of the problem of small oscillations is given in the appendix, section 10.8.

The generalized momenta are

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}\, \dot{q}_{\sigma'} \ , \tag{10.3}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} - \frac{\partial U}{\partial q_\sigma} \ . \tag{10.4}$$

The Euler-Lagrange equations are then $\dot{p}_\sigma = F_\sigma$, or

$$T_{\sigma\sigma'}\, \ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma} \right) \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma} \tag{10.5}$$

which is a set of coupled nonlinear second order ODEs. Here we are using the Einstein 'summation convention', where we automatically sum over any and all repeated indices.

10.2 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar{q}_1, \ldots, \bar{q}_n\}$, which satisfies the n nonlinear equations

$$\frac{\partial U}{\partial q_\sigma}\bigg|_{q=\bar{q}} = 0 \ . \tag{10.6}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_\sigma \equiv \bar{q}_\sigma + \eta_\sigma \ . \tag{10.7}$$

The coordinates $\{\eta_1, \ldots, \eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2}\mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\mathrm{V}_{\sigma\sigma'}\,\eta_\sigma\eta_{\sigma'} \ , \tag{10.8}$$

where

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{q}_\sigma\,\partial\dot{q}_{\sigma'}}\bigg|_{q=\bar{q}} \quad , \qquad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \ . \tag{10.9}$$

Writing $\boldsymbol{\eta}^{\mathrm{t}}$ for the row-vector $(\eta_1, \ldots, \eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathrm{t}}\,\mathrm{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathrm{t}}\,\mathrm{V}\,\boldsymbol{\eta} \ , \tag{10.10}$$

where T and V are the constant matrices of eqn. 10.9.

10.3 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_\sigma = \mathrm{A}_{\sigma i}\,\xi_i \ , \tag{10.11}$$

where both σ and i run from 1 to n. The $n \times n$ matrix $\mathrm{A}_{\sigma i}$ is known as the *modal matrix.* With the substitution $\boldsymbol{\eta} = \mathrm{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathrm{t}} = \boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}$, where $\mathrm{A}^{\mathrm{t}}_{i\sigma} = \mathrm{A}_{\sigma i}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\xi}}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A}\,\dot{\boldsymbol{\xi}} - \tfrac{1}{2}\,\boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A}\,\boldsymbol{\xi} \ . \tag{10.12}$$

We now choose the matrix A such that

$$\begin{aligned} \mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A} &= \mathbb{I} \\ \mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A} &= \mathrm{diag}\left(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\right) \ . \end{aligned} \tag{10.13}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2}\sum_{i=1}^{n}\left(\dot{\xi}_i^2 - \omega_i^2\,\xi_i^2\right) \ , \tag{10.14}$$

with the solution

$$\xi_i(t) = C_i \cos(\omega_i t) + D_i \sin(\omega_i t) \ , \tag{10.15}$$

where $\{C_1, \ldots, C_n\}$ and $\{D_1, \ldots, D_n\}$ are $2n$ constants of integration, and where no sum is implied on i. Note that

$$\boldsymbol{\xi} = \mathrm{A}^{-1}\boldsymbol{\eta} = \mathrm{A}^{\mathrm{t}}\mathrm{T}\,\boldsymbol{\eta} \ . \tag{10.16}$$

In terms of the original generalized displacements, the solution is

$$\eta_\sigma(t) = \sum_{i=1}^{n} \mathrm{A}_{\sigma i} \Big\{ C_i \cos(\omega_i t) + D_i \sin(\omega_i t) \Big\} \ , \tag{10.17}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$\begin{aligned} C_i &= \mathrm{A}^{\mathrm{t}}_{i\sigma}\, \mathrm{T}_{\sigma\sigma'}\, \eta_{\sigma'}(0) \\ D_i &= \omega_i^{-1} \mathrm{A}^{\mathrm{t}}_{i\sigma}\, \mathrm{T}_{\sigma\sigma'}\, \dot\eta_{\sigma'}(0) \ , \end{aligned} \tag{10.18}$$

again with no implied sum on i on the RHS of the second equation, and where we have used $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}}\,\mathrm{T}$, from eqns. **??**. (The implied sums in eqn. 10.18 are over σ and σ'.)

Note that the normal coordinates have unusual dimensions: $[\boldsymbol{\xi}] = \sqrt{M} \cdot L$, where L is length and M is mass.

10.3.1 Can you really just choose an A so that this works?

Yes.

10.3.2 Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per **??**. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $\mathcal{O}_1 \in \mathrm{O}(n)$ such that

$$\mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1 = \mathrm{T}_{\mathrm{d}} \ , \tag{10.19}$$

  where T_{d} is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $\mathrm{T}_{\mathrm{d}}^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of T_{d}. Consider the linear transformation $\mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}$. Its effect on T is

$$\mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2} = 1 \ . \tag{10.20}$$

- Since $\mathcal{O}_1$ and $\mathrm{T_d}$ are wholly derived from T, the only thing we know about

$$\widetilde{\mathrm{V}} \equiv \mathrm{T_d^{-1/2}}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T_d^{-1/2}} \tag{10.21}$$

is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $\mathcal{O}_2 \in \mathrm{O}(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\begin{aligned} \mathcal{O}_2^{\mathrm{t}}\, \mathrm{T_d^{-1/2}}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T_d^{-1/2}}\, \mathcal{O}_2 &= 1 \\ \mathcal{O}_2^{\mathrm{t}}\, \mathrm{T_d^{-1/2}}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T_d^{-1/2}}\, \mathcal{O}_2 &= \mathrm{diag}\,(\omega_1^2, \ldots, \omega_n^2)\ . \end{aligned} \tag{10.22}$$

All that remains is to identify the modal matrix $\mathrm{A} = \mathcal{O}_1\, \mathrm{T_d^{-1/2}}\, \mathcal{O}_2$.

Note that it is *not possible* to simultaneously diagonalize *three* symmetric matrices in general.

10.3.3 Finding the modal matrix

While the above proof allows one to construct A by finding the two orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

We start with the equations of motion, $\mathrm{T}\,\ddot{\boldsymbol{\eta}} + \mathrm{V}\,\boldsymbol{\eta} = 0$. In component notation, we have

$$\mathrm{T}_{\sigma\sigma'}\, \ddot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\, \eta_{\sigma'} = 0\ . \tag{10.23}$$

We now assume that $\boldsymbol{\eta}(t)$ oscillates with a single frequency ω, *i.e.* $\eta_\sigma(t) = \psi_\sigma\, e^{-i\omega t}$. This results in a set of linear algebraic equations for the components ψ_σ:

$$\big(\omega^2\, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\, \psi_{\sigma'} = 0\ . \tag{10.24}$$

These are n equations in n unknowns: one for each value of $\sigma = 1, \ldots, n$. Because the equations are homogeneous and linear, there is always a trivial solution $\boldsymbol{\psi} = 0$. In fact one might think this is the only solution, since

$$\big(\omega^2\, \mathrm{T} - \mathrm{V}\big)\, \boldsymbol{\psi} = 0 \qquad \stackrel{?}{\Longrightarrow} \qquad \boldsymbol{\psi} = \big(\omega^2\, \mathrm{T} - \mathrm{V}\big)^{-1}\, 0 = 0\ . \tag{10.25}$$

However, this fails when the matrix $\omega^2\, \mathrm{T} - \mathrm{V}$ is defective[1], *i.e.* when

$$\det\big(\omega^2\, \mathrm{T} - \mathrm{V}\big) = 0\ . \tag{10.26}$$

Since T and V are of rank n, the above determinant yields an n^{th} order polynomial in ω^2, whose n roots are the desired squared eigenfrequencies $\{\omega_1^2, \ldots, \omega_n^2\}$.

[1] The label *defective* has a distastefully negative connotation. In modern parlance, we should instead refer to such a matrix as *determinantally challenged*.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\sum_{\sigma'=1}^{n} \left(\omega_i^2 \, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\right) \psi_{\sigma'}^{(i)} = 0 \tag{10.27}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^{(i)}$. That is, there are n equations ($\sigma = 1, \ldots, n$), but one linear dependency since $\det\,(\omega_i^2\,\mathrm{T} - \mathrm{V}) = 0$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship,

$$\psi_{\sigma}^{(i)}\, \mathrm{T}_{\sigma\sigma'}\, \psi_{\sigma'}^{(j)} = \delta_{ij} \ . \tag{10.28}$$

To see this, let us duplicate eqn. 10.27, replacing i with j, and multiply both equations as follows:

$$\begin{aligned} \psi_{\sigma}^{(j)} \times \left(\omega_i^2\, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\right) \psi_{\sigma'}^{(i)} &= 0 \\ \psi_{\sigma}^{(i)} \times \left(\omega_j^2\, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\right) \psi_{\sigma'}^{(j)} &= 0 \ . \end{aligned} \tag{10.29}$$

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_i^2 - \omega_j^2) \sum_{\sigma,\sigma'=1}^{n} \psi_{\sigma}^{(i)}\, \mathrm{T}_{\sigma\sigma'}\, \psi_{\sigma'}^{(j)} = 0 \ , \tag{10.30}$$

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$ corresponding to distinct eigenvalues $\omega_i^2 \neq \omega_j^2$ are orthogonal: $(\boldsymbol{\psi}^{(i)})^{\mathrm{t}}\,\mathrm{T}\,\boldsymbol{\psi}^{(j)} = 0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector – are fixed by imposing the condition of normalization:

$$\psi_{\sigma}^{(i)} \to \psi_{\sigma}^{(i)} \Big/ \sqrt{\psi_{\mu}^{(i)}\, \mathrm{T}_{\mu\mu'}\, \psi_{\mu'}^{(i)}} \qquad \Longrightarrow \qquad \psi_{\sigma}^{(i)}\, \mathrm{T}_{\sigma\sigma'}\, \psi_{\sigma'}^{(j)} = \delta_{ij} \ . \tag{10.31}$$

The modal matrix is just the matrix of eigenvectors: $\mathrm{A}_{\sigma i} = \psi_{\sigma}^{(i)}$.

