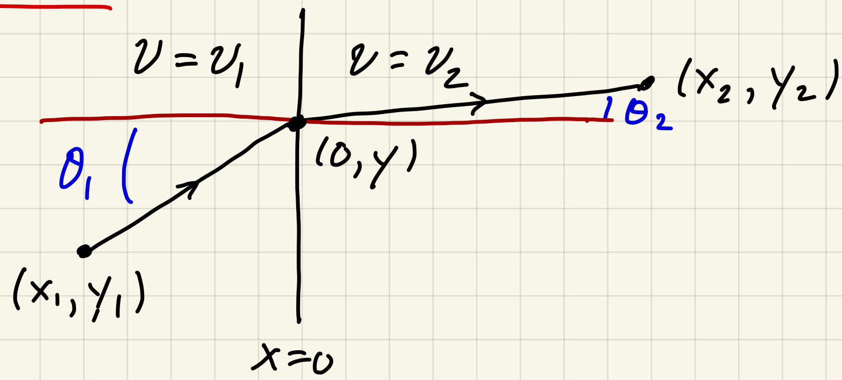
200A Lecture 1

Snell's law:



$$T(y) = \frac{1}{v_1}\sqrt{x_1^2 + (y - y_1)^2} + \frac{1}{v_2}\sqrt{x_2^2 + (y_2 - y)^2}$$

$$\frac{dT}{dy} = \frac{1}{v_1}\frac{y - y_1}{\sqrt{x_1^2 + (y - y_1)^2}} - \frac{1}{v_2}\frac{y_2 - y}{\sqrt{x_2^2 + (y_2 - y)^2}} \equiv 0$$

$$= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \equiv 0$$

Thus with $v_j = c/n_j$ we have $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Now consider a sequence of slabs with differing v_j.

We must have

$$\frac{\sin\theta_j}{v_j} = \frac{\sin\theta_{j+1}}{v_{j+1}} \xrightarrow[\text{limit}]{\text{continuum}} \frac{\sin\theta(x)}{v(x)} = P = \text{constant}$$

We'll see that P corresponds to conserved momentum in mechanics. Note that

$$\sin\theta(x) = \frac{y'(x)}{\sqrt{1 + [y'(x)]^2}} = P\,v(x)$$

which yields

$$y' = \frac{Pv}{\sqrt{1 - P^2 v^2}} \Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} ds\, \frac{P\,v(s)}{\sqrt{1 - P^2 v^2(s)}}$$

$$\frac{d}{dx}\frac{y'}{v\sqrt{1+(y')^2}} = \frac{y''}{v\sqrt{1+(y')^2}} - \frac{y'^2 y''}{v(1+(y')^2)^{3/2}} - \frac{v' y'}{v^2\sqrt{1+(y')^2}}$$

$$= \frac{1}{v\left[1+(y')^2\right]^{3/2}}\left\{ y'' - \frac{v'}{v}\left(1+(y')^2\right)y'\right\} = 0$$

Thus,

$$y'' - (\ln v)'\left[1+(y')^2\right]y' = 0$$

Of course this may be integrated once to yield

$$\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} = P v(x)$$

Functional calculus

- Functions : eat numbers, excrete numbers

  e.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

  extremization : demand $df = 0$ to lowest order in dx

$$f(x^* + dx) = f(x^*) + \underbrace{f'(x^*)\,dx + \frac{1}{2} f''(x^*)(dx)^2 + \ldots}_{df}$$

Thus, $df = 0$ in $dx \to 0$ limit says $f'(x^*) = 0$, i.e. if $f'(x^*) = 0$ then x^* is an extremum. To second order,

$$f''(x^*) > 0 \Rightarrow \text{minimum}, \quad f''(x^*) < 0 \Rightarrow \text{maximum},$$

$$f''(x^*) = 0 \Rightarrow \text{inflection}$$

Multivariable functions: $f(\overbrace{x_1, \ldots, x_n}^{\vec{x}})$ $\quad f: \mathbb{R}^n \to \mathbb{R}$

$$f(\vec{x}^* + d\vec{x}) = f(\vec{x}^*) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\bigg|_{\vec{x}^*} dx_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}\bigg|_{\vec{x}^*} dx_j\, dx_k + \cdots$$

Extremum $\Rightarrow \dfrac{\partial f}{\partial x_j}\bigg|_{\vec{x}^*} = 0 \quad \forall\, j = 1, \ldots, n$

Hessian matrix: $H_{jk} = \dfrac{\partial^2 f}{\partial x_j \partial x_k}\bigg|_{\vec{x}^*}$ real, symmetric

eigenvalues of H: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

All $\lambda_j > 0 \Rightarrow \vec{x}^*$ local minimum

All $\lambda_j < 0 \Rightarrow \vec{x}^*$ local maximum

Some positive, some negative eigenvalues $\Rightarrow \vec{x}^*$ inflection pt

- **Functionals:** functionals eat functions, excrete numbers

Typically, functionals are integrals, e.g.

$$F[y(x)] = \int_{x_L}^{x_R} dx \left\{ \frac{1}{2} K \left(\frac{dy}{dx}\right)^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4 \right\}$$

Consider a class of functionals of the form

$$F[y(x)] = \int_{x_L}^{x_R} dx\, L(y, y', x)$$

where $L(y, y', x)$ is a specified function of three variables, e.g.

$$L = \frac{1}{2} K (y')^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4$$

Note this class may be extended to

$$G[y(x)] = \int_{x_L}^{x_R} dx\, L(y, y', y'', x)$$

Etc.

