A Quick Look at Closures

⇒ Turbulence, so far:
- satisfied, from "physicist's perspective"?
- ① scalings – rooted in phenomenology?!

  ② mixing length models – also rooted in phenomenology!? $\nu_T = u_T \ell$

  $\langle V_1 V_2 \rangle = -\nu_T \frac{\partial u}{\partial x}$

⇒ where have Navier-Stokes Equations gone?

⇒ Might one:

- <u>derive</u> eddy viscosity
- <u>derive</u> K41 spectrum

from some systematic procedure starting from NSE?

⇒ Apply to more complex problems → MHD, stratified turbulence etc.

⇒ <u>Framework</u>.

References on Closure:

- Kraichnan 59 → Basics of DIA
- Kraichnan 61 → Random Coupling Model
- Kraichnan 76 → Test Field Model

→ Forster, Nelson, Stephen '77 → Forced Burgers Turbulence

→ Hunt '90 → Rapid Distortion Theory.

Spectral Equation

so

viscous damping — turbulent viscous damping

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \langle \tilde{V}^2 \rangle_k + 2 \sum_{k'} (k+k')^2 \Theta_{\substack{k,k' \\ k+k'}} \langle \tilde{V}^2 \rangle_{k'} \langle \tilde{V}^2 \rangle_k$$

$$= S_k + 2 \sum_{p,q} (p+q)^2 \Theta_{k,p,q} \langle \tilde{V}^2 \rangle_p \langle \tilde{V}^2 \rangle_q$$

random stirring

mode-coupling induced stirring — nonlinear noise. ($\gg S_k$ in inertial range)

- structure is that of Langevin equation, with noise and drag renormalized, of course same origin.

i.e. $\frac{\partial \tilde{V}}{\partial t} + \mu \tilde{V} = \tilde{f}$ — Stokes drag; thermal noise

$\mu = \frac{6\pi \eta}{\rho} a$

$\Rightarrow$

NL noise

$$\frac{\partial}{\partial t} E_k + \nu^T \langle \tilde{V}^2 \rangle_k = S^T_{p+q=k}$$

turbulent viscosity

- energetics — does renormalized theory respect primitive equations?

$$\sum_k T_k \overset{?}{=} 0$$

$$\sum_k T_k = \sum_k \sum_{k'} 2(k+k')^2 \Theta_{\substack{k,k' \\ k+k'}} \langle \tilde{V}^2 \rangle_{k'} \langle \tilde{V}^2 \rangle_{k}$$

$$- \sum_k \sum_{\substack{p,q \\ p+q=k}} 2(p+q)^2 \Theta_{\substack{p,q \\ p+q=k}} \langle \tilde{V}^2 \rangle_p \langle \tilde{V}^2 \rangle_q$$

$= 0$ (re-label) !

RPA, $\Theta_{k,p,q} \rightarrow$ Molec. Chaos

Comment: Equilibrium closure as $C(f) \rightarrow \Theta$ at time $\Leftrightarrow$ $\Rightarrow$ relaxation to equipartition spectra (stat. mech).

N.B.: Upon summation, coherent damping conserves energy vs. incoherent emission.

i.e. cascade as sequence of coherent damping $\rightarrow$ incoherent emission $\rightarrow$ coherent damping $\rightarrow$ ...., ala' band models.

Closure Zoology: based upon use of coupled response fctn, spectral eqns

① i.e. $\frac{\delta V}{\delta f}$ response fctn $\leftrightarrow$ depends on spectra $\langle \tilde{V}^2 \rangle_{k'}$

② $\frac{\partial}{\partial t} \langle \tilde{V}^2 \rangle_k$ depends on C_k, L_k, etc.

$\Rightarrow$

DIA: Solve coupled equations for $\delta V/\delta f$ and $\langle \tilde{V}^2 \rangle_k$

EDQNM : parametrize C_k in terms $\langle \tilde{v}^2 \rangle_k$, yielding spectral equation

Eddy viscosity models / R. B. T. : $\partial V/\partial t$ equation

Weak Turbulence : neglect $C_{k+k'}$ in $L_{k+k'}$.