With the eigenvectors $\psi_{\sigma}^{(i)}$ thusly normalized, we have

$$\begin{aligned} 0 &= \psi_{\sigma}^{(i)} \left(\omega_j^2\, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\right) \psi_{\sigma'}^{(j)} \\ &= \omega_j^2\, \delta_{ij} - \psi_{\sigma}^{(i)}\, \mathrm{V}_{\sigma\sigma'}\, \psi_{\sigma'}^{(j)} \ , \end{aligned} \tag{10.32}$$

with no sum on j. This establishes the result

$$\mathrm{A}^{\mathrm{t}}\, \mathrm{V}\, \mathrm{A} = \mathrm{diag}\left(\omega_1^2\, , \, \ldots \, , \, \omega_n^2\right) \ . \tag{10.33}$$

10.4 Example: Masses and Springs

Two blocks and three springs are configured as in Fig. 10.1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

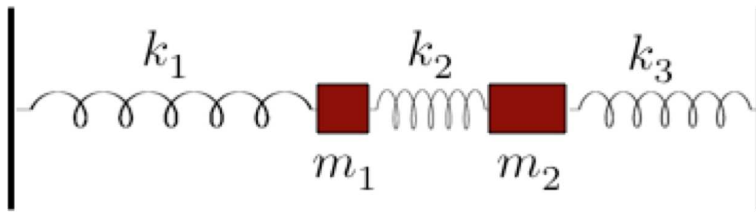


Figure 10.1: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.

(b) Find the T and V matrices.

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.

(d) Find the normal modes of oscillation.

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2} m_1 \, \dot{x}_1^2 + \tfrac{1}{2} m_2 \, \dot{x}_2^2 - \tfrac{1}{2} k_1 \, x_1^2 - \tfrac{1}{2} k_2 \, (x_2 - x_1)^2 - \tfrac{1}{2} k_3 \, x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i \, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda \, \omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \ .$$

The determinant is

$$\begin{aligned}\det\left(\omega^2 \mathrm{T} - \mathrm{V}\right) &= \left(2\lambda^2 - 11\lambda + 14\right) k^2 \\ &= \left(2\lambda - 7\right)\left(\lambda - 2\right) k^2 \ .\end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V})\, \vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)}\, \mathrm{T}_{ij}\, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \quad , \quad \mathcal{C}_+ = \frac{1}{\sqrt{6m}} \ . \tag{10.34}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \ .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 \ .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) \ . \end{aligned}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega_- t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega_- t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

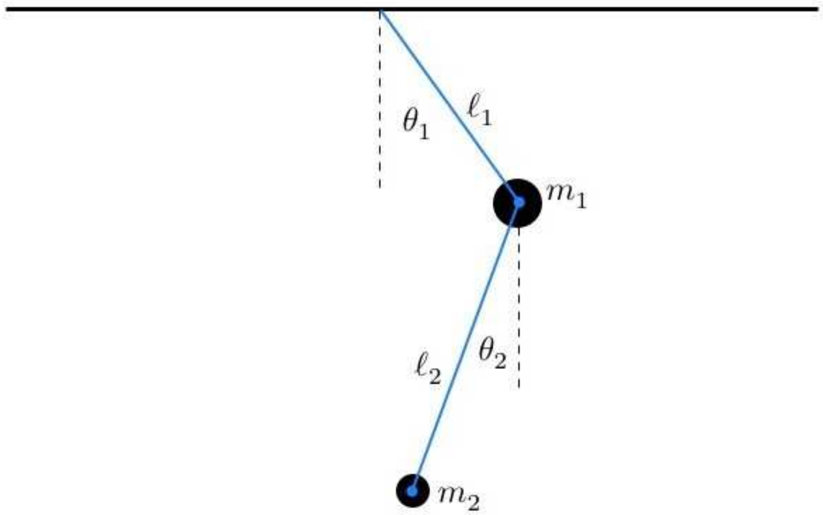


Figure 10.2: The double pendulum.

10.5 Example: Double Pendulum

As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$\begin{aligned} T &= m\ell^2\dot\theta_1^2 + m\ell^2\cos(\theta_1 - \theta_1)\,\dot\theta_1\dot\theta_2 + \tfrac{1}{2}m\ell^2\dot\theta_2^2 \\ V &= -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2 \ , \end{aligned} \tag{10.35}$$

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \ . \tag{10.36}$$

Then

$$\omega^2\mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \ , \tag{10.37}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2 \ . \tag{10.38}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathrm{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \ , \tag{10.39}$$

where $C_\pm$ are constants. One can check $\mathrm{T}_{\sigma\sigma'}\,\psi^{(i)}_\sigma\,\psi^{(j)}_{\sigma'}$ vanishes for $i \neq j$. We then normalize by demanding $\mathrm{T}_{\sigma\sigma'}\,\psi^{(i)}_\sigma\,\psi^{(i)}_{\sigma'} = 1$ (no sum on i), which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$\mathrm{A}=\begin{pmatrix}\psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^-\end{pmatrix}=\frac{1}{2\sqrt{m\ell^2}}\begin{pmatrix}\sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}}\end{pmatrix}\ . \tag{10.40}$$

10.6 Zero Modes

Recall Noether's theorem, which says that for every continuous one-parameter family of coordinate transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q,\zeta) \qquad , \qquad \tilde{q}_\sigma(q,\zeta=0)=q_\sigma\ , \tag{10.41}$$

which leaves the Lagrangian invariant, *i.e.* $dL/d\zeta=0$, there is an associated conserved quantity,

$$\Lambda=\sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \qquad \text{satisfies} \qquad \frac{d\Lambda}{dt}=0\ . \tag{10.42}$$

For small oscillations, we write $q_\sigma=\bar{q}_\sigma+\eta_\sigma$, hence

$$\Lambda_k=\sum_\sigma C_{k\sigma}\,\dot{\eta}_\sigma\ , \tag{10.43}$$

where k labels the one-parameter families (in the event there is more than one continuous symmetry), and where

$$C_{k\sigma}=\sum_{\sigma'}\mathrm{T}_{\sigma\sigma'}\,\frac{\partial \tilde{q}_{\sigma'}}{\partial \zeta_k}\bigg|_{\zeta=0}\ . \tag{10.44}$$

Therefore, we can define the (unnormalized) normal mode

$$\xi_k=\sum_\sigma C_{k\sigma}\,\eta_\sigma\ , \tag{10.45}$$

which satisfies $\ddot{\xi}_k=0$. Thus, in systems with continuous symmetries, to each such continuous symmetry there is an associated zero mode of the small oscillations problem, *i.e.* a mode with $\omega_k^2=0$.

10.6.1 Example of zero mode oscillations

The simplest example of a zero mode would be a pair of masses m_1 and m_2 moving frictionlessly along a line and connected by a spring of force constant k. We know from our study of central forces that the Lagrangian may be written

$$\begin{aligned} L&=\tfrac{1}{2}m_1\dot{x}_1^2+\tfrac{1}{2}m_2\dot{x}_2^2-\tfrac{1}{2}k(x_1-x_2)^2 \\ &=\tfrac{1}{2}M\dot{X}^2+\tfrac{1}{2}\mu\dot{x}^2-\tfrac{1}{2}kx^2\ , \end{aligned} \tag{10.46}$$

where $X=(m_1x_1+m_2x_2)/(m_1+m_2)$ is the center of mass position, $x=x_1-x_2$ is the relative coordinate, $M=m_1+m_2$ is the total mass, and $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass. The relative coordinate

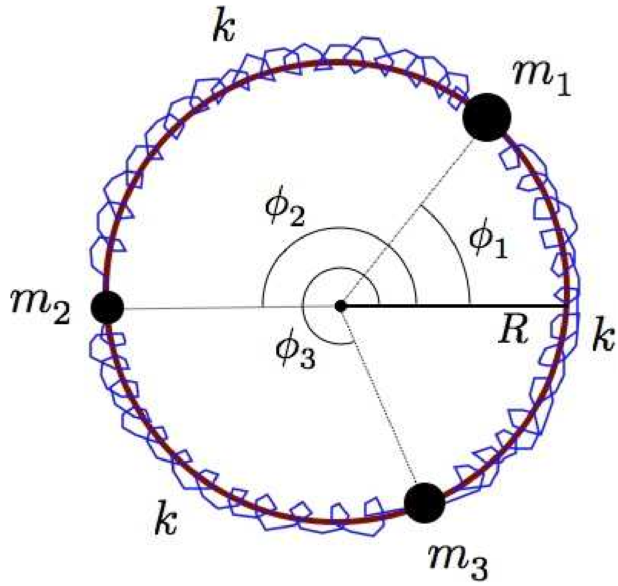


Figure 10.3: Coupled oscillations of three masses on a frictionless hoop of radius R. All three springs have the same force constant k, but the masses are all distinct.

obeys $\ddot{x} = -\omega_0^2\, x$, where the oscillation frequency is $\omega_0 = \sqrt{k/\mu}$. The center of mass coordinate obeys $\ddot{X} = 0$, *i.e.* its oscillation frequency is zero. The center of mass motion is a zero mode.

Another example is furnished by the system depicted in fig. 10.3, where three distinct masses m_1, m_2, and m_3 move around a frictionless hoop of radius R. The masses are connected to their neighbors by identical springs of force constant k. We choose as generalized coordinates the angles ϕ_σ ($\sigma = 1, 2, 3$), with the convention that

$$\phi_1 \le \phi_2 \le \phi_3 \le 2\pi + \phi_1 \ . \tag{10.47}$$

Let $R\chi$ be the equilibrium length for each of the springs. Then the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2} k R^2 \Big\{ (\phi_2 - \phi_1 - \chi)^2 + (\phi_3 - \phi_2 - \chi)^2 + (2\pi + \phi_1 - \phi_3 - \chi)^2 \Big\} \\ &= \tfrac{1}{2} k R^2 \Big\{ (\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2 + (2\pi + \phi_1 - \phi_3)^2 + 3\chi^2 - 4\pi\chi \Big\} \ . \end{aligned} \tag{10.48}$$

Note that the equilibrium angle χ enters only in an additive constant to the potential energy. Thus, for the calculation of the equations of motion, it is irrelevant. It doesn't matter whether or not the equilibrium configuration is unstretched ($\chi = 2\pi/3$) or not ($\chi \neq 2\pi/3$).