We now compute the functional variation by computing

$$\delta F = F[y(x)+\delta y(x)] - F[y(x)]$$

$$= \int_{x_L}^{x_R} dx \left\{ L(y'+\delta y', y+\delta y, x) - L(y', y, x) \right\}$$

$$= \int_{x_L}^{x_R} dx \left\{ \frac{\partial L}{\partial y'}\delta y' + \frac{\partial L}{\partial y}\delta y + \ldots \right\} \qquad \delta y' = \frac{d}{dx}\delta y$$

$$= \int_{x_L}^{x_R} dx \left\{ \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\delta y\right) + \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y \right\}$$

$$= \left.\frac{\partial L}{\partial y'}\right|_{x_R} \delta y(x_R) - \left.\frac{\partial L}{\partial y'}\right|_{x_L} \delta y(x_L) + \int_{x_L}^{x_R} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y$$

Suppose $y(x)$ is fixed at the endpoints, in which case

$$\delta y(x_L) = \delta y(x_R) = 0$$

Then since $\delta y(x)$ elsewhere on $[x_L, x_R]$ is arbitrary, we conclude that

$$\frac{\delta F}{\delta y(x)} = \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]_x = 0 \quad \forall \; x \in [x_L, x_R]$$

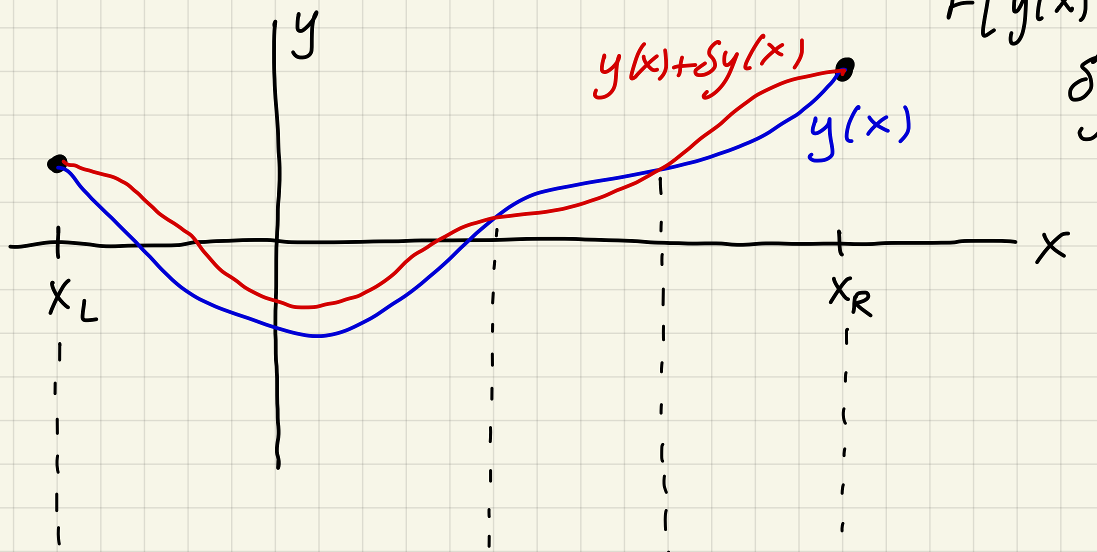
Since $L = L(y', y, x)$, the above equation is a second order ODE, known as the Euler-Lagrange equation. NB: If $y(x_{L,R})$ are not fixed, then we also require

$$\left.\frac{\partial L}{\partial y'}\right|_{x_{L,R}} = 0 \quad \text{as well as} \quad \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

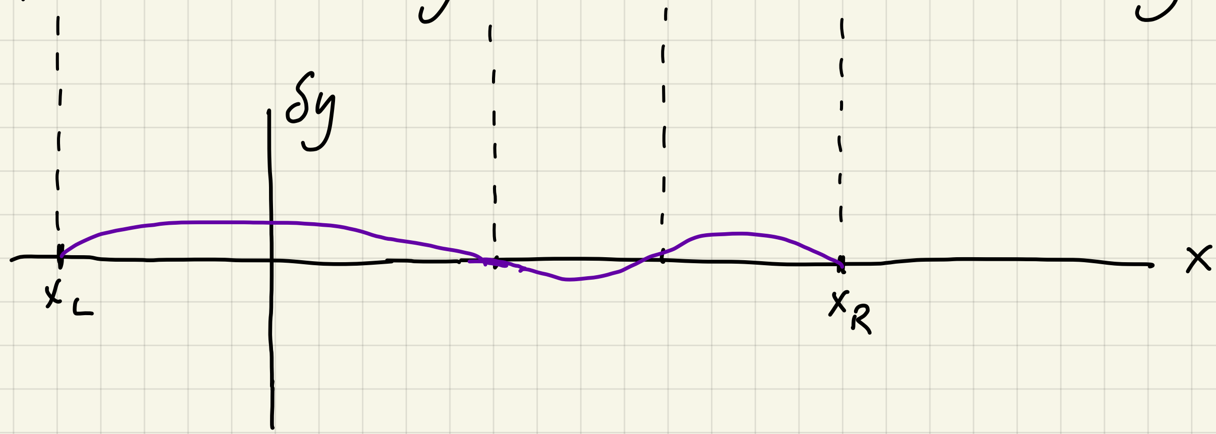
in order that $\delta F = 0$.

