PHYSICS 200A : CLASSICAL MECHANICS
MIDTERM EXAMINATION SOLUTIONS

Normative time limit: four hours (consecutive!)

You are allowed to consult the online PHYS 200A course materials.

All problems are worth a total of 50 points each.

[**1**] A uniformly dense ladder of mass m and length 2ℓ leans against a block of mass M, as shown in Fig. 1. Choose as generalized coordinates the horizontal position X of the right end of the block, the angle θ the ladder makes with respect to the floor, and the coordinates (x, y) of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder is $T_{\text{CM}} + T_{\text{rot}}$, where $T_{\text{CM}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\text{rot}} = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertial. For a uniformly dense ladder of length 2ℓ, the moment of inertia is $I = \frac{1}{3}m\ell^2$.

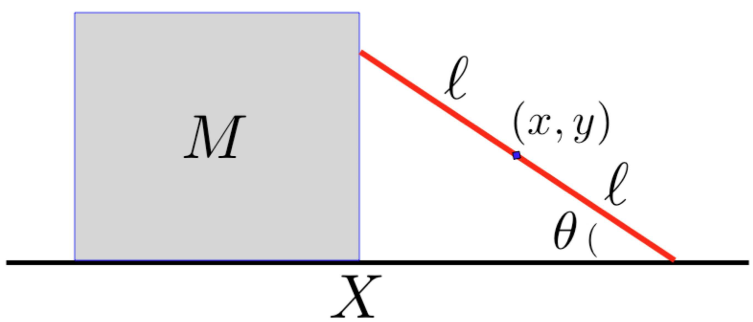

Figure 1: A ladder of length 2ℓ leaning against a massive block. All surfaces are frictionless.

(a) Write down the Lagrangian for this system in terms of the coordinates X, θ, x, y, and their time derivatives. `[10 points]`

(b) Write down all the equations of constraint. `[10 points]`

(c) Write down all the equations of motion. `[10 points]`

(d) Find all conserved quantities. `[10 points]`

(e) Find an equation relating the angle θ^* at which the ladder detaches from the block and the initial angle of inclination θ_0. Your equation should only include θ^*, θ_0, and the dimensionless ratios M/m and $I/m\ell^2$, but not $\dot{\theta}$ or $\ddot{\theta}$. *Hint: Find the energy of the system at the moment of detachment.* `[10 points]`

[**2**] Two identical semi-infinite lengths of string are joined at a point of mass m which moves vertically along a thin wire, as depicted in fig. 2. The mass moves with friction coefficient γ, *i.e.* its equation of motion is

$$m\ddot{z} + \gamma\dot{z} = F \quad ,$$

where z is the vertical displacement of the mass, and F is the force on the mass due to the string segments on either side. In this problem, gravity is to be neglected. It may be convenient to define $K \equiv 2\tau/mc^2$ and $Q \equiv \gamma/mc$.

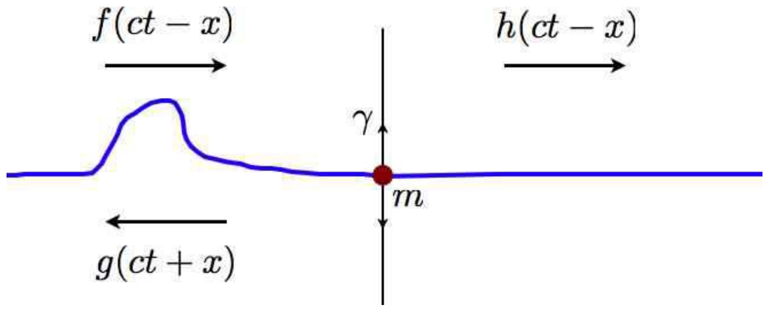

Figure 2: A point mass m joining two semi-infinite lengths of identical string moves vertically along a thin wire with friction coefficient γ.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct-x) + g(ct+x) & (x<0) \\ h(ct-x) & (x>0) \end{cases} \quad .$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$. `[15 points]`

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \quad ,$$

et cetera for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\big|r(k)\big|^2 + \big|t(k)\big|^2$. Show that this sum is always less than or equal to unity, and interpret this fact. `[15 points]`

(c) Find an expression which is a functional of $f(x)$ or $\hat{f}(k)$, for the total energy change ΔE of the string due to the friction acting on the mass point. *Hint: You can compute* ΔE *by computing the net outgoing energy current at* $x = 0^{\pm}$ *and then integrating over time.* `[10 points]`

(d) For an incident wave whose characteristic wavelength λ satisfies $K\lambda \gg 1$ and $Q\lambda \gg 1$, find the ratio $|\Delta E|/E_0$, where E_0 is the initial energy in the string. `[10 points]`

[3] Consider the map

$$\begin{aligned} q_{n+1} &= q_n + f(q_n, p_{n+1}) \\ p_{n+1} &= p_n + g(q_n, p_{n+1}) \quad . \end{aligned}$$

(a) Under what conditions does this map generate a canonical transformation $(q_n, p_n) \rightarrow (q_{n+1}, p_{n+1})$? `[10 points]`

(b) Show that the conditions in part (a) are satisfied if f and g are expressed as first (partial) derivatives of a function $R(q_n, p_{n+1})$. [10 points]

(c) For the map

$$\begin{aligned} q_{n+1} &= q_n + b\, q_n + c\, p_{n+1} \\ p_{n+1} &= p_n - a\, q_n - b\, p_{n+1} \quad , \end{aligned}$$

where a, b, and c are constants, what is the function $R(q_n, p_{n+1})$ from part (b)?
[10 points]

(d) Express the map in part (c) as $\boldsymbol{\varphi}_{n+1} = \hat{T}\boldsymbol{\varphi}_n$, where $\boldsymbol{\varphi}_n = \begin{pmatrix} q_n \\ p_n \end{pmatrix}$. Find an explicit expression for $\hat{T}$. [10 points]

(e) For fixed $b > 0$, plot the phase diagram in the (a, c) plane, identifying regions where $|\hat{T}^n \boldsymbol{\varphi}_0|$ grows exponentially with n (for generic initial conditions $\boldsymbol{\varphi}_0$), and regions where it is bounded. Sketch your results. [10 points]

[4] Consider the Hamiltonian for one-dimensional particle motion in a gravitational field,

$$H(z,p) = \overbrace{\frac{p^2}{2m} + mgz}^{H_0} + \overbrace{\varepsilon \alpha z^3}^{\varepsilon H_1} \quad ,$$

where ε is small. The particle is constrained such that $z \geq 0$. It msy be useful to consult §15.5.5 of the Lecture Notes.

(a) Find the unperturbed Hamiltonian $\widetilde{H}_0(J_0)$ and the unperturbed frequency $\nu_0(J_0)$.
[15 points]

(b) Find the unperturbed frequencies $\nu_0(h)$, where h is the amplitude of the z motion. Your result should look familiar. [15 points]

(c) Find the energy $E(J)$ to lowest nontrivial order in ε. [20 points]