The kinetic energy is simple:

$$T = \tfrac{1}{2} R^2 \Big(m_1\, \dot{\phi}_1^2 + m_2\, \dot{\phi}_2^2 + m_3\, \dot{\phi}_3^2 \Big) \ . \tag{10.49}$$

The T and V matrices are then

$$\mathrm{T} = \begin{pmatrix} m_1 R^2 & 0 & 0 \\ 0 & m_2 R^2 & 0 \\ 0 & 0 & m_3 R^2 \end{pmatrix} \quad , \qquad \mathrm{V} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix} \ . \tag{10.50}$$

We then have

$$\omega^2\,\mathrm{T} - \mathrm{V} = kR^2 \begin{pmatrix} \frac{\omega^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\Omega_3^2} - 2 \end{pmatrix} . \tag{10.51}$$

We compute the determinant to find the characteristic polynomial:

$$\begin{aligned} P(\omega) &= \det(\omega^2\,\mathrm{T} - \mathrm{V}) \\ &= \frac{\omega^6}{\Omega_1^2\,\Omega_2^2\,\Omega_3^2} - 2\left(\frac{1}{\Omega_1^2\,\Omega_2^2} + \frac{1}{\Omega_2^2\,\Omega_3^2} + \frac{1}{\Omega_1^2\,\Omega_3^2}\right)\omega^4 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right)\omega^2 \ , \end{aligned} \tag{10.52}$$

where $\Omega_i^2 \equiv k/m_i$. The equation $P(\omega) = 0$ yields a cubic equation in ω^2, but clearly ω^2 is a factor, and when we divide this out we obtain a quadratic equation. One root obviously is $\omega_1^2 = 0$. The other two roots are solutions to the quadratic equation:

$$\omega_{2,3}^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \pm \sqrt{\tfrac{1}{2}\big(\Omega_1^2 - \Omega_2^2\big)^2 + \tfrac{1}{2}\big(\Omega_2^2 - \Omega_3^2\big)^2 + \tfrac{1}{2}\big(\Omega_1^2 - \Omega_3^2\big)^2} \ . \tag{10.53}$$

To find the eigenvectors and the modal matrix, we set

$$\begin{pmatrix} \frac{\omega_j^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\Omega_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0 \ , \tag{10.54}$$

Writing down the three coupled equations for the components of $\boldsymbol{\psi}^{(j)}$, we find

$$\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)\psi_1^{(j)} = \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)\psi_2^{(j)} = \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)\psi_3^{(j)} \ . \tag{10.55}$$

We therefore conclude

$$\boldsymbol{\psi}^{(j)} = \mathcal{C}_j \begin{pmatrix} \left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-1} \end{pmatrix} . \tag{10.56}$$

The normalization condition $\psi_\sigma^{(i)}\,\mathrm{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)} = \delta_{ij}$ then fixes the constants $\mathcal{C}_j$:

$$\left[m_1\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-2} + \ m_2\left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-2} + \ m_3\left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-2} \right] \big|\mathcal{C}_j\big|^2 = 1 \ . \tag{10.57}$$

The Lagrangian is invariant under the one-parameter family of transformations

$$\phi_\sigma \longrightarrow \phi_\sigma + \zeta \tag{10.58}$$

for all $\sigma = 1, 2, 3$. The associated conserved quantity is

$$\begin{aligned} \varLambda &= \sum_\sigma \frac{\partial L}{\partial \dot\phi_\sigma} \frac{\partial \tilde\phi_\sigma}{\partial \zeta} \\ &= R^2 \left(m_1 \dot\phi_1 + m_2 \dot\phi_2 + m_3 \dot\phi_3 \right) \ , \end{aligned} \tag{10.59}$$

which is, of course, the total angular momentum relative to the center of the ring. Thus, from $\dot\varLambda = 0$ we identify the zero mode as ξ_1, where

$$\xi_1 = \mathcal{C} \left(m_1 \phi_1 + m_2 \phi_2 + m_3 \phi_3 \right) \ , \tag{10.60}$$

where $\mathcal{C}$ is a constant. Recall the relation $\eta_\sigma = \mathrm{A}_{\sigma i}\, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i. We can invert this relation to obtain

$$\xi_i = \mathrm{A}^{-1}_{i\sigma}\, \eta_\sigma = \mathrm{A}^{\mathrm{t}}_{i\sigma}\, \mathrm{T}_{\sigma\sigma'}\, \eta_{\sigma'} \ . \tag{10.61}$$

Here we have used the result $\mathrm{A}^{\mathrm{t}}\, \mathrm{T}\, \mathrm{A} = 1$ to write

$$\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}}\, \mathrm{T} \ . \tag{10.62}$$

This is a convenient result, because it means that if we ever need to express the normal coordinates in terms of the generalized displacements, we don't have to invert any matrices – we just need to do one matrix multiplication. In our case here, the T matrix is diagonal, so the multiplication is trivial. From eqns. 10.60 and 10.61, we conclude that the matrix $\mathrm{A}^{\mathrm{t}}\,\mathrm{T}$ must have a first *row* which is proportional to (m_1, m_2, m_3). Since these are the very diagonal entries of T, we conclude that A^{t} itself must have a first row which is proportional to $(1, 1, 1)$, which means that the first *column* of A is proportional to $(1, 1, 1)$. But this is confirmed by eqn. 10.55 when we take $j = 1$, since $\omega^2_{j=1} = 0$: $\psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3$.

10.7 Chain of Mass Points

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2} m \sum_n \dot x_n^2 - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\ &= \tfrac{1}{2} m \sum_n \dot u_n^2 - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \ , \end{aligned} \tag{10.63}$$

where $u_n \equiv x_n - na - b$ is the displacement from equilibrium of the n^{th} mass. The constant b is arbitrary. The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot u_n}\right) = m \ddot u_n &= \frac{\partial L}{\partial u_n} \\ &= k(u_{n+1} - u_n) - k(u_n - u_{n-1}) \\ &= k(u_{n+1} + u_{n-1} - 2u_n) \ . \end{aligned} \tag{10.64}$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$
\begin{aligned}
u_n &= \frac{1}{\sqrt{N}} \sum_q e^{iqan}\, \hat{u}_q \\
\hat{u}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, u_n \ ,
\end{aligned}
\tag{10.65}
$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat{u}_q\}$, the equations of motion become

$$
\begin{aligned}
\ddot{\hat{u}}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot{u}_n \\
&= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(u_{n+1} + u_{n-1} - 2u_n\right) \\
&= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(e^{-iqa} + e^{+iqa} - 2\right) u_n \\
&= -\frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \hat{u}_q
\end{aligned}
\tag{10.66}
$$

Thus, the $\{\hat{u}_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$
\omega_q = 2\sqrt{\frac{k}{m}}\, \left|\sin\left(\tfrac{1}{2}qa\right)\right| \ . \tag{10.67}
$$

This means that the modal matrix is

$$
\mathrm{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan} \ , \tag{10.68}
$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. (The normal modes themselves are then $\xi_q = \mathrm{A}^\dagger_{qn} \mathrm{T}_{nn'} u_{n'} = \sqrt{m}\, \hat{u}_q$. For complex A, the normalizations are $\mathrm{A}^\dagger \mathrm{T} \mathrm{A} = \mathbb{I}$ and $\mathrm{A}^\dagger \mathrm{V} \mathrm{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$
\begin{aligned}
\mathrm{T}_{nn'} &= m\, \delta_{n,n'} \\
\mathrm{V}_{nn'} &= 2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}
\end{aligned}
\tag{10.69}
$$

and that

$$
\begin{aligned}
(\mathrm{A}^\dagger \mathrm{T} \mathrm{A})_{qq'} &= \sum_{n=1}^{N} \sum_{n'=1}^{N} \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^{N} \sum_{n'=1}^{N} e^{-iqan}\, m\, \delta_{nn'}\, e^{iq'an'} \\
&= \frac{1}{N} \sum_{n=1}^{N} e^{i(q'-q)an} = \delta_{qq'} \ ,
\end{aligned}
\tag{10.70}
$$

and

$$\begin{aligned}
(\mathrm{A}^\dagger \mathrm{V} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan} \Big(2k\,\delta_{n,n'} - k\,\delta_{n,n'+1} - k\,\delta_{n,n'-1} \Big)\, e^{iq'an'} \\
&= \frac{k}{m} \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} \big(2 - e^{-iq'a} - e^{iq'a} \big) \\
&= \frac{4k}{m} \sin^2\!\big(\tfrac{1}{2}qa\big)\, \delta_{qq'} = \omega_q^2\, \delta_{qq'}
\end{aligned} \tag{10.71}$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = \frac{2\pi}{a}$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbb{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[2]. One natural set to choose is $q \in \big[-\frac{\pi}{a}, \frac{\pi}{a}\big]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}_q^*\, \dot{\hat{u}}_q - k \sum_q (1 - \cos qa)\, \hat{u}_q^*\, \hat{u}_q \ , \tag{10.72}$$

where the sum is over q in the first Brillouin zone. Note that

$$\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}_q^* \ . \tag{10.73}$$

This means that we can restrict the sum to half the Brillouin zone:

$$L = \tfrac{1}{2} m \sum_{q \in [0, \frac{\pi}{a}]} \bigg\{ \dot{\hat{u}}_q^*\, \dot{\hat{u}}_q - \frac{4k}{m} \sin^2\!\big(\tfrac{1}{2}qa\big)\, \hat{u}_q^*\, \hat{u}_q \bigg\} \ . \tag{10.74}$$

Now $\hat{u}_q$ and $\hat{u}_q^*$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}_q^*$ gives

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\hat{u}}_q^*}\bigg) = \frac{\partial L}{\partial \hat{u}_q^*} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2\, \hat{u}_q \ . \tag{10.75}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

10.7.1 Continuum limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write

$$u_n(t) \longrightarrow u(x = na, t) \tag{10.76}$$

[2]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

in which case

$$\begin{aligned} T &= \tfrac{1}{2} m \sum_n \dot{u}_n^2 \qquad \longrightarrow \qquad \tfrac{1}{2} m \int \frac{dx}{a} \left(\frac{\partial u}{\partial t}\right)^2 \\ V &= \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \qquad \longrightarrow \qquad \tfrac{1}{2} k \int \frac{dx}{a} \left(\frac{u(x+a) - u(x)}{a}\right)^2 a^2 \end{aligned} \tag{10.77}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int \! dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2} \, \mu \left(\frac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2} \, \tau \left(\frac{\partial u}{\partial x}\right)^2 , \end{aligned} \tag{10.78}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[3]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[4]. The action is

$$S\big[u(x,t)\big] = \int_{t_a}^{t_b} \! dt \int_{x_a}^{x_b} \! dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \ , \tag{10.79}$$

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t u\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \left(\partial_x u\right)}\right) = 0 \ . \tag{10.80}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \tag{10.81}$$

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C \, e^{iqx} \, e^{-i\omega t} \ , \tag{10.82}$$

where

$$\omega = cq \ . \tag{10.83}$$

Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m} \sin^2\!\big(\tfrac{1}{2} qa\big) \longrightarrow \frac{ka^2}{m} \, q^2 = c^2 \, q^2 \ , \tag{10.84}$$

and so the results agree.

[3]For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.

[4]$\mathcal{L}$ may also depend explicitly on x and t.