Graphical representation:

$$F[y(x)] = F$$
$$F[y(x)+\delta y(x)] = F+\delta F$$
$$\delta y(x_{L,R}) \equiv 0$$



The variation $\delta y(x)$ resembles the following



$$\delta F[y(x)] = F[y(x)+\delta y(x)] - F[y(x)]$$

$$\delta y' = \frac{d}{dx}\delta y = \delta \frac{dy}{dx} \quad , \text{ i.e. } [\delta, d] = 0$$

$$\frac{\partial L}{\partial y'}\delta y' = \frac{\partial L}{\partial y'}\frac{d}{dx}\delta y = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\delta y\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\delta y$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} : \quad \frac{d}{dx} = \frac{\partial}{\partial x} + y''\frac{\partial}{\partial y'} + y'\frac{\partial}{\partial y}$$

We now consider two important special cases:

① $\frac{\partial L}{\partial y} = 0$, i.e. $L(y, y', x)$ independent of y

Then EL eqn says $\cancel{\frac{\partial L}{\partial y}} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$,

which may be integrated once to yield $\frac{\partial L}{\partial y'} = P$, where P = constant. This is then a first order ODE in $y(x)$. Example: $L = \frac{1}{v(x)} \sqrt{1+(y')^2}$. Then

$$P = \frac{\partial L}{\partial y'} = \frac{y'}{v\sqrt{1+(y')^2}} \equiv \frac{1}{v_0}$$

(momentum conservation in mechanics)

$$\Rightarrow \frac{dy}{dx} = \frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \quad \text{with } v_0 \equiv 1/P$$

② $\frac{\partial L}{\partial x} = 0$, i.e. $L(y, y', x)$ independent of x

Define $H \equiv y' \frac{\partial L}{\partial y'} - L$. Then

(energy conservation in mechanics)

$$\frac{dH}{dx} = \frac{d}{dx}\left\{ y' \frac{\partial L}{\partial y'} - L \right\}$$

$$= \cancel{y'' \frac{\partial L}{\partial y'}} + y' \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \cancel{\frac{\partial L}{\partial y'} y''} - \frac{\partial L}{\partial y} y' - \cancel{\frac{\partial L}{\partial x}}$$

$$= y' \left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} \right] = 0 \quad \text{if EL satisfied}$$

Thus, $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dH}{dx} = 0 \Rightarrow H$ is constant

$$y' \frac{\partial L}{\partial y'} - L = H \quad \text{again a first order ODE}$$

③ If $L(y,y',x) = L_0(y,y',x) + \frac{d}{dx}\Delta(y,x)$, then

$$F[y(x)] = \int_{x_L}^{x_R} dx\, L_0(y,y',x) + \Delta(y(x_R),x_R) - \Delta(y(x_L),x_L)$$

If $\delta y(x_{L,R}) = 0$ (fixed endpoints), then the Δ term makes no contribution to the EL eqns, which are then

$$\frac{\partial L_0}{\partial y} - \frac{d}{dx}\left(\frac{\partial L_0}{\partial y'}\right) = 0$$

- Functional Taylor series :

$$\begin{aligned} F[y+\delta y] = F[y] &+ \int_{x_L}^{x_R} dx_1\, K_1(x_1)\, \delta y(x_1) \\ &+ \frac{1}{2!}\int_{x_L}^{x_R} dx_1 \int_{x_L}^{x_R} dx_2\, K_2(x_1,x_2)\, \delta y(x_1)\, \delta y(x_2) \\ &+ \frac{1}{3!}\int_{x_L}^{x_R} dx_1 \int_{x_2}^{x_R} dx_2 \int_{x_L}^{x_R} dx_3\, K_3(x_1,x_2,x_3)\, \delta y(x_1)\, \delta y(x_2)\, \delta y(x_3) \\ &+ \mathcal{O}(\delta y^4) \end{aligned}$$

Thus,

$$K_n(x_1,\ldots,x_n) = \frac{\delta^n F}{\delta y(x_1)\cdots\delta y(x_n)} = n^{th} \text{ functional derivative}$$

$$\left.\begin{array}{l} \bullet \text{ Examples : §3.3 in the lecture notes} \\ \bullet \text{ More on functionals : §3.4} \end{array}\right\} \text{READ!}$$

Mechanics

Lecture 2 (Oct 7)

Hamilton's principle : $\delta S = 0$ where

$$S[q(t)] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) = \text{action functional}$$

with $q = \{q_1, \dots, q_n\}$ = set of generalized coordinates

The function $L(q, \dot{q}, t)$ is the Lagrangian, and is given by $L = T - U$, where T = kinetic energy and U = potential energy. Typically $T = T(q, \dot{q})$ is a quadratic form in the generalized velocities $\{\dot{q}_\sigma\}$, i.e. $T(q, \dot{q}) = T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'}$. For example

$$T = \frac{1}{2} m \dot{\vec{x}}^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad \text{Cartesian } (x, y, z)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) \qquad \text{polar } (r, \theta, \phi)$$

The potential energy U is most often a function of q, but $U = U(q, \dot{q})$ applies, e.g., for charged particles in a magnetic field, where

$$U(\vec{x}, \dot{\vec{x}}) = \underbrace{q}_{\text{charge}}\, \underbrace{\phi(\vec{x})}_{\text{scalar potential}} - \frac{q}{c} \underbrace{\vec{A}(\vec{x})}_{\text{vector potential}} \cdot \frac{d\vec{x}}{dt}$$

Free particle $\Rightarrow L = \frac{1}{2} m \vec{v}^2$ (§3.6.3)

- NB : In general $L = \frac{1}{2} T_{\sigma\sigma'}(q, t)\, \dot{q}_\sigma \dot{q}_{\sigma'} - U(q, \dot{q}, t)$

Equations of motion: $\underbrace{F_\sigma}$ $\equiv \dfrac{\partial L}{\partial q_\sigma}$ = generalized force

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q_\sigma}}\Big) = \frac{\partial L}{\partial q_\sigma}, \quad \sigma \in \{1, \dots, n\} \qquad \begin{matrix} y \to q_\sigma \\ y' \to \dot q_\sigma \end{matrix}$$

$p_\sigma \equiv \dfrac{\partial L}{\partial \dot q_\sigma}$ = generalized momentum

Thus, $\dot p_\sigma = F_\sigma$, i.e. Newton's second law.