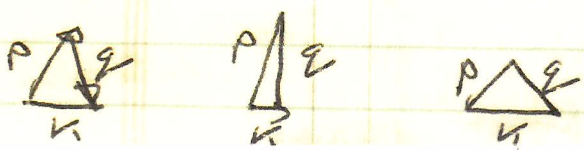
⋮

Comments on Closures:

- consistent with conservation laws, albeit trivially;
- based upon <u>assumed</u> weak coupling / RPA hypothesis (The Swindle Occurs Here!)

  $N \sim C_{k+k'} V_{k+k'} + (\,) V_k V_{k'}$, etc.
- $\omega_{\substack{\text{triad} \\ \text{decorrel.}}} = \sum (\omega_{\text{decorr}})_{\text{triad}}$
- no restriction on shape of interacting triad, i.e. $\rightarrow$ confusion of $\left\{ \begin{array}{l} \text{sweeping} \\ \text{straining} \end{array} \right.$!

$\underline{p} + \underline{q} = \underline{k}$



, etc. $\Leftrightarrow$ sweeping!

→ Foundations of the D.I.A. and Issues in Turbulence Closure (R.H. Kraichnan, J. Math Phys. 2, 124 (1961)).

① – reprise of the D.I.A. and the D.I.A. propagator for N.-S. T.

② – stochastic oscillator models ↔ general structure

③ – random coupling model and the problem of realizability

① Reprise

Recall D.I.A. → coupled equations for $\left\{ \begin{array}{l} \text{propagator} \\ \text{spectrum} \end{array} \right.$.

Interesting to note:

→ essential physics is nonlinear scrambling in triad coherence (i.e. sets coherence time)

→ $\left. \begin{array}{l} \text{spectrum} \\ \text{propagator} \end{array} \right\}$ sets scrambling time ⇒ cascade dynamics

sets response

Useful to note for later that for N.-St. Eq. 1;
D.I.A. for propagator evolution gives;

$$\underbrace{\frac{\partial g(k,\tau)}{\partial \tau}}_{\text{propagator fctn.}} + \overbrace{\nu}^{\text{molecular viscosity}} k^2 g(k,\tau)$$

$$\underline{k} + \underline{p} + \underline{q} = 0$$

$$= -\frac{k}{2}\int dp \int dq \, \frac{p}{q} \underbrace{b(k,p,q)}_{\text{coupling coeff}} \underbrace{E(q)}_{\text{background energy}} \overbrace{\int_0^\tau ds}^{\text{non-Markovian structure}} \underbrace{g(k,\tau-s)\, g(p,s)}_{\substack{\text{test-wave response propagator} \\ \text{(from closure)}}} \underbrace{r(q,s)}_{\substack{\text{self-correlation of} \\ \text{backgnd. spectrum}}}$$

Can simplify using:

a) $E(q)$ largest at small q $\rightarrow$ energy containing range.

b) and $\underline{p} + \underline{q} = \underline{k}$ $\Rightarrow$ $|\underline{p}| \sim |\underline{k}| \gg q$ (selection rule)

c) $r(q,s) \cong r(q,0)$ $\rightarrow$ i.e. large eddys have long lifetimes, treat as slow, relative to high k response.

so ...

$$\frac{\partial g(k,\tau)}{\partial \tau} + \nu k^2 g(k,\tau) = -k^2 V_0^2 \int_0^\tau g(k,\tau-s)\, g(k,s)\, ds = 0$$

↳ non-Markovian – convolution

$$\underbrace{k^2 V_0^2}_{\text{effective straining/sweeping (?) time}} = +\frac{k}{2}\int dp \int dq \; \frac{p}{q}\, b(k,p,q)\, E(q)$$

Can solve via Laplace transform (n.b. convolution!)
so:

$$\boxed{g(k,\tau) = e^{-\nu k^2 \tau}\, J_1(2kV_0\tau)/kV_0\tau}$$

"trademark" D.I.A. propagator

Some observations:

i.) sweeping vs straining → physics of eddy lifetime !? → V_0 ?!?

ii.) $g(k,\tau)$ oscillates ?! – physical meaning ?

iii.) ultimately gives $E(k) \sim k^{-3/2}$, not $E(k) \sim k^{-5/3}$.

see 59a for Refs.