10.8 Appendix I : General Formulation

In the development in section 10.1, we assumed that the kinetic energy T is a homogeneous function of degree 2, and the potential energy U a homogeneous function of degree 0, in the generalized velocities $\dot{q}_\sigma$. However, we've encountered situations where this is not so: problems with time-dependent holonomic constraints, such as the mass point on a rotating hoop, and problems involving charged particles moving in magnetic fields. The general Lagrangian is of the form

$$L = \tfrac{1}{2}\, T_{2\,\sigma\sigma'}(q)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} + T_{1\,\sigma}(q)\, \dot{q}_\sigma + T_0(q) - U_{1\,\sigma}(q)\, \dot{q}_\sigma - U_0(q) \ , \tag{10.85}$$

where the subscript 0, 1, or 2 labels the degree of homogeneity of each term in the generalized velocities. The generalized momenta are then

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{2\,\sigma\sigma'}\, \dot{q}_{\sigma'} + T_{1\,\sigma} - U_{1\,\sigma} \tag{10.86}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{\partial (T_0 - U_0)}{\partial q_\sigma} + \frac{\partial (T_{1\,\sigma'} - U_{1\,\sigma'})}{\partial q_\sigma}\, \dot{q}_{\sigma'} + \frac{1}{2}\, \frac{\partial T_{2\,\sigma'\sigma''}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} \ , \tag{10.87}$$

and the equations of motion are again $\dot{p}_\sigma = F_\sigma$. Once we solve

In equilibrium, we seek a time-independent solution of the form $q_\sigma(t) = \bar{q}_\sigma$. This entails

$$\frac{\partial}{\partial q_\sigma}\bigg|_{q=\bar{q}} \Big(U_0(q) - T_0(q) \Big) = 0 \ , \tag{10.88}$$

which give us n equations in the n unknowns $(q_1, \ldots, q_n)$. We then write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand in the notionally small quantities η_σ. It is important to understand that we assume η and all of its time derivatives as well are small. Thus, we can expand L to quadratic order in $(\eta, \dot{\eta})$ to obtain

$$L = \tfrac{1}{2}\, \mathrm{T}_{\sigma\sigma'}\, \dot{\eta}_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{B}_{\sigma\sigma'}\, \eta_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{V}_{\sigma\sigma'}\, \eta_\sigma\, \eta_{\sigma'} \ , \tag{10.89}$$

where

$$\mathrm{T}_{\sigma\sigma'} = T_{2\,\sigma\sigma'}(\bar{q}) \quad , \quad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 \big(U_0 - T_0 \big)}{\partial q_\sigma\, \partial q_{\sigma'}}\bigg|_{q=\bar{q}} \quad , \quad \mathrm{B}_{\sigma\sigma'} = 2\, \frac{\partial \big(U_{1\,\sigma'} - T_{1\,\sigma'} \big)}{\partial q_\sigma}\bigg|_{q=\bar{q}} \ . \tag{10.90}$$

Note that the T and V matrices are symmetric. The $\mathrm{B}_{\sigma\sigma'}$ term is new.

Now we can always write $\mathrm{B} = \frac{1}{2}(\mathrm{B}^{\mathrm{s}} + \mathrm{B}^{\mathrm{a}})$ as a sum over symmetric and antisymmetric parts, with $\mathrm{B}^{\mathrm{s}} = \mathrm{B} + \mathrm{B}^{\mathrm{t}}$ and $\mathrm{B}^{\mathrm{a}} = \mathrm{B} - \mathrm{B}^{\mathrm{t}}$. Since,

$$\mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\, \eta_\sigma\, \dot{\eta}_{\sigma'} = \frac{d}{dt}\Big(\tfrac{1}{2}\, \mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\, \eta_\sigma\, \eta_{\sigma'} \Big) \ , \tag{10.91}$$

any symmetric part to B contributes a total time derivative to L, and thus has no effect on the equations of motion. Therefore, we can project B onto its antisymmetric part, writing

$$\mathrm{B}_{\sigma\sigma'} = \left(\frac{\partial\big(U_{1\,\sigma'} - T_{1\,\sigma'}\big)}{\partial q_\sigma} - \frac{\partial\big(U_{1\,\sigma} - T_{1\,\sigma}\big)}{\partial q_{\sigma'}}\right)_{q=\bar{q}} \ . \tag{10.92}$$

We now have

$$p_\sigma = \frac{\partial L}{\partial \dot{\eta}_\sigma} = \mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \tfrac{1}{2}\,\mathrm{B}_{\sigma\sigma'}\,\eta_{\sigma'} \ , \tag{10.93}$$

and

$$F_\sigma = \frac{\partial L}{\partial \eta_\sigma} = -\tfrac{1}{2}\,\mathrm{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} - \mathrm{V}_{\sigma\sigma'}\,\eta_{\sigma'} \ . \tag{10.94}$$

The equations of motion, $\dot{p}_\sigma = F_\sigma$, then yield

$$\mathrm{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathrm{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \ . \tag{10.95}$$

Let us write $\boldsymbol{\eta}(t) = \boldsymbol{\eta}\,e^{-i\omega t}$. We then have

$$\big(\omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V}\big)\,\boldsymbol{\eta} = 0 \ . \tag{10.96}$$

To solve eqn. 10.96, we set $P(\omega) = 0$, where $P(\omega) = \det\big[\mathrm{Q}(\omega)\big]$, with

$$\mathrm{Q}(\omega) \equiv \omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V} \ . \tag{10.97}$$

Since T, B, and V are real-valued matrices, and since $\det(M) = \det(M^{\mathrm{t}})$ for any matrix M, we can use $\mathrm{B}^{\mathrm{t}} = -\mathrm{B}$ to obtain $P(-\omega) = P(\omega)$ and $P(\omega^*) = \big[P(\omega)\big]^*$. This establishes that if $P(\omega) = 0$, *i.e.* if ω is an eigenfrequency, then $P(-\omega) = 0$ and $P(\omega^*) = 0$, *i.e.* $-\omega$ and ω^* are also eigenfrequencies (and hence $-\omega^*$ as well).

10.9 Appendix II : Additional Examples

10.9.1 Right Triatomic Molecule

A molecule consists of three identical atoms located at the vertices of a 45° right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Consider only planar motion of this molecule.

(a) Find three 'zero modes' for this system (*i.e.* normal modes whose associated eigenfrequencies vanish).

(b) Find the remaining three normal modes.

Solution

It is useful to choose the following coordinates:

$$\begin{aligned}(X_1, Y_1) &= (x_1\ ,\ y_1)\\ (X_2, Y_2) &= (a + x_2\ ,\ y_2)\\ (X_3, Y_3) &= (x_3\ ,\ a + y_3)\ .\end{aligned} \tag{10.98}$$

The three separations are then

$$\begin{aligned}d_{12} &= \sqrt{(a + x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= a + x_2 - x_1 + \ldots\\[2ex] d_{23} &= \sqrt{(-a + x_3 - x_2)^2 + (a + y_3 - y_2)^2}\\ &= \sqrt{2}\,a - \tfrac{1}{\sqrt{2}}\big(x_3 - x_2\big) + \tfrac{1}{\sqrt{2}}\big(y_3 - y_2\big) + \ldots\\[2ex] d_{13} &= \sqrt{(x_3 - x_1)^2 + (a + y_3 - y_1)^2}\\ &= a + y_3 - y_1 + \ldots\ .\end{aligned} \tag{10.99}$$

The potential is then

$$\begin{aligned}U &= \tfrac{1}{2}k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}k\,\big(d_{23} - \sqrt{2}\,a\big)^2 + \tfrac{1}{2}k\,\big(d_{13} - a\big)^2\\ &= \tfrac{1}{2}k\big(x_2 - x_1\big)^2 + \tfrac{1}{4}k\big(x_3 - x_2\big)^2 + \tfrac{1}{4}k\big(y_3 - y_2\big)^2\\ &\qquad - \tfrac{1}{2}k\big(x_3 - x_2\big)\big(y_3 - y_2\big) + \tfrac{1}{2}k\big(y_3 - y_1\big)^2\end{aligned} \tag{10.100}$$

Defining the row vector

$$\boldsymbol{\eta}^{\rm t} \equiv \big(x_1\ ,\ y_1\ ,\ x_2\ ,\ y_2\ ,\ x_3\ ,\ y_3\big)\ , \tag{10.101}$$

we have that U is a quadratic form:

$$U = \tfrac{1}{2}\eta_\sigma \mathrm{V}_{\sigma\sigma'}\eta_{\sigma'} = \tfrac{1}{2}\boldsymbol{\eta}^{\rm t}\,\mathrm{V}\,\boldsymbol{\eta}, \tag{10.102}$$

with

$$\mathrm{V} = \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\rm eq.} = k\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -1\\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2}\end{pmatrix} \tag{10.103}$$

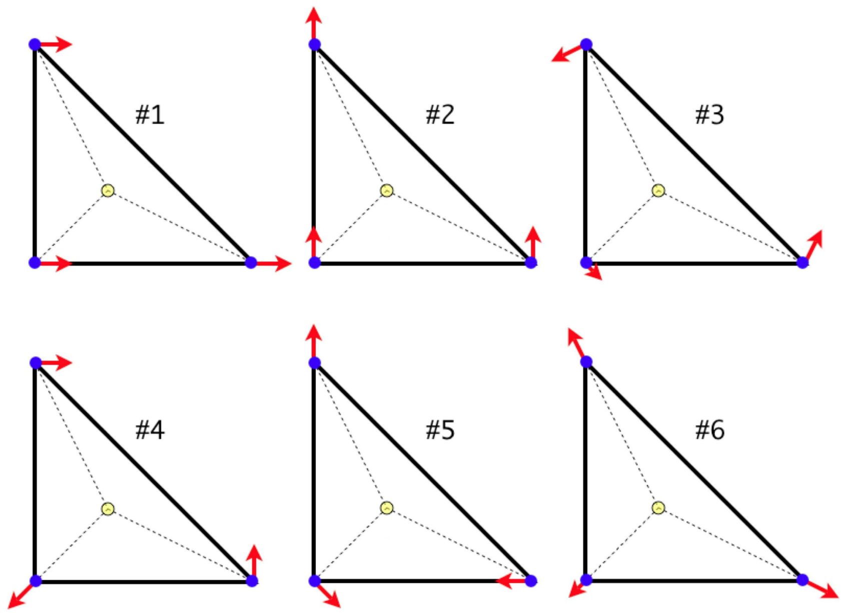


Figure 10.4: Normal modes of the 45° right triangle. The yellow circle is the location of the CM of the triangle.