- Conservation laws:

Most general setting: to be discussed (Noether's theorem)

For now, recall results from COV:

① $\dfrac{\partial L}{\partial q_\sigma} = 0 \Rightarrow p_\sigma = \dfrac{\partial L}{\partial \dot q_\sigma}$ = constant $(\dot p_\sigma = 0)$

Momentum p_σ is conserved because the force $F_\sigma = 0$

Example: $T = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$, $U = mgz$

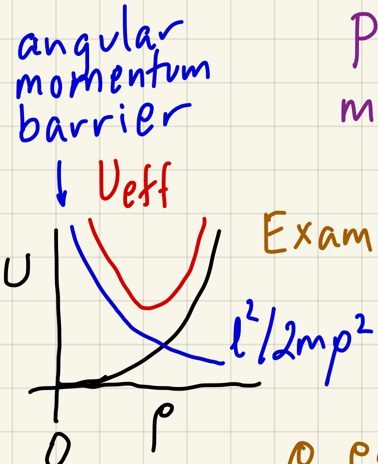
$$\Rightarrow F_x = \frac{\partial L}{\partial x} = 0, \quad F_y = \frac{\partial L}{\partial y} = 0$$

$$p_x = m\dot x \Rightarrow x(t) = x(0) + \frac{p_x}{m} t$$

$$p_y = m\dot y \Rightarrow y(t) = y(0) + \frac{p_y}{m} t$$

$$p_z = m\dot z, \quad F_z = -\frac{\partial U}{\partial z} = -mg$$

$$m\ddot z = -mg \Rightarrow z(t) = z(0) + \dot z(0)t - \tfrac{1}{2}gt^2$$



Example': $L = \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2) - U(\rho)$ (2D polar)

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi = \text{angular momentum} \equiv \ell$$

ρ eqn: $m\ddot\rho = m\rho\dot\phi^2 - U'(\rho) = \dfrac{\ell^2}{m\rho^3} - U'(\rho) = -U'_{eff}(\rho)$

$$U_{eff}(\rho) = \frac{\ell^2}{2m\rho^2} + U(\rho)$$

Scratch

$$L = \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2) - U(\rho)$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi = \ell$$

IMPORTANT: Can substitute $\dot\phi = \frac{\ell}{m\rho^2}$ in eqns of motion but <u>not</u> in Lagrangian itself!

WRONG:

$$L = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\phi^2 - U(\rho)$$

$$= \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\left(\frac{\ell}{m\rho^2}\right)^2 - U(\rho)$$

$$= \frac{1}{2}m\dot\rho^2 + \frac{\ell^2}{2m\rho^2} - U(\rho)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\rho}\right) = m\ddot\rho = \frac{\partial L}{\partial \rho} = -\frac{\ell^2}{m\rho^3} - U'(\rho)$$

wrong sign!

RIGHT:

$$L = \frac{1}{2}m\dot\rho^2 + \frac{1}{2}m\rho^2\dot\phi^2 - U(\rho)$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi = \ell \text{ constant } (\dot p_\phi = 0)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\rho}\right) = m\ddot\rho = \frac{\partial L}{\partial \rho} = m\rho\dot\phi^2 - U'(\rho)$$

$$= m\rho\left(\frac{\ell}{m\rho^2}\right)^2 - U'(\rho)$$

$$= +\frac{\ell^2}{m\rho^3} - U'(\rho) = -U'_{eff}(\rho)$$

right sign!

② $\frac{\partial L}{\partial t} = 0 \Rightarrow H = \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} - L = \dot{q}_\sigma p_\sigma - L$ conserved

implied summation on repeated indices

See it again:

$$\frac{dH}{dt} = \cancel{\ddot{q}_\sigma p_\sigma} + \cancel{\dot{q}_\sigma \dot{p}_\sigma} - \cancel{\underbrace{\frac{\partial L}{\partial q_\sigma}}_{F_\sigma} \dot{q}_\sigma} - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma} - \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_\sigma}\frac{dq_\sigma}{dt} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{q}_\sigma}{dt} + \frac{\partial L}{\partial t}$$

Thus, $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$, and for

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{x}}_j^2 - U(\vec{x}_1, \ldots, \vec{x}_N)$$

we have that $H = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{x}}_j^2 + U(\vec{x}_1, \ldots, \vec{x}_N)$

is a constant of the motion.

- In general, $H = \dot{q}_\sigma p_\sigma - L(q, \dot{q}, t)$ is a Legendre transform of L:

$$dH = \cancel{p_\sigma d\dot{q}_\sigma} + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma} - \frac{\partial L}{\partial t} dt$$

and hence $H = H(q, p, t)$ with

$$\frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -F_\sigma, \quad \frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

We then have Hamilton's equations of motion:

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \dot{\xi}_\alpha = J_{\alpha\beta} \frac{\partial H}{\partial \xi_\beta}$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} q \\ p \end{pmatrix}$$

Procedure

(i) Choose a set of generalized coordinates

(ii) Find KE $T(q,\dot{q},t)$ and PE $U(q,t)$ or $U(q,\dot{q},t)$ and thus the Lagrangian $L(q,\dot{q},t) = T - U$.

(iii) Find the canonical momenta $p_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma}$ and the generalized forces $F_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma}$.