Closures and Renormalization - Overview

Closures

Refs. → W.D. McComb "The Physics of Fluid Turbulence"
"Renormalization - A Guide for Beginners."

and see postings

⇒ object of closure to derive equations for observables of turbulence from Navier-Stokes Eqn. - dynamics, not just geometry ...
(contrast fractology)

→ observables typically:
- response function (effective eddy viscosity)
- spectrum (time scale)

i.e. (low order moments) ↔ not full pdf ...

→ procedure is perturbative / RPT
Cala' QLT, mean field theory)

⇒ closure methodology usually involves:

a.) RPA / weak coupling approximation
Test field model

i.e. $\frac{\partial}{\partial t} a_{\underline{k}} + \gamma_{\underline{k}} a_{\underline{k}} + \sum_{\underline{k}'} C_{\underline{k},\underline{k}'} \, a_{\underline{k}+\underline{k}'} \, a_{-\underline{k}'} = f_{\underline{k}}$

generic NL model eqn.

$|a_{\underline{k}}|^2 = E(\underline{k})$

$$\frac{\partial}{\partial t} E(\underline{k}) + \gamma_{\underline{k}} E(\underline{k}) + \sum_{\underline{k}'} C_{\underline{k},\underline{k}'} \, a_{-\underline{k}} \, a_{-\underline{k}'} \, a_{\underline{k}+\underline{k}'}$$

i.e. $\frac{\partial}{\partial t} \langle a^2 \rangle_{\underline{k}} \sim \langle a a a \rangle$ i.e. coupled moment hierarchy

Key issue → how treat.

and moment hierarchy $\Rightarrow$

$$\frac{\partial}{\partial t}\langle a^3\rangle \sim \langle aaaa\rangle$$
$$\sim \langle a^2\rangle\langle a^2\rangle$$

- application of RPA to $\langle a^4\rangle$
- on $\langle a^4\rangle \sim \langle cc\,aaaa\rangle$

quasi-Gaussian $\sim |c|^2\langle a^2\rangle^2$

(rndm coupling)

b) to renormalization

$$\langle a^3\rangle \sim \tau_c\langle a^2\rangle\langle a^2\rangle$$

what controls this?

- if simple perturbation theory, ($\rightarrow$ is this physical?)

$1/\tau_c \sim \nu k^2$, necessarily

$\Rightarrow$ $\tau_c \sim (\nu k^2)^{-1} \rightarrow \infty$, relative to (inertial range time scales $O(Re)$

so

$$\frac{\partial}{\partial t}\langle a^2\rangle \sim \tau_c\langle a^2\rangle\langle a^2\rangle$$

too much energy transfer due to spectral depletion

transfer unphysically large, due to long correlation times (also unphysical) response time $\rightarrow$ eddy visc.

- mindless perturbation theory yields unphysically long correlations $\Rightarrow$ $\frac{\partial \langle a^2 \rangle}{\partial t}$ large $\Rightarrow$ $E < 0$ results
  $\therefore$ unphysical ! $\rightarrow$ "realizability problem" $\rightarrow$ model

must "renormalize" $\tau_c \rightarrow (\nu_T k^2)^{-1}$ (i.e. treat
$\rightarrow$ time-scale self-consistently), so that modal coherence consistent with inertial range scrambling rate !

Example: Burgerlence (Driven Burgers/KPZ Equation)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \nu \partial_x^2 v = f$$

f ↑ stochastic forcing

n.b.: perturbative closure will completely miss shock formation physics. [pdf(v') asymmetry]. c.f. Saffman '68

$$\boxed{\frac{d}{dt} v = -v^2}$$

a.) Response function

NL Langevin Equation

$$\frac{\partial v_k}{\partial t} + \frac{ik}{2}\sum_{k'} v_{-k'} v_{k+k'} + \nu k^2 v_k = f_k(t)$$

Eddy viscosity !

now, seek $\delta v_k / \delta f_k$ $\Rightarrow$ response function for mode k.

key physics: space/time scales.