The kinetic energy is simply

$$T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \ , \tag{10.104}$$

which entails

$$\mathrm{T}_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'} \ . \tag{10.105}$$

(b) The three zero modes correspond to x-translation, y-translation, and rotation. Their eigenvectors, respectively, are

$$\boldsymbol{\psi}_1 = \frac{1}{\sqrt{3m}}\begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{3m}}\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{2\sqrt{3m}}\begin{pmatrix}1\\-1\\1\\2\\-2\\-1\end{pmatrix} \ . \tag{10.106}$$

To find the unnormalized rotation vector, we find the CM of the triangle, located at $\big(\frac{a}{3}\,,\,\frac{a}{3}\big)$, and sketch orthogonal displacements $\hat{\boldsymbol{z}} \times (\boldsymbol{R}_i - \boldsymbol{R}_{\rm CM})$ at the position of mass point i.

(c) The remaining modes may be determined by symmetry, and are given by

$$\boldsymbol{\psi}_4 = \frac{1}{2\sqrt{m}}\begin{pmatrix}-1\\-1\\0\\1\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{\psi}_5 = \frac{1}{2\sqrt{m}}\begin{pmatrix}1\\-1\\-1\\0\\0\\1\end{pmatrix} \quad , \quad \boldsymbol{\psi}_6 = \frac{1}{2\sqrt{3m}}\begin{pmatrix}-1\\-1\\2\\-1\\-1\\2\end{pmatrix} \ , \tag{10.107}$$

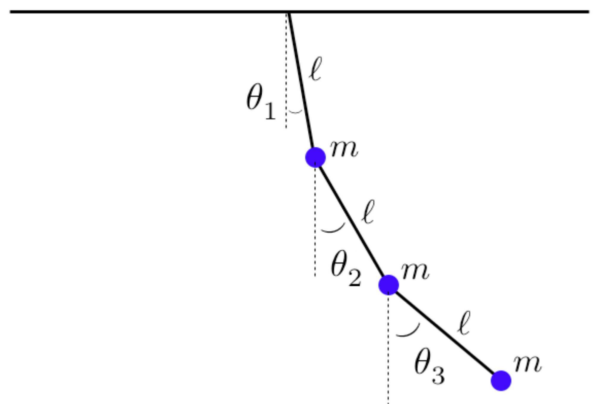


Figure 10.5: The triple pendulum.

with

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad , \qquad \omega_2 = \sqrt{\frac{2k}{m}} \qquad , \qquad \omega_3 = \sqrt{\frac{3k}{m}} \ . \tag{10.108}$$

Since $\mathrm{T} = m \cdot 1$ is a multiple of the unit matrix, the orthogonormality relation $\psi^a_i \, \mathrm{T}_{ij} \, \psi^b_j = \delta^{ab}$ entails that the eigenvectors are mutually orthogonal in the usual dot product sense, with $\boldsymbol{\psi}_a \cdot \boldsymbol{\psi}_b = m^{-1} \, \delta_{ab}$. One can check that the eigenvectors listed here satisfy this condition.

The simplest of the set $\{\boldsymbol{\psi}_4, \boldsymbol{\psi}_5, \boldsymbol{\psi}_6\}$ to find is the uniform dilation $\boldsymbol{\psi}_6$, sometimes called the 'breathing' mode. This must keep the triangle in the same shape, which means that the deviations at each mass point are proportional to the distance to the CM. Next, it is simplest to find $\boldsymbol{\psi}_4$, in which the long and short sides of the triangle oscillate out of phase. Finally, the mode $\boldsymbol{\psi}_5$ must be orthogonal to all the remaining modes. No heavy lifting (*e.g.* `Mathematica`) is required!

10.9.2 Triple Pendulum

Consider a triple pendulum consisting of three identical masses m and three identical rigid massless rods of length ℓ, as depicted in Fig. 10.5.

(a) Find the T and V matrices.

(b) Find the equation for the eigenfrequencies.

(c) Numerically solve the eigenvalue equation for ratios ω_a^2/ω_0^2, where $\omega_0 = \sqrt{g/\ell}$. Find the three normal modes.

Solution

The Cartesian coordinates for the three masses are

$$
\begin{aligned}
x_1 &= \ell \sin\theta_1 & y_1 &= -\ell \cos\theta_1 \\
x_2 &= \ell \sin\theta_1 + \ell \sin\theta_2 & y_2 &= -\ell \cos\theta_1 - \ell \cos\theta_2 \\
x_3 &= \ell \sin\theta_1 + \ell \sin\theta_2 + \ell \sin\theta_3 & y_3 &= -\ell \cos\theta_1 - \ell \cos\theta_2 - \ell \cos\theta_3 \ .
\end{aligned}
$$

By inspection, we can write down the kinetic energy:

$$
\begin{aligned}
T &= \tfrac{1}{2} m \big(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2 + \dot x_3^3 + \dot y_3^2\big) \\
&= \tfrac{1}{2} m\, \ell^2 \Big\{ 3\,\dot\theta_1^2 + 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 \\
&\qquad + 2\,\cos(\theta_1 - \theta_3)\,\dot\theta_1\,\dot\theta_3 + 2\,\cos(\theta_2 - \theta_3)\,\dot\theta_2\,\dot\theta_3 \Big\}
\end{aligned}
$$

The potential energy is

$$
U = -mg\ell \Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\} \ ,
$$

and the Lagrangian is $L = T - U$:

$$
\begin{aligned}
L = \tfrac{1}{2} m\, \ell^2 \Big\{ & 3\,\dot\theta_1^2 + 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 + 2\,\cos(\theta_1 - \theta_3)\,\dot\theta_1\,\dot\theta_3 \\
& + 2\,\cos(\theta_2 - \theta_3)\,\dot\theta_2\,\dot\theta_3 \Big\} + mg\ell\,\Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\} \ .
\end{aligned}
$$

The canonical momenta are given by

$$
\begin{aligned}
\pi_1 &= \frac{\partial L}{\partial \dot\theta_1} = m\,\ell^2 \Big\{ 3\,\dot\theta_1 + 2\,\dot\theta_2\,\cos(\theta_1 - \theta_2) + \dot\theta_3\,\cos(\theta_1 - \theta_3) \Big\} \\
\pi_2 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ 2\,\dot\theta_2 + 2\,\dot\theta_1\,\cos(\theta_1 - \theta_2) + \dot\theta_3\,\cos(\theta_2 - \theta_3) \Big\} \\
\pi_3 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ \dot\theta_3 + \dot\theta_1\,\cos(\theta_1 - \theta_3) + \dot\theta_2\,\cos(\theta_2 - \theta_3) \Big\} \ .
\end{aligned}
$$

The only conserved quantity is the total energy, $E = T + U$.

(a) As for the T and V matrices, we have

$$
\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = m\ell^2 \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}
$$

and

$$
\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = mg\ell \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .
$$

(b) The eigenfrequencies are roots of the equation $\det(\omega^2\,\mathrm{T}-\mathrm{V})=0$. Defining $\omega_0\equiv\sqrt{g/\ell}$, we have

$$\omega^2\,\mathrm{T}-\mathrm{V}=m\ell^2\begin{pmatrix}3(\omega^2-\omega_0^2) & 2\,\omega^2 & \omega^2\\ 2\,\omega^2 & 2(\omega^2-\omega_0^2) & \omega^2\\ \omega^2 & \omega^2 & (\omega^2-\omega_0^2)\end{pmatrix}$$

and hence

$$\begin{aligned}\det(\omega^2\mathrm{T}-\mathrm{V}) &= 3(\omega^2-\omega_0^2)\cdot\Big[2(\omega^2-\omega_0^2)^2-\omega^4\Big]-2\,\omega^2\cdot\Big[2\,\omega^2(\omega^2-\omega_0^2)-\omega^4\Big]\\ &\qquad+\omega^2\cdot\Big[2\,\omega^4-2\,\omega^2(\omega^2-\omega_0^2)\Big]\\ &=6\,(\omega^2-\omega_0^2)^3-9\,\omega^4\,(\omega^2-\omega_0^2)+4\omega^6\\ &=\omega^6-9\,\omega_0^2\,\omega^4+18\,\omega_0^4\,\omega^2-6\,\omega_0^6\ .\end{aligned}$$

(c) The equation for the eigenfrequencies is

$$\lambda^3-9\,\lambda^2+18\,\lambda-6=0\ , \tag{10.109}$$

where $\omega^2=\lambda\,\omega_0^2$. This is a cubic equation in λ. Numerically solving for the roots, one finds

$$\omega_1^2=0.415774\,\omega_0^2\quad,\quad\omega_2^2=2.29428\,\omega_0^2\quad,\quad\omega_3^2=6.28995\,\omega_0^2\ . \tag{10.110}$$

I find the (unnormalized) eigenvectors to be

$$\boldsymbol{\psi}_1=\begin{pmatrix}1\\1.2921\\1.6312\end{pmatrix}\quad,\quad\boldsymbol{\psi}_2=\begin{pmatrix}1\\0.35286\\-2.3981\end{pmatrix}\quad,\quad\boldsymbol{\psi}_3=\begin{pmatrix}1\\-1.6450\\0.76690\end{pmatrix}\ . \tag{10.111}$$

10.9.3 Equilateral Linear Triatomic Molecule

Consider the vibrations of an equilateral triangle of mass points, depicted in figure 10.6 . The system is confined to the (x,y) plane, and in equilibrium all the strings are unstretched and of length a.

(a) Choose as generalized coordinates the Cartesian displacements (x_i,y_i) with respect to equilibrium. Write down the exact potential energy.

(b) Find the T and V matrices.

(c) There are three normal modes of oscillation for which the corresponding eigenfrequencies all vanish: $\omega_a=0$. Write down these modes explicitly, and provide a physical interpretation for why $\omega_a=0$. Since this triplet is degenerate, there is no unique answer – any linear combination will also serve as a valid 'zero mode'. However, if you think physically, a natural set should emerge.

(d) The three remaining modes all have finite oscillation frequencies. They correspond to distortions of the triangular shape. One such mode is the "breathing mode" in which the triangle uniformly expands

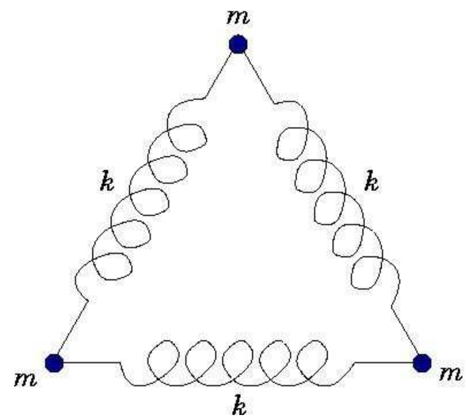


Figure 10.6: An equilateral triangle of identical mass points and springs.

and contracts. Write down the eigenvector associated with this normal mode and compute its associated oscillation frequency.