$\longrightarrow p_\sigma = p_\sigma(q,\dot{q},t)$

(iv) Identify any conserved quantities (later: Noether's thm)

(v) Evaluate $\dot{p}_\sigma$ (carefully!) and write $\dot{p}_\sigma = F_\sigma$

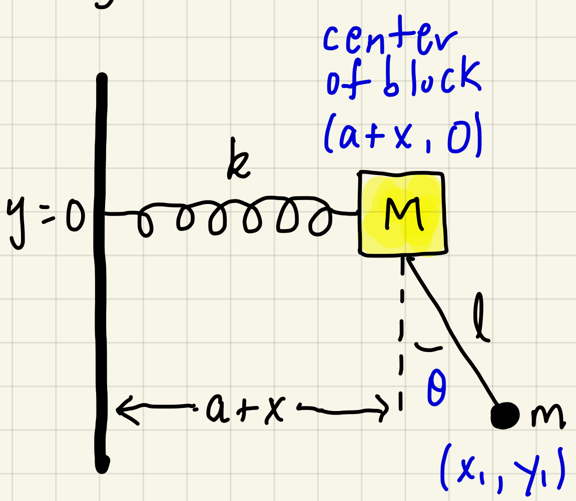
(vi) Integrate the equations of motion to get $\{q_\sigma(t)\}$, the motion of the system.

2n constants of integration $\{q_\sigma(0), \dot{q}_\sigma(0)\}$

§ 3.8 : Cartesian, cylindrical, and polar coordinates

§ 3.10 : Examples

§ 3.10.4 : Pendulum attached to mass on a spring



a = unstretched length of spring

coordinates of mass m : (x_1, y_1)

$$x_1 = a + x + l\sin\theta\ , \qquad y_1 = -l\cos\theta$$

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$$

$$= \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\dot{\theta}$$

$$U = \tfrac{1}{2}kx^2 + mgy_1$$

$$= \tfrac{1}{2}kx^2 - mgl\cos\theta$$

Lagrangian:

$$L = T - U$$
$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mgl\cos\theta$$

Generalized momenta:

- $p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\,\dot{\theta}$
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml\cos\theta\,\dot{x} + ml^2\dot{\theta}$

Generalized forces:

- $F_x = \frac{\partial L}{\partial x} = -kx$
- $F_\theta = \frac{\partial L}{\partial \theta} = -ml\sin\theta\,\dot{x}\dot{\theta} - mgl\sin\theta$

Equations of motion:

- $\dot{p}_x = F_x \Rightarrow (M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = -kx$
- $\dot{p}_\theta = F_\theta \Rightarrow ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} \cancel{- ml\sin\theta\,\dot{x}\dot{\theta}}$
  $= \cancel{- ml\sin\theta\,\dot{x}\dot{\theta}} - mgl\sin\theta$

Conserved quantities:

Only $H = \dot{x}p_x + \dot{\theta}p_\theta - L$

$$= \left[(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta}\right] + \left[ml\cos\theta\,\dot{x}\dot{\theta} + ml^2\dot{\theta}^2\right]$$
$$- \frac{1}{2}(M+m)\dot{x}^2 - \frac{1}{2}ml^2\dot{\theta}^2 - ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}kx^2 - mgl\cos\theta$$
$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}kx^2 - mgl\cos\theta$$
$$= T + U \equiv E$$

Small oscillations: linearize the equations of motion

- $(M+m)\ddot{x} + m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,\dot{\theta}^2 = -kx$
- $m\ell\cos\theta\,\ddot{x} + m\ell^2\ddot{\theta} = -mg\ell\sin\theta$

$$\Longrightarrow \quad \begin{aligned} &\bullet\ (M+m)\ddot{x} + m\ell\ddot{\theta} = -kx \\ &\bullet\ \ddot{x} + \ell\ddot{\theta} = -g\theta \end{aligned} \qquad \left(\begin{aligned}&\text{expand about } x=\theta=0 \\ &\text{assume } x,\theta,\dot{x},\dot{\theta} \text{ small}\end{aligned}\right)$$

The five parameters (M,m,ℓ,k,g) may be reduced to three:

$$u \equiv \frac{x}{\ell}\ ,\quad \alpha \equiv \frac{m}{M}\ ,\quad \omega_0^2 \equiv \frac{k}{M}\ ,\quad \omega_1^2 \equiv \frac{g}{\ell}$$

Then we have

$$\begin{aligned} &\bullet\ (1+\alpha)\ddot{u} + \alpha\ddot{\theta} + \omega_0^2 u = 0 \\ &\bullet\ \ddot{u} + \ddot{\theta} + \omega_1^2\theta = 0 \end{aligned}$$

This linear system may be solved by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{-i\omega t} \qquad \frac{d^2}{dt^2} \longrightarrow -\omega^2$$

$$\Rightarrow \begin{pmatrix} \omega_0^2 - (1+\alpha)\omega^2 & -\alpha\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} = 0$$

A nontrivial sol$^{\underline{n}}$ requires that the determinant vanish:

$$\omega^4 - \left[\omega_0^2 + (1+\alpha)\omega_1^2\right]\omega^2 + \omega_0^2\omega_1^2 = 0$$

$$\omega_\pm^2 = \tfrac{1}{2}\left[\omega_0^2 + (1+\alpha)\omega_1^2\right] \pm \tfrac{1}{2}\sqrt{\left[\omega_0^2 - (1+\alpha)\omega_1^2\right]^2 + 4\alpha\,\omega_0^2\omega_1^2}$$

There are two eigenvalues for ω^2, given by

$$\omega_\pm^2 = \frac{1}{2}\left[\omega_0^2+(1+\alpha)\omega_1^2\right] \pm \frac{1}{2}\sqrt{\left[\omega_0^2-(1+\alpha)\omega_1^2\right]^2 + 4\alpha\omega_0^2\omega_1^2}$$