for $Re \ll 1$,

$$\frac{\partial}{\partial t} V_k + \nu k^2 V_k + i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'} = f_k(t)$$

$$(-i\omega + \nu k^2) V_k = f_{k,\omega}$$

$$R_{k,\omega} = \delta V_{k,\omega} / \delta f_{k,\omega} = 1/-i\omega + \nu k^2$$

$\Rightarrow$ time scale set by viscosity !?

for $Re \gg 1$ $\Rightarrow$ idiotic! $\rightarrow$ need faster time scale

$\therefore$ - need extract effective time-scale from nonlinearity

- physics is time scale of nonlinear scrambling / coupling - NL response $\rightarrow$ how calculate?

i.e. $$\frac{\partial}{\partial t} V_k + \nu k^2 V_k + C_k V_k = f_k(t)$$

i.e. seek response of test mode wave, interacting with rest of turbulent spectrum.....

- reflects physics $i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'}$

- phase coherent with f_k

$$C_k V_k \Leftrightarrow i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'}$$

so, in lowest order

$C_k \sim |V|^2$ (C phase independent)

Now, to calculate C_k;

$$\left(-i\omega + \nu k^2\right) V_{k \atop \omega} + \frac{i k}{2} \sum_{k'} V_{-k' \atop -\omega'} V_{k+k' \atop \omega+\omega'} = f_{k \atop \omega}$$

$V_{k+k' \atop \omega+\omega'} \rightarrow V^{(2)}_{k+k' \atop \omega+\omega'} \Leftrightarrow$ V driven by direct ~~bect~~ interaction of V_k, $V_{k'}$ (hence DIA)

$$\left(-i\omega + \nu k^2\right) V_{k \atop \omega} + i k \sum_{k'} V_{-k' \atop -\omega'} V^{(2)}_{k+k' \atop \omega+\omega'} = f_{k \atop \omega}$$

where: $i k \sum_{k'} V_{-k' \atop -\omega'} V^{(2)}_{k+k' \atop \omega+\omega'} \equiv C_{k \atop \omega} V_{k \atop \omega}$ (5.7)

so, when calculated:

$$\left(\partial f_{k,\omega} / \partial V_{k,\omega}\right)^{-1} = 1 \Big/ \left(-i\omega + \overbrace{\nu k^2 + C_{k,\omega}}^{\text{dressed viscosity}}\right)$$

bare

limits resp. $\longrightarrow$ reflects inertial range scrambling

[fast field hypothesis]

Now, to calculate: NL scrambling rate (self-consistent) $\rightarrow$ all other interactions than those selected

$$\left(-i(\omega+\omega') + \nu (k+k')^2 + C_{k+k' \atop \omega+\omega'}\right) V^{(2)}_{k+k' \atop \omega+\omega'}$$

$$= -i \frac{(k+k')}{2} \left(V_{k'} V_{k} + V_{k} V_{k'} \right) = -i(k+k')(V_{k'} V_{k})$$

Now, define

NL interaction

$$L^{-1}_{\substack{k+k' \\ \omega+\omega'}} = -i(\omega+\omega') + \nu(k+k')^2 + C_{\substack{k+k' \\ \omega+\omega'}}$$

(renormalized propagator)

$$V^{(2)}_{\substack{k+k' \\ \omega+\omega'}} = L_{\substack{k+k' \\ \omega+\omega'}} \left(-i(k+k')\right) V_{\substack{k' \\ \omega'}} V_{\substack{k \\ \omega}}$$

so, self consistently,

$$C_{\substack{k \\ \omega}} V_{\substack{k \\ \omega}} = ik \sum_{\substack{k' \\ \omega'}} V_{\substack{-k' \\ -\omega'}} L_{\substack{k+k' \\ \omega+\omega'}} (-i)(k+k') V_{\substack{k' \\ \omega'}} V_{\substack{k \\ \omega}}$$

$$= \left(k^2 \sum_{k',\omega'} |V_{\substack{k' \\ \omega'}}|^2 L_{\substack{k+k' \\ \omega+\omega'}} \left(1+\frac{k'}{k}\right)\right) V_{\substack{k \\ \omega}}$$

so

$$\begin{cases} dV_{k,\omega}/df_{k,\omega} = 1\Big/ -i\omega + \nu k^2 + C_{\substack{k \\ \omega}} \rightarrow \text{renormalized response function} \\ C_{k,\omega} = \nu_{\substack{k \\ \omega}} k^2 \equiv k^2 \sum_{k',\omega'} |V_{\substack{k' \\ \omega'}}|^2 L_{\substack{k+k' \\ \omega+\omega'}} \left(1+\frac{k'}{k}\right) \end{cases}$$

renormalized turbulent viscosity

$\nu \to \nu + \nu_{k,\omega}$.