(e) The fifth and sixth modes are degenerate. They must be orthogonal (with respect to the inner product defined by T) to all the other modes. See if you can figure out what these modes are, and compute their oscillation frequencies. As in (a), any linear combination of these modes will also be an eigenmode.

(f) Write down your full expression for the modal matrix A_{ai}, and check that it is correct by using `Mathematica`.

Solution

Choosing as generalized coordinates the Cartesian displacements relative to equilibrium, we have the following:

$$\begin{aligned}
&\#1 : \big(x_1, y_1\big) \\
&\#2 : \big(a + x_2, y_2\big) \\
&\#3 : \big(\tfrac{1}{2}a + x_3, \tfrac{\sqrt{3}}{2}a + y_3\big) \ .
\end{aligned}$$

Let d_{ij} be the separation of particles i and j. The potential energy of the spring connecting them is then $\frac{1}{2}\,k\,(d_{ij} - a)^2$.

$$\begin{aligned}
d_{12}^2 &= \big(a + x_2 - x_1\big)^2 + \big(y_2 - y_1\big)^2 \\
d_{23}^2 &= \big(-\tfrac{1}{2}a + x_3 - x_2\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_2\big)^2 \\
d_{13}^2 &= \big(\tfrac{1}{2}a + x_3 - x_1\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_1\big)^2 \ .
\end{aligned}$$

The full potential energy is

$$U = \tfrac{1}{2}\,k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{23} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{13} - a\big)^2 \ . \tag{10.112}$$

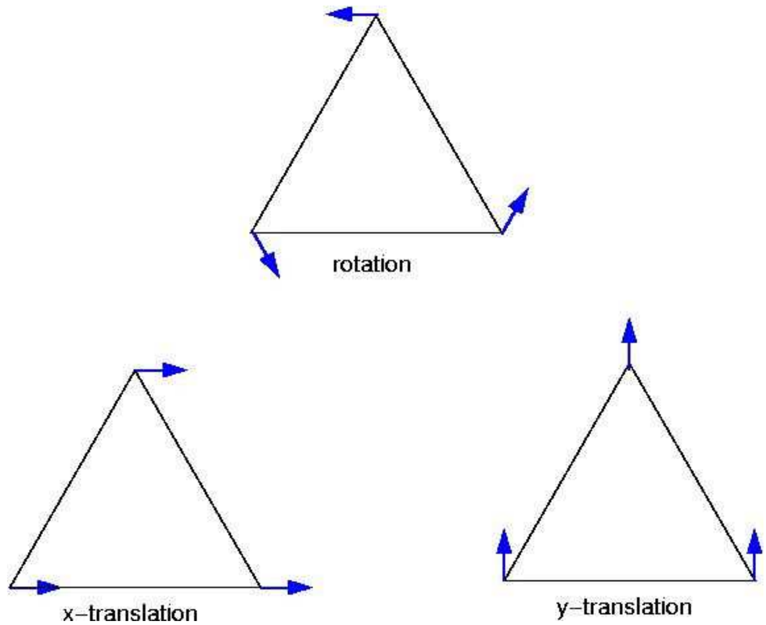


Figure 10.7: Zero modes of the mass-spring triangle.

This is a cumbersome expression, involving square roots.

To find T and V, we need to write T and V as quadratic forms, neglecting higher order terms. Therefore, we must expand $d_{ij} - a$ to linear order in the generalized coordinates. This results in the following:

$$\begin{aligned} d_{12} &= a + \big(x_2 - x_1\big) + \ldots \\ d_{23} &= a - \tfrac{1}{2}\,\big(x_3 - x_2\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_2\big) + \ldots \\ d_{13} &= a + \tfrac{1}{2}\,\big(x_3 - x_1\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_1\big) + \ldots\ . \end{aligned}$$

Thus,

$$\begin{aligned} U = \tfrac{1}{2}\,k\,\big(x_2 - x_1\big)^2 &+ \tfrac{1}{8}\,k\,\big(x_2 - x_3 - \sqrt{3}\,y_2 + \sqrt{3}\,y_3\big)^2 \\ &+ \tfrac{1}{8}\,k\,\big(x_3 - x_1 + \sqrt{3}\,y_3 - \sqrt{3}\,y_1\big)^2 + \text{higher order terms}\ . \end{aligned}$$

Defining

$$\big(q_1, q_2, q_3, q_4, q_5, q_6\big) = \big(x_1, y_1, x_2, y_2, x_3, y_3\big)\ ,$$

we may now read off

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\bar{\mathrm{q}}} = k \begin{pmatrix} 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 \end{pmatrix}$$

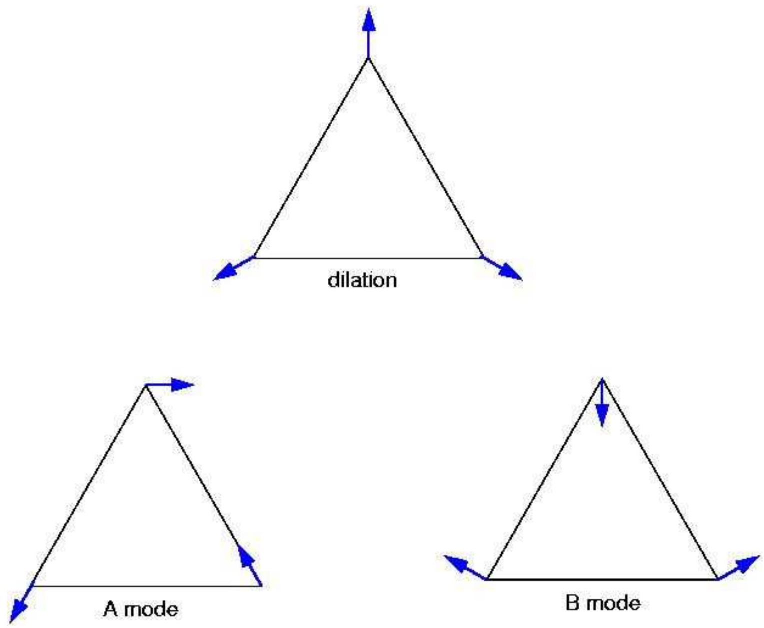


Figure 10.8: Finite oscillation frequency modes of the mass-spring triangle.

The T matrix is trivial. From

$$T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \ .$$

we obtain

$$\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \, \partial \dot{q}_j} = m \, \delta_{ij} \ ,$$

and $\mathrm{T} = m \cdot \mathbb{I}$ is a multiple of the unit matrix.

The zero modes are depicted graphically in figure 10.7. Explicitly, we have

$$\boldsymbol{\xi}_x = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_y = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\rm rot} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ -1 \\ 0 \end{pmatrix} \ .$$

That these are indeed zero modes may be verified by direct multiplication:

$$\mathrm{V}\, \boldsymbol{\xi}_{x.y} = \mathrm{V}\, \boldsymbol{\xi}_{\rm rot} = 0 \ . \tag{10.113}$$

The three modes with finite oscillation frequency are depicted graphically in figure 10.8. Explicitly, we

Figure 10.9: *John Henry*, statue by Charles O. Cooper (1972). "Now the man that invented the steam drill, he thought he was mighty fine. But John Henry drove fifteen feet, and the steam drill only made nine." - from *The Ballad of John Henry*.

have

$$\boldsymbol{\xi}_{\rm A} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\rm B} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ 1/2 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\rm dil} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} .$$

The oscillation frequencies of these modes are easily checked by multiplying the eigenvectors by the matrix V. Since $\mathrm{T} = m \cdot \mathbb{I}$ is diagonal, we have $\mathrm{V}\,\boldsymbol{\xi}_a = m\omega_a^2\,\boldsymbol{\xi}_a$. One finds

$$\omega_{\rm A} = \omega_{\rm B} = \sqrt{\frac{3k}{2m}} \quad , \quad \omega_{\rm dil} = \sqrt{\frac{3k}{m}} .$$

`Mathematica`? I don't need no stinking `Mathematica`.

10.10 Aside : Christoffel Symbols

The coupled equations in eqn. 10.5 may be written in the form

$$\ddot{q}_\sigma + \Gamma^\sigma_{\mu\nu}\,\dot{q}_\mu\,\dot{q}_\nu = F_\sigma \ , \qquad (10.114)$$

with

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, T^{-1}_{\sigma\alpha} \left(\frac{\partial T_{\alpha\mu}}{\partial q_\nu} + \frac{\partial T_{\alpha\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\alpha} \right) \tag{10.115}$$

and

$$F_\sigma = -T^{-1}_{\sigma\alpha}\, \frac{\partial U}{\partial q_\alpha} \ . \tag{10.116}$$

The components of the rank-three tensor $\Gamma^{\sigma}_{\alpha\beta}$ are known as *Christoffel symbols*, in the case where $T_{\mu\nu}(q)$ defines a *metric* on the space of generalized coordinates.

Chapter 11

Elastic Collisions

11.1 Center of Mass Frame

A collision or 'scattering event' is said to be *elastic* if it results in no change in the internal state of any of the particles involved. Thus, no internal energy is liberated or captured in an elastic process.

Consider the elastic scattering of two particles. Recall the relation between laboratory coordinates $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ and the CM and relative coordinates $\{\boldsymbol{R}, \boldsymbol{r}\}$:

$$\begin{aligned} \boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} \qquad &, \qquad \boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1+m_2}\,\boldsymbol{r} \\ \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad &, \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1+m_2}\,\boldsymbol{r} \end{aligned} \tag{11.1}$$

If external forces are negligible, the CM momentum $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ is constant, and therefore the frame of reference whose origin is tied to the CM position is an inertial frame of reference. In this frame,

$$\boldsymbol{v}_1^{\rm CM} = \frac{m_2\,\boldsymbol{v}}{m_1+m_2} \qquad , \qquad \boldsymbol{v}_2^{\rm CM} = -\frac{m_1\,\boldsymbol{v}}{m_1+m_2} \ , \tag{11.2}$$

where $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_1^{\rm CM} - \boldsymbol{v}_2^{\rm CM}$ is the relative velocity, which is the same in both L and CM frames. Note that the CM momenta satisfy

$$\begin{aligned} \boldsymbol{p}_1^{\rm CM} &= m_1\boldsymbol{v}_1^{\rm CM} = \mu\boldsymbol{v} \\ \boldsymbol{p}_2^{\rm CM} &= m_2\boldsymbol{v}_2^{\rm CM} = -\mu\boldsymbol{v} \ , \end{aligned} \tag{11.3}$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. Thus, $\boldsymbol{p}_1^{\rm CM} + \boldsymbol{p}_2^{\rm CM} = 0$ and the total momentum in the CM frame is zero. We may then write

$$\boldsymbol{p}_1^{\rm CM} \equiv p_0\hat{\boldsymbol{n}} \qquad , \qquad \boldsymbol{p}_2^{\rm CM} \equiv -p_0\hat{\boldsymbol{n}} \qquad \Rightarrow \qquad E^{\rm CM} = \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} = \frac{p_0^2}{2\mu} \ . \tag{11.4}$$

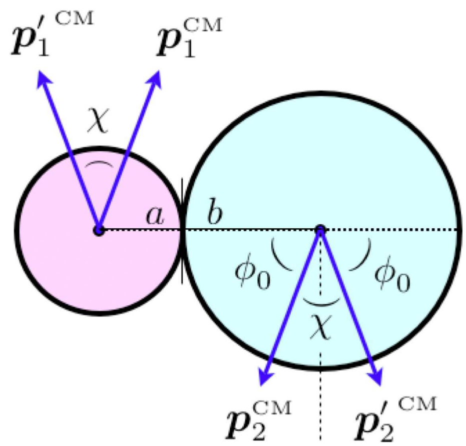


Figure 11.1: The scattering of two hard spheres of radii a and b. The scattering angle is χ.