The general sol$\underline{^n}$ is then

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} u_0^+ \\ \theta_0^+ \end{pmatrix} e^{-i\omega_+ t} + \begin{pmatrix} u_0^- \\ \theta_0^- \end{pmatrix} e^{-i\omega_- t}\right]$$

solution must be real

where

$$\begin{pmatrix} \omega_0^2-(1+\alpha)\omega_\pm^2 & -\alpha\omega_\pm^2 \\ -\omega_\pm^2 & \omega_1^2-\omega_\pm^2 \end{pmatrix}\begin{pmatrix} u_0^\pm \\ \theta_0^\pm \end{pmatrix} = 0$$

normal modes

$$\begin{pmatrix} u_0^\pm \\ \theta_0^\pm \end{pmatrix} e^{-i\omega_\pm t}$$

This fixes the ratios $\dfrac{u_0^\pm}{\theta_0^\pm} = \left(\dfrac{\omega_1^2}{\omega_\pm^2} - 1\right) \in \mathbb{R}$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \begin{matrix} u_2/u_1 \\ \vdots \\ u_n/u_1 \end{matrix}$$

Thus, we are free to choose $\theta_0^\pm$, which are two <u>complex</u> constants $\Rightarrow$ four real parameters.

We fix them via the initial conditions,

$$\begin{pmatrix} u(0) \\ \theta(0) \end{pmatrix} \text{ and } \begin{pmatrix} \dot{u}(0) \\ \dot{\theta}(0) \end{pmatrix} \Rightarrow \text{four real pieces of initial data}$$

Here we have used the fact that if $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{-i\omega t}$ is a sol$\underline{^n}$, then so is $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{+i\omega t}$. In this sense, we might speak of <u>four</u> eigenfrequencies $\{\omega_+, \omega_-, -\omega_+, -\omega_-\}$ of which two are positive and two are negative.

Scratch

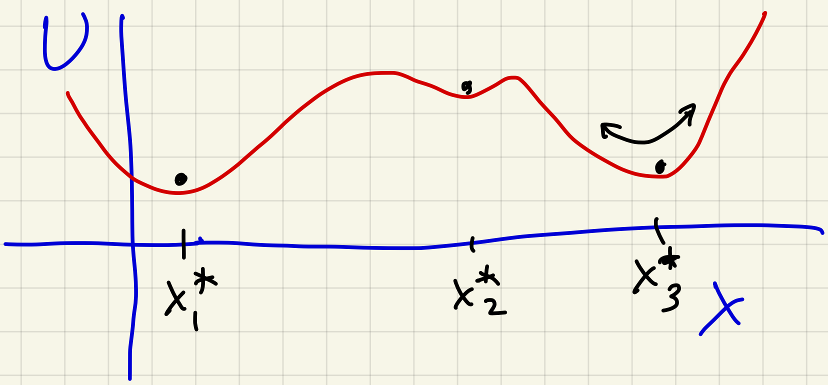
$$U(x) = \frac{1}{2}kx^2 + \frac{1}{4}bx^4$$

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - \frac{1}{4}bx^4$$

$$m\ddot{x} = -kx - \cancel{bx^3}$$

Eqbm @ $x = 0$, $\dot{x} = 0$

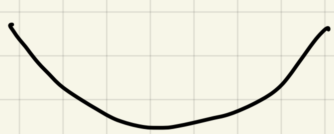


expand about $x = x_j^*$ (sol^{ns} to $U'(x^*) = 0$)

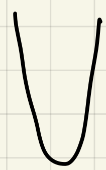
$$\Rightarrow \quad x = x_j^* + \delta x$$

$$\Rightarrow \quad m\ddot{\delta x} = -U''(x_j^*)\,\delta x$$

$$\omega_j^2 = \sqrt{U''(x_j^*)/m}$$



U'' small
w_j small



U'' big
w big

Virial Theorem

- formula describing time-averaged motion of a mechanical system

Define the virial $G(q,p) = \sum_\sigma q_\sigma p_\sigma$, for which

$$\frac{dG}{dt} = \sum_\sigma (\dot{q}_\sigma p_\sigma + \dot{p}_\sigma q_\sigma) = \sum_\sigma \left\{ \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} + q_\sigma \frac{\partial L}{\partial q_\sigma} \right\}$$

Suppose $T = \frac{1}{2} T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k=2$ in the generalized velocities, and that $\partial U / \partial \dot{q}_\sigma = 0$. Then

$$\sum_\sigma \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma \frac{\partial T}{\partial \dot{q}_\sigma} = 2T$$

Now consider the time average of $\dot{G}$ over $[0, \tau]$:

$$\left\langle \frac{dG}{dt} \right\rangle_\tau = \frac{1}{\tau} \int_0^\tau dt\, \frac{dG}{dt} = \frac{G(\tau) - G(0)}{\tau}$$

If G is bounded, then we have $\langle \dot{G} \rangle_\tau \to 0$ as $\tau \to \infty$. This is the case for any bounded motion, such as planetary orbits. In such cases,

$$2\langle T \rangle = -\left\langle \sum_{\sigma=1}^{n} q_\sigma F_\sigma \right\rangle \qquad n = \underset{\substack{\uparrow \\ \text{dim}^n \text{ of space}}}{d} \cdot N$$

$$= \left\langle \sum_{j=1}^{N} \vec{x}_j \cdot \frac{\partial}{\partial \vec{x}_j} U(\vec{x}_1, \ldots, \vec{x}_N) \right\rangle = k \langle U \rangle$$

if $U(\vec{x}_1, \ldots, \vec{x}_N)$ homogeneous of degree k in $\{x_j^\alpha\}$.