→ nonlinear scrambling rate

→ recursively defined.

About $\gamma_{k,\omega}$:

- at long wavelength } low frequency } $k \ll k'$, $\omega \ll \omega'$ $\Rightarrow$ quasilinear limit, Markovian (purity)

$$\gamma_{\substack{k\\ \omega}} \rightarrow \underbrace{\gamma^T}_{\text{effective transport coefficient (diffusion)}} \cong \sum_{k',\omega'} |V_{\substack{k'\\ \omega'}}|^2 L_{\substack{k'\\ \omega'}}$$

effective transport coefficient $\Leftrightarrow$ sets NL/turbulent time scale

$$\gamma^T \sim \langle \tilde{V}^2 \rangle \tau_c \sim \tilde{V}_{rms} l_c \qquad l_c \sim \tilde{V} \tau_c$$

$\gamma \rightarrow$ F.-P. E.

$\gamma_{\substack{k\\ \omega}} \rightarrow$ need Zwanzig-Mori theory.

- important to note:

$$\gamma_{\substack{k\\ \omega}} \rightarrow \gamma^T = \sum_{k',\omega'} |V_{\substack{k'\\ \omega'}}|^2 \left(k'^2 \gamma_{\substack{k'\\ \omega'}}\right) \Big/ \left\{\omega'^2 + \left(k'^2 \gamma_{\substack{k'\\ \omega'}}\right)^2\right\}$$

$\rightarrow$ response function in γ^T also renormalized $\Leftrightarrow$ (self-consistency) $\Leftrightarrow$ $\gamma^T \Leftrightarrow$ random Doppler shift

$\rightarrow$ irreversibility from inertial ranges $\underset{\sim}{\overset{\text{transfer}}{\text{mixing}}}$ / $\overset{\text{impact}}{\underset{\text{(RPA)}}{\Downarrow}}$

dissipation

i.e. contrast QLT with resonance, i.e.

$\rightarrow$ to estimate;

$$\boxed{D = \frac{q^2}{m^2} \sum_k |E_k|^2 \pi \delta(\omega - kv)}$$

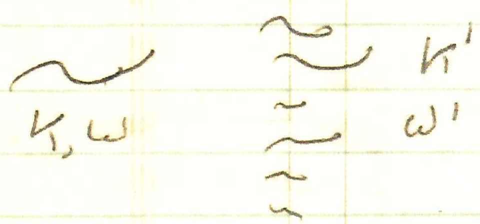
$$\begin{cases} \gamma^2 \sim \dfrac{1}{k^2} V_{rms}^2 \\ \gamma \sim \dfrac{1}{k} V_{rms} \end{cases}$$

$-\ \gamma^T_{k,\omega}$ vs. γ^T ?

$k, \omega \to 0$ if $k < k'$, $\omega < \omega'$

$\Rightarrow$ Markovian limit $\to$ no memory (a la' F.P.E.)
Fokker-Planck Eqn.

i.e. consider interaction of 'test wave' k, ω
with background k', ω'.



$\Leftrightarrow$ for $\tau', \lambda' \ll \tau, \lambda$

$\Rightarrow$ interaction appears as random, memory-less
kick, as in walk.

for $\tau', \lambda' \sim \tau, \lambda$

$\Rightarrow$ interaction is one of mutual sloshing, etc.
i.e. test wave "feels" space time
history of turbulence background.

