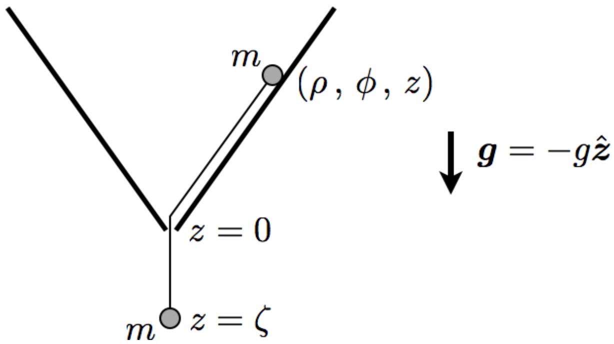
**PHYSICS 200A : CLASSICAL MECHANICS**
**FINAL EXAMINATION**
Do all problems

**[1]** Two identical point masses $m$ are connected by a massless string of length $b$. One point mass slides frictionlessly along the two-dimensional surface $z = f(\rho)$, where $(\rho, \phi, z)$ are cylindrical coordinates in three dimensions. The other end of the string is threaded through a small hole at the bottom of the surface, which may be presumed to lie at $\rho = 0$. The situation is depicted in the figure below (shown for the case $f''(\rho) = 0$).




(a) Using generalized coordinates $(\rho, \phi, z, \zeta)$ as shown in the figure, write down the Lagrangian. [4 points]

(b) Identify all constraints. [4 points]

(c) Write the equations of motion, retaining the Lagrange multipliers, for a general surface of rotation $z = f(\rho)$. [4 points]

(d) Identify all conserved quantities. [4 points]

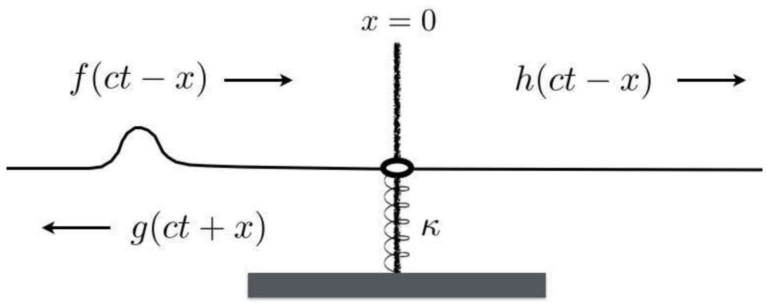
(e) Identify how the normal force $F_n$ supplied by the surface and the string tension $T$ are related to your Lagrange multipliers. [3 points]

(f) For the case of a cone, $f(\rho) = \rho \tan\alpha$, solve for the undetermined multipliers. [3 points]

(g) Eliminating the multipliers, find an equation of motion for $\rho$. [3 points]

**[2]** Two semi-infinite pieces of string, each with mass density $\mu$ and under tension $\tau$, meet at $x = 0$ in a massless ring which is attached to a spring of spring constant $K$, providing a restoring force $-Ku(t)$, and a pole which provides a frictional force $-\gamma\dot{u}(t)$, where $u(t)$ is the height of the ring relative to its equilibrium position, as depicted in the figure below.

(a) Write down the two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$. [5 points]

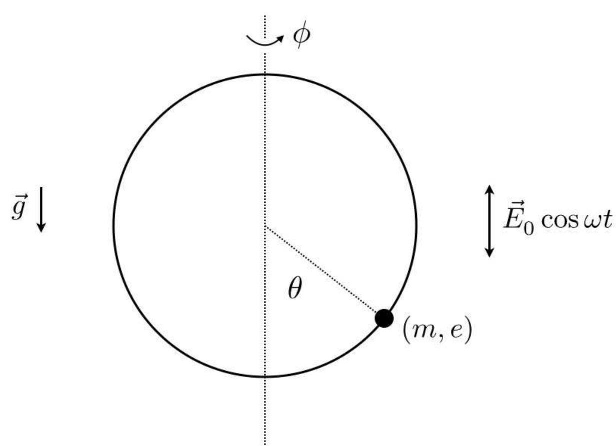



(b) Find the transmission coefficient $t(k) = \hat{h}(k)/\hat{f}(k)$, [5 points]

(c) Suppose $f(x) = y_0 \, e^{-|x|/\ell}$. Find the total energy $E$ of the wave. [5 points]

(d) Find $h(x)$. You may find it convenient to define $\lambda \equiv (2\tau + \gamma c)/K$, where $c \equiv \sqrt{\tau/\mu}$. [10 points]

[**3**] A particle of mass $m$ and charge $e$ moves frictionlessly along a massless hoop of radius $a$. The symmetry axis of the hoop rotates in the horizontal plane. The particle moves in the presence of both gravity $\vec{g} = -g\hat{\boldsymbol{z}}$ and a rapidly oscillating AC electric field $\vec{E}(t) = E_0 \hat{\boldsymbol{z}} \cos \omega t$. The setup is depicted in the figure below.




(a) Find the Lagrangian of the system $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t)$. [5 points]

(b) Find the Hamiltonian of the system $H(\theta, \phi, p_\theta, p_\phi, t)$. [5 points]

(c) Using the method of fast perturbations, and invoking conservation of $p_\phi$, find the effective Hamiltonian $K(P_\Theta, \Theta)$ for the slow time scale motion of the angle $\Theta(t)$, defined to be the slow component of $\theta(t)$. [5 points]

(d) When $p_\phi = 0$, show that the point $\Theta = \pi$ becomes stable for small oscillations if the electric field strength $|E_0|$ exceeds a critical value, $E_{0,c}$. Find $E_{0,c}$. [5 points]

(e) For general $p_\phi$, find an equation whose solution yields the equilibrium positions of $\Theta$. It may be convenient to define the quantities

$$\omega_0 = \sqrt{\frac{g}{a}} \quad , \quad \Omega = \sqrt{\frac{eE_0}{ma}} \quad , \quad \nu = \frac{p_\phi}{ma^2} \quad ,$$

all of which have dimensions of frequency. [5 points]

[4] *Provide short, accurate answers to each of the following.*

(a) A particle of mass $m$ moves in two dimensions $(x, y)$ subject to the potential $U(x, y) = mgy$ and the constraint

$$x\, e^{y/a} + y\, e^{x/a} = b \ ,$$

where $a$ and $b$ are constants. Find the equations of motion. [5 points]

(b) A particle of mass $m$ moves in three dimensions subject to a potential

$$U(x, y, z) = \frac{U_0}{(x + 2z)^2 + (y - x)^2} \ .$$

Find all conserved quantities. [5 points]

(c) In what sense do hurricanes rotate in the northern hemisphere, when viewed from above? Explain the basic physics which yields this result. [5 points]

(d) Under what conditions is the Hamiltonian of a mechanical system equal to the sum of its kinetic and potential energies? [5 points]

(e) A canonical transformation is generated by

$$F_2(q, P) = qP \cos\lambda + \tfrac{1}{2}(q^2 - P^2) \sin\lambda \ ,$$

where $\lambda$ is a dimensionless parameter. Find $Q(q, p)$ and $P(q, p)$ and show explicitly that the transformation is canonical. [5 points]