Scratch

Euler's thm for homogeneous functions:

$f(x_1, \dots, x_n)$ homogeneous of degree k if

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^k f(x_1, \dots, x_n)$$

examples

$$f(x,y) = x^5 + a x^4 y + b \frac{y^6}{x} \qquad k = 5$$

$$T(\dot{q}_1, \dots, \dot{q}_n) = \frac{1}{2} T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'} \qquad k = 2$$

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=1} f(\lambda x_1, \dots, \lambda x_n) = x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n}$$

$$= \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=1} \lambda^k f(x_1, \dots, x_n)$$

$$= k \lambda^{k-1} f(x_1, \dots, x_n) \Big|_{\lambda=1}$$

$$\therefore \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = k f$$

Check: $\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\left(x^5 + a x^4 y + b \frac{y^6}{x}\right) =$

$$x \cdot 5x^4 + x \cdot 4a x^3 y - x \cdot b \frac{y^6}{x^2} + y \cdot 0 + y \cdot a x^4 + y \cdot 6b \frac{y^5}{x}$$

$$= 5x^5 + 5a x^4 y + 5b \frac{y^6}{x} = 5\left(x^5 + a x^4 y + b \frac{y^6}{x}\right)$$

Since $T + U = E$ is conserved, we have

$$\langle T \rangle = \frac{kE}{k+2} \quad , \quad \langle U \rangle = \frac{2E}{k+2}$$

Application: Keplerian orbits, $k = -1$

$$\langle T \rangle = -E \, , \quad \langle U \rangle = 2E \quad ; \quad E < 0$$

Note then that a satellite losing energy due to frictional losses as it enters the atmosphere must <u>increase</u> its kinetic energy, i.e. it moves faster! (Think also about angular momentum conservation.)

<u>Noether's Theorem</u>

<u>Lecture 3 (Oct. 12)</u>

"To each independent, continuous one-parameter family of coordinate transformations which leave L invariant there corresponds an associated <u>conserved charge</u>."

(In fact, we only need require S is invariant. See §3.12.4 of the notes.)

<u>Proof</u>: Let $q_\sigma \to \bar{q}_\sigma(q, \zeta)$ be our one-parameter family of transformations with continuous parameter ζ, and with $\bar{q}_\sigma(q, \zeta = 0) = q_\sigma \;\forall\, \sigma$. Invariance of $L \Rightarrow$

$$\frac{d}{d\zeta}\bigg|_{\zeta=0} L(\bar{q}, \dot{\bar{q}}, t) = \frac{\partial L}{\partial \bar{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\bar{q}}_\sigma} \frac{\partial \dot{\bar{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \frac{\partial \bar{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{d}{dt}\left(\frac{\partial \bar{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = \frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\right)_{\zeta=0}}^{\text{(conserved charge)}} = 0$$

evaluate along motion of system

Thus, $\Lambda = \sum_{\sigma=1}^{n} \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\Big|_{\zeta=0} = \sum_{\sigma=1}^{n} p_\sigma \frac{\partial \bar{q}_\sigma}{\partial \zeta}\Big|_{\zeta=0}$ is conserved!

Examples

- $L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - U(y)$. Then let

$$\left.\begin{aligned} \bar{x}(x,y,\zeta) &= x + \zeta \\ \bar{y}(x,y,\zeta) &= y \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{\bar{x}} &= \dot{x} \\ \dot{\bar{y}} &= \dot{y} \end{aligned}$$

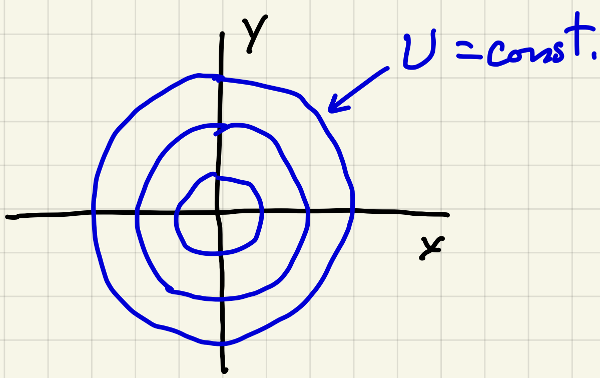
Clearly $\frac{d}{d\zeta} L(\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}) = 0$, and the associated conserved charge is

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{\partial \bar{x}}{\partial \zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \bar{y}}{\partial \zeta}\Big|_{\zeta=0} = \frac{\partial L}{\partial \dot{x}} = p_x$$

i.e. $p_x = m\dot{x}$ is a "constant of the motion".

- $L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - U(\sqrt{x^2+y^2})$

$$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) - U(\rho)$$



Define $\bar{\rho}(\rho, \phi, \zeta) = \rho$

$\bar{\phi}(\rho, \phi, \zeta) = \phi + \zeta$

Again $dL/d\zeta = 0$ and we have

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}} \frac{\partial \bar{\rho}}{\partial \zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{\partial \phi}{\partial \zeta}\Big|_{\zeta=0}$$

$$= p_\phi = m\rho^2 \dot{\phi} \quad \text{(angular momentum conserved)}$$

In Cartesian coordinates, this invariance is expressed as

$$\left.\begin{aligned}\bar{x}(\zeta) &= x\cos\zeta - y\sin\zeta \\ \bar{y}(\zeta) &= x\sin\zeta + y\cos\zeta\end{aligned}\right\} \quad \frac{\partial \bar{x}}{\partial \zeta} = -\bar{y}\,,\ \frac{\partial \bar{y}}{\partial \zeta} = +\bar{x}$$

$$\Lambda = \frac{\partial L}{\partial \dot{x}}\left.\frac{\partial \bar{x}}{\partial \zeta}\right|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}}\left.\frac{\partial \bar{y}}{\partial \zeta}\right|_{\zeta=0}$$

$$= m\dot{x}(-y) + m\dot{y}(+x) = m(x\dot{y} - y\dot{x})$$

$$= \hat{z}\cdot\vec{\rho}\times(m\dot{\vec{\rho}}) = m\rho^2\dot{\phi} = p_\phi$$