- also,

$$\nu^T k^2 V_{k,\omega} \to -\nu \partial^2 V$$ eddy viscosity

$$\nu^T_{k,\omega} k^2 V_{k,\omega} \to + \int dx' \int d\tau \; C(x-x', t-\tau) \; V(x', \tau)$$

memory convolution (space / time)

- why "renormalization":

i.e. QED $\frac{1}{p - m_0}$ → Fermion propagator (bare)

electron

↳ bare mass, electron

"renorm" ⇒ $\frac{1}{\not p - m_0 + \Sigma} \to \frac{1}{\not p - m}$ (renormalized)

↳ self-energy; due electron interaction with vacuum polarization cloud (ambient fluctuations)

turbulence:

$\frac{1}{-i\omega + \nu_0 k^2}$ → V propagator

↳ bare (collisional) viscosity

renorm.

$$\Rightarrow \frac{1}{\left[-i\omega+(\nu_0+\nu_{\underset{\omega}{k}})k^2\right]} \rightarrow \text{v propagator}$$

renormalized viscosity (dressing)

↳ interaction of mode/ eddy with turbulence (dressing)

$\Sigma \Leftrightarrow \nu_{\underset{\omega}{k}}$

D.I.A. is procedure for calculation of self-energy.

→ Aside: Candidate Time Scales for Model Interaction

① νk^2 → viscous damping rate

② γ_{NL} → nonlinear energy transfer rate

③ $\left|\left(\frac{\omega}{k} - \frac{\partial\omega}{\partial k}\right)\Delta k\right|$ → wave-(resonant particle) autocorrelation rate

④ $|\Delta\omega_{MM}|$ → wave-wave autocorrelation rate, set by mis-match dispersion

⑤ $\Delta\omega_{\underline{k}}$ → nonlinear scrambling rate (NL acts on self)

Examples:

i.) Weak Turbulence Theory → Wave-Wave (includes weak wave turbulence)

④ < ②, ⑤

Wave-Particle → ③ < ②, ⑤, $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t}$

(ii) N.S.T. $\to$ no resonances, $Re \gg 1$

② , ③ , ④ $\to 0$

① < ⑤ $\Rightarrow$ renormalization needed.

→ Spectral Equation — spectrum is goal.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \nu\frac{\partial^2 v}{\partial x^2} = \tilde{f}$$

$$\frac{\partial \langle v^2\rangle}{\partial t} + \left\langle v^2 \frac{\partial v}{\partial x}\right\rangle + \langle \nu(\partial_x v)^2\rangle = \langle \tilde{f}\tilde{v}\rangle$$

$$\left\langle \frac{\partial}{\partial x}\left(\frac{v^3}{3}\right)\right\rangle \to 0 \text{ end points}$$

→ [NL conserves energy (to boundary terms)]

∴ have energy balance:

$$\underbrace{\frac{\partial}{\partial t}\langle v^2\rangle}_{\text{net K.E.}} = \underbrace{\langle \tilde{v}\tilde{f}\rangle}_{\substack{\text{source}\\ \text{(forcing)}\\ S}} - \nu\underbrace{\langle(\partial_x v)^2\rangle}_{\substack{\text{viscous}\\ \text{dissipation}}}$$

in k:

$$\frac{\partial}{\partial t}\langle \tilde{v}^2\rangle_k = S_k - \nu k^2\langle\tilde{v}^2\rangle_k + \overset{\substack{\text{nonlinear transfer}\\ \downarrow}}{T_k}$$

↑ inertial range interaction

where $\sum_k T_k = 0$ ⟺ NL transfer conserves energy i.e. expect T_k is sum of two cancelling terms (upon summation) or is anti-symmetric in k.

→ renormalized theory must respect symmetry, conservation laws of original, primitive eqn.