The energy is evaluated when the particles are asymptotically far from each other, in which case the potential energy is assumed to be negligible. After the collision, energy and momentum conservation require

$$\boldsymbol{p}_1'^{\,\text{CM}} \equiv p_0\hat{\boldsymbol{n}}' \quad , \quad \boldsymbol{p}_2'^{\,\text{CM}} \equiv -p_0\hat{\boldsymbol{n}}' \quad \Rightarrow \quad E'^{\text{CM}} = E^{\text{CM}} = \frac{p_0^2}{2\mu} \ . \tag{11.5}$$

The angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is the *scattering angle* χ:

$$\boldsymbol{n} \cdot \boldsymbol{n}' \equiv \cos\chi \ . \tag{11.6}$$

The value of χ depends on the details of the scattering process, *i.e.* on the interaction potential $U(r)$. As an example, consider the scattering of two hard spheres, depicted in Fig. 11.1. The potential is

$$U(r) = \begin{cases} \infty & \text{if } r \le a+b \\ 0 & \text{if } r > a+b \ . \end{cases} \tag{11.7}$$

Clearly the scattering angle is $\chi = \pi - 2\phi_0$, where ϕ_0 is the angle between the initial momentum of either sphere and a line containing their two centers at the moment of contact.

There is a simple geometric interpretation of these results, depicted in Fig. 11.2. We have

$$\begin{aligned} \boldsymbol{p}_1 &= m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}} \quad , \quad & \boldsymbol{p}_1' &= m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}}' \\ \boldsymbol{p}_2 &= m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}} \quad , \quad & \boldsymbol{p}_2' &= m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}}' \ . \end{aligned} \tag{11.8}$$

So draw a circle of radius p_0 whose center is the origin. The vectors $p_0\hat{\boldsymbol{n}}$ and $p_0\hat{\boldsymbol{n}}'$ must both lie along this circle. We define the angle ψ between $\boldsymbol{V}$ and $\boldsymbol{n}$:

$$\hat{\boldsymbol{V}} \cdot \boldsymbol{n} = \cos\psi \ . \tag{11.9}$$

It is now an exercise in geometry, using the law of cosines, to determine everything of interest in terms

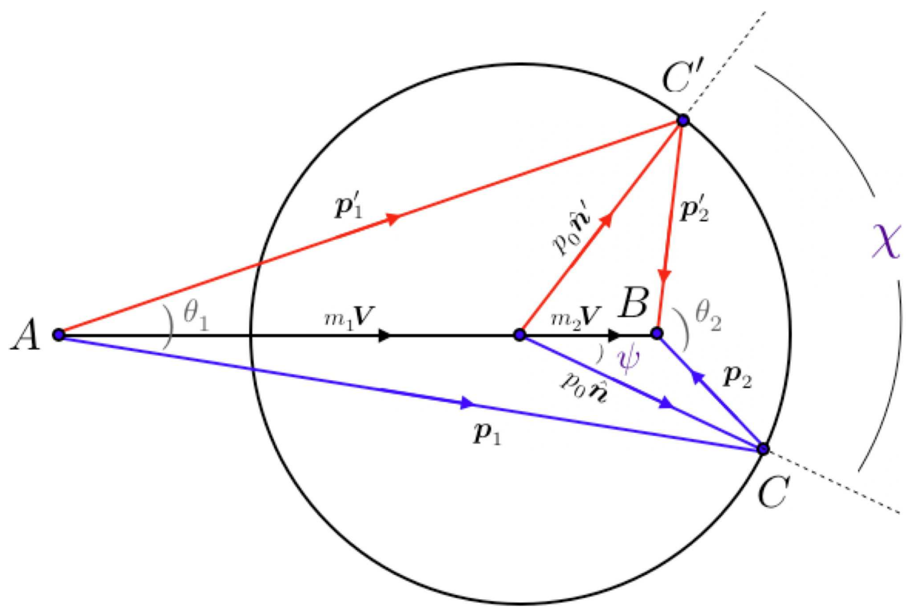


Figure 11.2: Scattering of two particles of masses m_1 and m_2. The scattering angle χ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

of the quantities V, v, ψ, and χ. For example, the momenta are

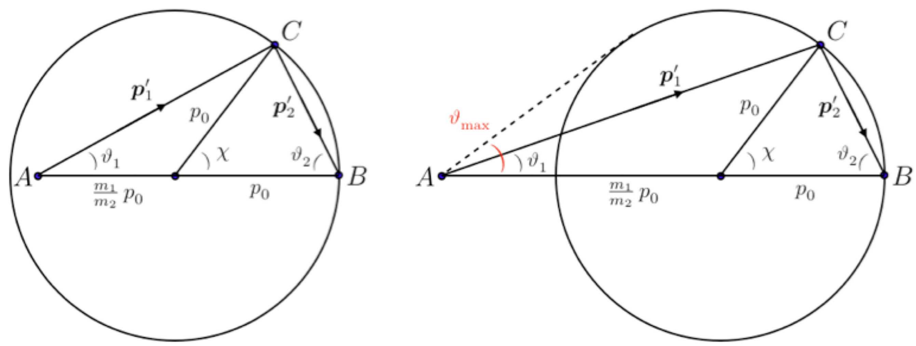
$$\begin{aligned}
p_1 &= \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1\mu V v\cos\psi} \\
p_1' &= \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1\mu V v\cos(\chi-\psi)} \\
p_2 &= \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2\mu V v\cos\psi} \\
p_2' &= \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2\mu V v\cos(\chi-\psi)} \ ,
\end{aligned} \tag{11.10}$$

and the scattering angles are

$$\begin{aligned}
\theta_1 &= \tan^{-1}\!\left(\frac{\mu v\sin\psi}{\mu v\cos\psi + m_1 V}\right) + \tan^{-1}\!\left(\frac{\mu v\sin(\chi-\psi)}{\mu v\cos(\chi-\psi) + m_1 V}\right) \\
\theta_2 &= \tan^{-1}\!\left(\frac{\mu v\sin\psi}{\mu v\cos\psi - m_2 V}\right) + \tan^{-1}\!\left(\frac{\mu v\sin(\chi-\psi)}{\mu v\cos(\chi-\psi) - m_2 V}\right) .
\end{aligned} \tag{11.11}$$

If particle 2, say, is initially at rest, the situation is somewhat simpler. In this case, $\boldsymbol{V} = m_1\boldsymbol{V}/(m_1+m_2)$ and $m_2\boldsymbol{V} = \mu\boldsymbol{v}$, which means the point B lies on the circle in Fig. 11.3 ($m_1 \neq m_2$) and Fig. 11.4 ($m_1 = m_2$). Let $\vartheta_{1,2}$ be the angles between the directions of motion after the collision and the direction $\boldsymbol{V}$ of impact. The scattering angle χ is the angle through which particle 1 turns in the CM frame. Clearly

$$\tan\vartheta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \quad , \quad \vartheta_2 = \tfrac{1}{2}(\pi - \chi) \ . \tag{11.12}$$



(a) $m_1 < m_2$ (b) $m_1 > m_2$

Figure 11.3: Scattering when particle 2 is initially at rest.

We can also find the speeds v_1' and v_2' in terms of v and χ, from

$$p_1'^{\,2} = p_0^2 + \left(\tfrac{m_1}{m_2}\,p_0\right)^2 - 2\,\tfrac{m_1}{m_2}\,p_0^2\,\cos(\pi - \chi) \tag{11.13}$$

and

$$p_2^2 = 2\,p_0^2\,(1 - \cos\chi)\ . \tag{11.14}$$

These equations yield

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\chi}}{m_1 + m_2}\,v \qquad , \qquad v_2' = \frac{2m_1 v}{m_1 + m_2}\,\sin(\tfrac{1}{2}\chi)\ . \tag{11.15}$$

The angle $\vartheta_{\rm max}$ from Fig. 11.3(b) is given by $\sin\vartheta_{\rm max} = \frac{m_2}{m_1}$. Note that when $m_1 = m_2$ we have $\vartheta_1 + \vartheta_2 = \pi$. A sketch of the orbits in the cases of both repulsive and attractive scattering, in both the laboratory and CM frames, in shown in Fig. 11.5.

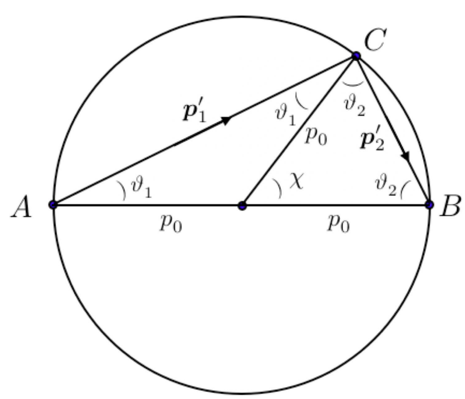


Figure 11.4: Scattering of identical mass particles when particle 2 is initially at rest.

