PHYSICS 200A : CLASSICAL MECHANICS
PROBLEM SET #5

[1] A string of uniform mass density and length ℓ hangs under its own weight in the earth's gravitational field. Consider small transverse displacements $u(x,t)$ in a plane.

(a) Compute the equilibrium tension in the string $\tau(x)$, where x is the distance from the point of suspension.

(b) Show that the normal modes satisfy Bessel's equation.

(c) What are the boundary conditions?

(d) What are the normal mode frequencies?

(e) What are the normal modes?

(f) Construct the general solution to the initial value problem.

[2] A string of length $2a$ is stretched to a constant tension τ with its ends fixed. The mass density of the string is given by

$$\sigma(x) = \sigma_0 \left(1 - \frac{|x|}{a}\right) .$$

(a) Use a zero-parameter trial function to derive a variational estimate of the lowest resonant frequency ω_1. Compare with the numerical value $\omega_1^2 \approx 3.477\,\tau/a^2\sigma_0$.

(b) Devise a one-parameter trial function and show that it leads to a better (*e.g.* lower frequency) estimate.

(c) Repeat part (a) for the next eigenfrequency ω_2, whose numerical value is $\omega_2^2 \approx 18.956\,\tau/a^2\sigma_0$.

[3] A wave travels along an infinite string stretched to a tension τ. The mass density of the string is σ_0 for $|x| > a$ and σ_1 for $|x| < a$.

(a) Solve the wave equation in the regions $|x| > a$ and $|x| < a$, respectively, to find exact expressions for the transmission and reflection amplitudes.

(b) Show that the energy transmission coefficient is given by

$$T = 1 - R = \left\{ 1 + \left(\frac{k_1^2 - k_0^2}{2k_0 k_1}\right)^2 \sin^2(2k_1 a) \right\}^{-1} ,$$

where $k_i = \sqrt{\sigma_i/\tau}\,\omega = \omega/c_i$. Discuss the frequency dependence of T, noting the position and widths of the transmission resonances (where $T = 1$).

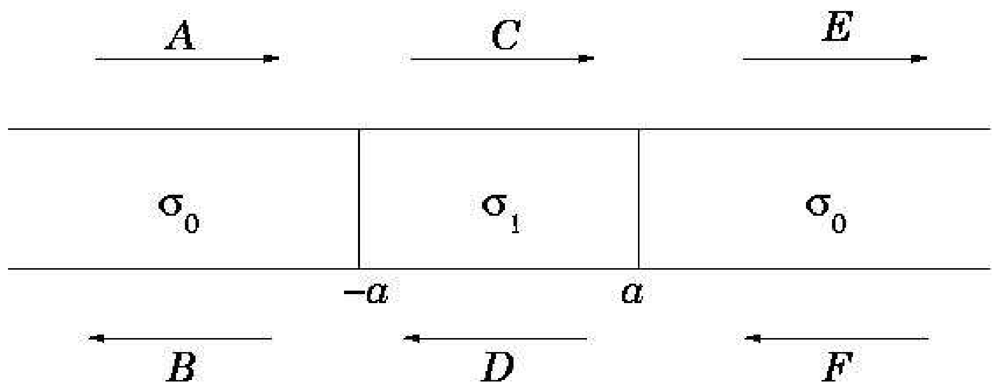

Figure 1: An infinite string of uniform mass density σ_0 with an inserted segment of length $2a$ with mass density σ_1. Waves $A\,e^{ik_0x}$ and $F\,e^{-ik_0x}$ are incident from the left and right, respectively, giving rise to outgoing waves $B\,e^{-ik_0x}$ and $E\,,e^{ik_0x}$.

[4] Consider a uniform circular membrane of radius a, areal mass density σ, and tension τ.

(a) A point mass m is attached at the center of the membrane. Show that the total density is now

$$\sigma(r,\phi) = \sigma + \frac{m}{\pi r}\,\delta(r) \quad .$$

(b) Show that the wave equation $-\nabla^\alpha \Big[\tau(\boldsymbol{r})\,\nabla^\alpha\,u(\boldsymbol{r})\Big] = \omega^2\,\sigma(\boldsymbol{r})\,u(\boldsymbol{r})$ may be recast as $K\psi = E\psi$, where $E = \omega^2$ and

$$K = -\frac{1}{\sqrt{\sigma(\boldsymbol{r})}}\nabla^\alpha\,\tau(\boldsymbol{r})\,\nabla^\alpha\,\frac{1}{\sqrt{\sigma(\boldsymbol{r})}} \quad .$$

Show that $K = K^\dagger$ is self-adjoint. Writing $\sigma(\boldsymbol{r}) = \sigma + \delta\sigma(\boldsymbol{r})$ and taking $\tau(\boldsymbol{r}) = \tau$, show that $K = K_0 + \delta K$ to first order in $\delta\sigma(\boldsymbol{r})$ and find an expression for δK.

(c) Show that the normalized eigenfunctions of K_0 are

$$\psi_{\ell,n}(r,\varphi) = \frac{1}{\sqrt{\pi}\,a\,J_{\ell+1}(x_{l,n})}\,J_\ell\big(x_{\ell,n}r/a\big)\,e^{i\ell\varphi} \quad ,$$

where $J_\ell(x_{\ell,n}) = 0$, *i.e.* $x_{\ell,n}$ is the $n^{\rm th}$ root of $J_\ell(x)$. You may find the following result to be useful:

$$\int\limits_0^a dr\;r\;J_\ell(x_{\ell,n}\,r/a)\,J_\ell(x_{\ell,n'}\,r/a) = \tfrac{1}{2}\,a^2\,\big[J_{\ell\pm1}(x_{\ell,n})\big]^2\,\delta_{n,n'} \quad ,$$

where one may take either sign in the $\pm$ symbol.

(d) Use first-order perturbation theory to show that only the circularly symmetric modes are affected by the point mass, in which case to first order in perturbation theory we have

$$\omega^2_{l,n} = \frac{c^2x^2_{l,n}}{a^2}\left\{1 - \frac{m\,\delta_{l,0}}{\pi\sigma a^2 J_1^2(x_{0,n})}\right\} \quad .$$

Discuss the behavior for large n and compare to the corresponding case of a point mass on a string.