Now!

$$T_k = \frac{1}{3}\left\langle \frac{\partial}{\partial x}\frac{V^3}{3}\right\rangle_k$$

coherent mode coupling → $\sim \nu \langle v^2\rangle$

$$= i\,2\sum_{k'} \tilde{V}_{-k}\left(\tilde{V}_{-k'}\,\tilde{V}^{(2)}_{k+k'}\,(k+k')\right)$$

$$-2i\sum_{\substack{p,q\\ p+q=k}} \tilde{V}_{-p}\,\tilde{V}_{-q}\,\tilde{V}^{(2)}_{p+q}\,(p+q)$$

incoherent mode coupling (nonlinear noise ↔ I.R. cascade)

i.e. coherent:

$$\cong \tilde{V}_{-k}\left(C_k\,\tilde{V}_k\right)$$ → (~) same as renormalized response function

$$\cong C_k\langle \tilde{V}_k^2\rangle$$ → dissipation of $\langle \tilde{V}^2\rangle_k$ due turbulent viscosity (death)

incoherent:

$$\cong -\sum_{p+q=k}\langle \tilde{V}^2\rangle_p\langle \tilde{V}^2\rangle_q$$ → (birth) nonlinear noise emission into k via mode coupling

Now,

must treat beat / virtual mode self-consistently $\leftrightarrow$ include NL mixing in time response history $\leftrightarrow$ self-consistent field

$$* \frac{\partial}{\partial t} \tilde{V}^{(2)}_{k+k'} + \left[\nu (k+k')^2 + C_{k+k'}\right] \tilde{V}^{(2)}_{k+k'}$$

$$= -(i)(k+k') \left[\tilde{V}_{k'} \tilde{V}_{k}\right]$$

$\Rightarrow$

$$\tilde{V}^{(2)}_{k+k'} = -i(k+k') \int L^{+1}_{k+k'} \tilde{V}_{k'} \tilde{V}_{k} \, d\tau$$

$$\tilde{V}^{(2)}_{p+q} = -i(p+q) \int L^{+1}_{p+q} \tilde{V}_{p} \tilde{V}_{q} \, d\tau$$

$$T^{C}_{k} = 2i \sum_{k'} \tilde{V}_{-k}(t) \tilde{V}_{-k'}(t) (k+k')(-i(k+k')) *$$
$$\int_0^\infty L^{+1}_{k+k'}(\tau) \tilde{V}_{k}(t-\tau) \tilde{V}_{k}(t-\tau) d\tau$$

need model of temporal self-coherence!

$$= 2 \sum_{k'} (k+k')^2 \left\langle \tilde{V}_{-k}(t) \tilde{V}_{-k}(t) \int_0^\infty L^{+1}_{k+k'}(\tau) \tilde{V}_{k}(t-\tau) \tilde{V}_{k}(t-\tau) \right\rangle d\tau$$

Now, take:

$$\langle \tilde{V}(t) \tilde{V}(t+\tau) \rangle_k = |\tilde{V}_k|^2 e^{-C_k \tau}$$

$\rightarrow$ self-correlation decays at rate given by response time (neglect νk^2 for convenience) so

$$\underset{\substack{\downarrow \\ \text{coherent}}}{T_k^C} = 2\sum_{k'} (k+k')^2 \int_0^\infty d\tau \exp\left[-(c_{k+k'} + c_k + c_{k'})\tau\right] * \underbrace{\langle \tilde{v}^2 \rangle_{k'} \langle \tilde{v}^2 \rangle_k}_{\text{slow time residual}}$$

$$\underset{\substack{\downarrow \\ \text{triad coherence time}}}{\Theta_{k,k',k+k'}} = \int_0^\infty d\tau \exp\left[-(c_{k+k'} + c_k + c_{k'})\tau\right]$$

triad coherence time → set by modal decorrelation rates.

Similarly,

$$\underset{\substack{\downarrow \\ \text{incoherent}}}{T_k^I} = 2\sum_{\substack{p,q \\ p+q \\ =k}} (p+q)^2 \Theta_{p,q,k} \langle \tilde{v}^2 \rangle_p \langle \tilde{v}^2 \rangle_q$$

⇒ energy equation becomes:

$$\frac{\partial}{\partial t} \langle \tilde{v}^2 \rangle_k + \nu k^2 \langle \tilde{v}^2 \rangle_k + T_k = S_k$$

$$T_k = 2\sum_{k'} (k+k')^2 \Theta_{k,k',k+k'} \langle \tilde{v}^2 \rangle_{k'} \langle \tilde{v}^2 \rangle_k - 2\sum_{p,q} (p+q)^2 \Theta_{p,q,k} \langle \tilde{v}^2 \rangle_p \langle \tilde{v}^2 \rangle_q$$