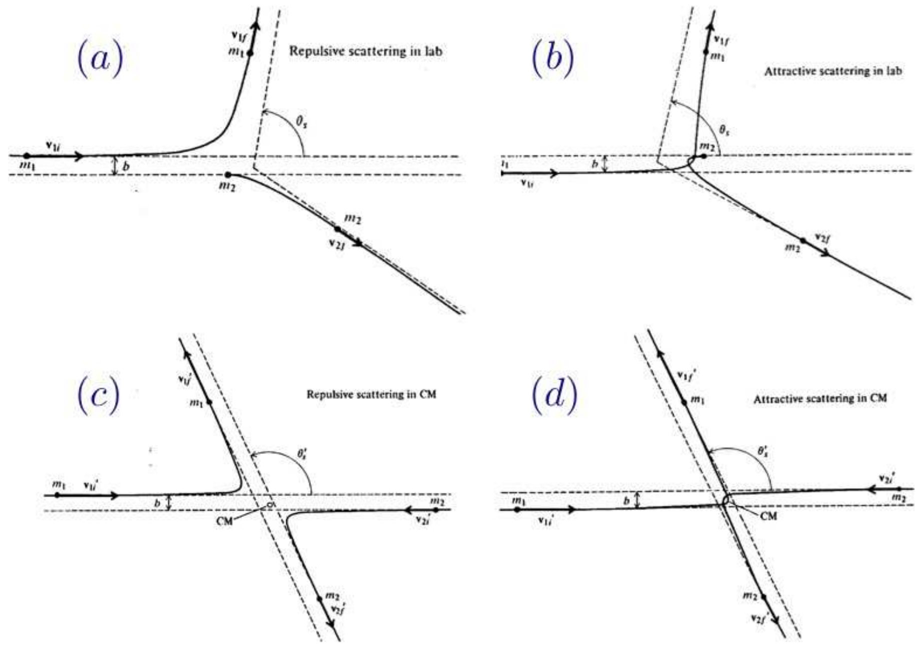


Figure 11.5: Repulsive (A,C) and attractive (B,D) scattering in the lab (A,B) and CM (C,D) frames, assuming particle 2 starts from rest in the lab frame. (From Barger and Olsson.)

11.2 Central Force Scattering

Consider a single particle of mass μ movng in a central potential $U(r)$, or a two body central force problem in which μ is the reduced mass. Recall that

$$\frac{dr}{dt} = \frac{d\phi}{dt} \cdot \frac{dr}{d\phi} = \frac{\ell}{\mu r^2} \cdot \frac{dr}{d\phi} \ , \tag{11.16}$$

and therefore

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r) \ . \end{aligned} \tag{11.17}$$

Solving for $\frac{dr}{d\phi}$, we obtain

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2\mu r^4}{\ell^2}\big(E - U(r)\big) - r^2} \ , \tag{11.18}$$

Consulting Fig. 11.6, we have that

$$\phi_0 = \frac{\ell}{\sqrt{2\mu}} \int\limits_{r_{\rm p}}^{\infty} \frac{dr}{r^2\sqrt{E - U_{\rm eff}(r)}} \ , \tag{11.19}$$

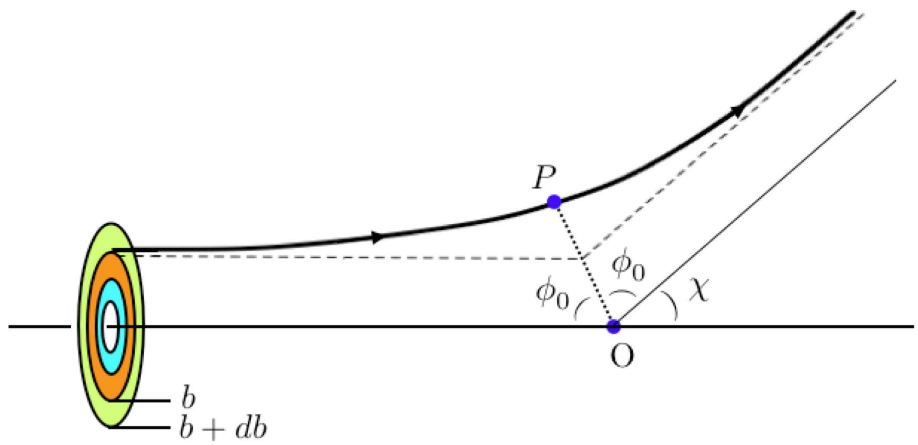


Figure 11.6: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

where $r_{\rm p}$ is the radial distance at periapsis, and where

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \tag{11.20}$$

is the effective potential, as before. From Fig. 11.6, we conclude that the scattering angle is

$$\chi = \left|\pi - 2\phi_0\right| \ . \tag{11.21}$$

It is convenient to define the *impact parameter* b as the distance of the asymptotic trajectory from a parallel line containing the force center. The geometry is shown again in Fig. 11.6. Note that the energy and angular momentum, which are conserved, can be evaluated at infinity using the impact parameter:

$$E = \tfrac{1}{2}\mu v_\infty^2 \qquad , \qquad \ell = \mu v_\infty b \ . \tag{11.22}$$

Substituting for $\ell(b)$, we have

$$\phi_0(E,b) = \int\limits_{r_{\rm p}}^{\infty} \frac{dr}{r^2} \, \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U(r)}{E}}} \ , \tag{11.23}$$

In physical applications, we are often interested in the deflection of a beam of incident particles by a scattering center. We define the *differential scattering cross section* $d\sigma$ by

$$d\sigma = \frac{\text{\# of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \ . \tag{11.24}$$

Now for particles of a given energy E there is a unique relationship between the scattering angle χ and the impact parameter b, as we have just derived in eqn. 11.23. The differential solid angle is given by $d\Omega = 2\pi \sin\chi \, d\chi$, hence

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi} \left|\frac{db}{d\chi}\right| = \left|\frac{d\,(\frac{1}{2}b^2)}{d\cos\chi}\right| \ . \tag{11.25}$$

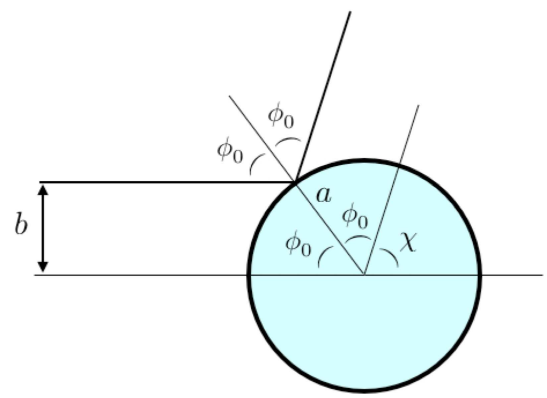


Figure 11.7: Geometry of hard sphere scattering.

Note that $\frac{d\sigma}{d\Omega}$ has dimensions of area. The integral of $\frac{d\sigma}{d\Omega}$ over all solid angle is the *total scattering cross section*,

$$\sigma_{\rm T} = 2\pi \int_0^\pi d\chi \, \sin\chi \, \frac{d\sigma}{d\Omega} \ . \tag{11.26}$$

11.2.1 Hard sphere scattering

Consider a point particle scattering off a hard sphere of radius a, or two hard spheres of radii a_1 and a_2 scattering off each other, with $a \equiv a_1 + a_2$. From the geometry of Fig. 11.7, we have $b = a \sin\phi_0$ and $\phi_0 = \frac{1}{2}(\pi - \chi)$, so

$$b^2 = a^2 \sin^2\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\chi\right) = \tfrac{1}{2} a^2 \left(1 + \cos\chi\right) \ . \tag{11.27}$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{d\left(\frac{1}{2}b^2\right)}{d\cos\chi} = \tfrac{1}{4}\, a^2 \tag{11.28}$$

and $\sigma_{\rm T} = \pi a^2$. The total scattering cross section is simply the area of a sphere of radius a projected onto a plane perpendicular to the incident flux.

11.2.2 Rutherford scattering

Consider scattering by the Kepler potential $U(r) = -\frac{k}{r}$. We assume that the orbits are unbound, *i.e.* they are Keplerian hyperbolae with $E > 0$, described by the equation

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon \, \cos\phi} \qquad \Rightarrow \qquad \cos\phi_0 = \pm\, \frac{1}{\varepsilon} \ . \tag{11.29}$$

Recall that the eccentricity is given by

$$\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} = 1 + \left(\frac{\mu b v_\infty}{k}\right)^2 \ . \tag{11.30}$$

We then have

$$\begin{aligned}\left(\frac{\mu b v_\infty}{k}\right)^2 &= \varepsilon^2 - 1 \\ &= \sec^2\!\phi_0 - 1 = \tan^2\!\phi_0 = \text{ctn}^2\big(\tfrac{1}{2}\chi\big) \ .\end{aligned} \tag{11.31}$$

Therefore

$$b(\chi) = \frac{k}{\mu v_\infty^2}\,\text{ctn}\big(\tfrac{1}{2}\chi\big) \tag{11.32}$$

We finally obtain

$$\begin{aligned}\frac{d\sigma}{d\Omega} = \frac{d\,\big(\frac{1}{2}b^2\big)}{d\cos\chi} &= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d\,\text{ctn}^2\big(\frac{1}{2}\chi\big)}{d\cos\chi} \\ &= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d}{d\cos\chi}\left(\frac{1+\cos\chi}{1-\cos\chi}\right) \\ &= \left(\frac{k}{2\mu v_\infty^2}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big) \ ,\end{aligned} \tag{11.33}$$

which is the same as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big) \ . \tag{11.34}$$

Since $\frac{d\sigma}{d\Omega} \propto \chi^{-4}$ as $\chi \to 0$, the total cross section σ_{T} diverges! This is a consequence of the long-ranged nature of the Kepler/Coulomb potential. In electron-atom scattering, the Coulomb potential of the nucleus is *screened* by the electrons of the atom, and the $1/r$ behavior is cut off at large distances.

11.2.3 Transformation to laboratory coordinates

We previously derived the relation

$$\tan\vartheta = \frac{\sin\chi}{\gamma + \cos\chi} \ , \tag{11.35}$$

where $\vartheta \equiv \vartheta_1$ is the scattering angle for particle 1 in the laboratory frame, and $\gamma = \frac{m_1}{m_2}$ is the ratio of the masses. We now derive the differential scattering cross section in the laboratory frame. To do so, we note that particle conservation requires

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{L}} \cdot\ 2\pi\sin\vartheta\,d\vartheta = \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} \cdot\ 2\pi\sin\chi\,d\chi \ , \tag{11.36}$$

which says

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{L}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} \cdot\ \frac{d\cos\chi}{d\cos\vartheta} \ . \tag{11.37}$$

From

$$\begin{aligned}\cos\vartheta &= \frac{1}{\sqrt{1+\tan^2\!\vartheta}} \\ &= \frac{\gamma + \cos\chi}{\sqrt{1+\gamma^2+2\gamma\cos\chi}} \ ,\end{aligned} \tag{11.38}$$

we derive

$$\frac{d\cos\vartheta}{d\cos\chi} = \frac{1+\gamma\cos\chi}{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}} \tag{11.39}$$

and, accordingly,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \frac{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}}{1+\gamma\cos\chi}\cdot\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} . \tag{11.40}$$