PHYSICS 200A : CLASSICAL MECHANICS
PROBLEM SET #3

[1] Two particles of masses m_1 and m_2 attract each other according to the logarithmic potential $U(r) = U_0 \ln(r/a)$.

(a) Write down and sketch the effective potential $U_{\text{eff}}(r)$.

(b) Find the radius r_0 and period τ_0 of a circular orbit.

(c) For small deviations about a circular orbit, write $r(t) = r_0 + \eta(t)$. Derive the equation of motion for the deviation $\eta(t)$ and solve this equation assuming η is small. (What do we mean by "small"?)

(d) What is the *geometric* equation of the perturbation $\eta(\phi)$? Is the perturbed orbit closed? Why or why not?

[2] A particle moves in a central force $\vec{f}(\vec{r}) = \hat{\boldsymbol{r}} f(r)$.

(a) Using the equation for the shape of the orbit, derive the force law under which the shape of the orbit is $r(\phi) = a/\cos(\phi - \phi_0)$. Explain why your answer makes excellent sense.

(b) What force law will result in the shape $r(\phi) = 2b/\phi^2$? Is the force attractive or repulsive? Sketch the orbit over the interval $\phi \in (0, \infty)$.

[3] Evil space aliens send a probe into our solar system to observe the earth. The probe orbits the sun with its perihelion at distance $r_{\text{p}} = \frac{3}{4}\, a_\oplus$, where its velocity is $v_{\text{p}} = \frac{4}{3}\, v_\oplus$. ($a_\oplus$ and $v_\oplus$ are the orbital radius and velocity of the earth, respectively.) The probe's orbit is coplanar with that of the earth, and you may neglect the interaction of the probe with all bodies other than the sun.

(a) Compute the eccentricity of the probe's orbit.

(b) Compute the probe's distance from the sun at aphelion, and its velocity at aphelion.

(c) Write down the geometric equation for the probe's orbit.

(d) Let perihelion occur when the azimuthal angle is $\phi = \pi$. At what value of ϕ does the probe cross the earth's orbit?

(e) Compute the period of the probe's orbit.

[4] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\,\big|\boldsymbol{r}_1 - \boldsymbol{r}_2\big|^{1/4}$.

(a) Sketch the effective potential $U_{\text{eff}}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.

(e) What value of n would result in a perturbed orbit shaped like that in fig. 1?

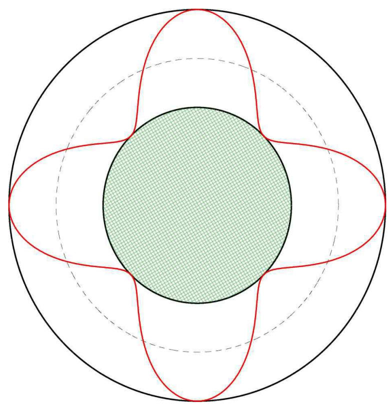

Figure 1: Closed precession in a central potential $U(r) = kr^n$.

[5] A symmetric top with one fixed point in a gravitational field moves with its symmetry axis nearly vertical ($\theta \ll 1$) and $p_\phi = p_\psi$.

(a) Expand the effective potential through terms of order θ^4.

(b) If $p_\psi^2 > 4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a minimum at $\theta = 0$. Sketch $U_{\text{eff}}(\theta)$ for small θ. Prove that the frequency of small oscillations about this configuration is given by

$$\Omega^2 = \frac{p_\psi^2 - 4I_1 Mg\ell}{4I_1^2} \ .$$

(c) If p_ψ^2 is slightly smaller than $4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a maximum at $\theta = 0$ and a minimum at some finite value θ^*. Find θ^*, and sketch $U_{\text{eff}}(\theta)$ for small θ, and find the frequency of small oscillations about $\theta = \theta^*$.