The Hamiltonian

Recall $H(q,p,t) = \sum_\sigma p_\sigma \dot{q}_\sigma - L$. We showed earlier that

$$dH = \sum_\sigma \left(\dot{q}_\sigma\, dp_\sigma - \dot{p}_\sigma\, dq_\sigma\right) - \frac{\partial L}{\partial t}\, dt$$

and therefore

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}\,, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \text{(Hamilton's eqns)}$$

as well as

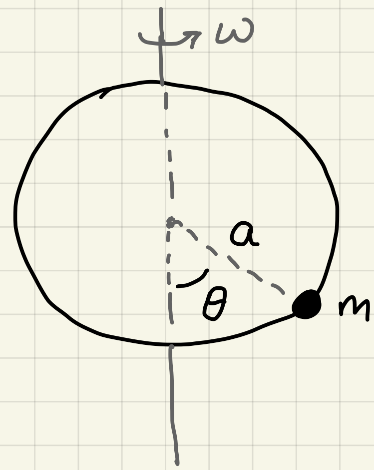
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- For $L = \frac{1}{2}m\dot{x}^2 - U(x)$, $p = m\dot{x}$ and $H = \frac{p^2}{2m} + U(x)$
- Read §§3.12.4, 3.13.2

If infinitesimal transformation $\delta t = A(q,t)\,\delta\zeta$, $\delta q_\sigma = B_\sigma(q,t)\,\delta\zeta$ leaves action $\int_{\bar{t}_1}^{\bar{t}_2} d\bar{t}\, L(\bar{q},\dot{\bar{q}},\bar{t})$ invariant, then

$$\Lambda = -H(q,p,t)\,A(q,t) + p_\sigma B_\sigma(q,t) \text{ is conserved.}$$

Example: Bead on a rotating hoop



Angular velocity about $\hat{z}$-axis is fixed to be ω. Thus

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2\right)$$
$$= \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right)$$

$$U = mga\,(1-\cos\theta)$$

Thus, $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$ and

$$H = \dot{\theta} p_\theta - L$$
$$= \frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2\theta + mga\,(1-\cos\theta)$$

NB: $H \neq T+U$ because T not homogeneous of degree 2 in $\dot{\theta}$.

Now we express $H(\theta, p_\theta)$:

$$H = \frac{p_\theta^2}{2ma^2} - \frac{1}{2} m a^2 \omega^2 \sin^2\theta + mga\,(1-\cos\theta)$$
$$= \frac{p_\theta^2}{2I} + U_{eff}(\theta)$$

where $I = ma^2$ = moment of inertia, and

$$U_{eff}(\theta) = -\frac{1}{2} m a^2 \omega^2 \sin^2\theta + mga\,(1-\cos\theta)$$

Hamilton's equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial U_{eff}}{\partial \theta}$$

Thus

$$I\ddot{\theta} = -\frac{\partial U_{eff}}{\partial \theta} \Rightarrow \ddot{\theta} = -\frac{1}{I}\frac{\partial U_{eff}}{\partial \theta} = -u'(\theta)$$

Define $\omega_0 \equiv (g/a)^{1/2}$ so

$$u(\theta) \equiv \frac{U_{eff}(\theta)}{I} = (1-\cos\theta)\,\omega_0^2 - \frac{1}{2}\sin^2\theta\,\omega^2$$

Equilibrium is achieved when $u'(\theta) = 0$:

$$u'(\theta) = \omega_0^2 \sin\theta - \omega^2 \sin\theta\cos\theta$$

with solutions

$$\theta^* = 0, \quad \theta^* = \pi, \quad \theta^* = +\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right), \quad \theta^* = -\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right)$$

$$\equiv \pm\theta_\omega \quad (\text{if } \omega^2 > \omega_0^2)$$

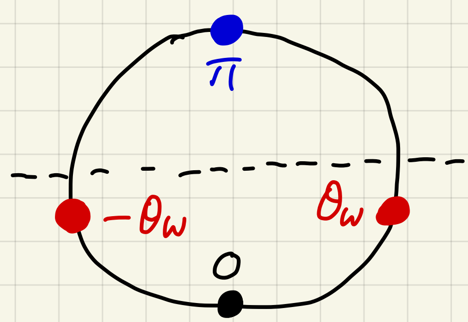
To assess stability, write $\theta = \theta^* + \delta\theta$, and

$$\delta\ddot{\theta} = -u''(\theta^*)\cdot\delta\theta$$

θ^* stable $\Rightarrow u''(\theta^*) > 0$

θ^* unstable $\Rightarrow u''(\theta^*) < 0$

$$\Omega_{osc} = \sqrt{u''(\theta^*)}$$



$$u''(\theta) = \omega_0^2\cos\theta - \omega^2\cos(2\theta)$$

$$= \begin{cases} \omega_0^2 - \omega^2 & \text{at } \theta^* = 0 \text{ — stable for } \omega^2 < \omega_0^2 \\ -\omega_0^2 - \omega^2 & \text{at } \theta^* = \pi \text{ — always unstable} \\ \omega^2 - \dfrac{\omega_0^4}{\omega^2} & \text{at } \theta^* = \pm\theta_\omega \ (\omega^2 > \omega_0^2) \text{ — stable for } \omega^2 > \omega_0^2 \end{cases}$$

Charged particle in EM fields

charge

Potential energy : $U(\vec{x},\dot{\vec{x}}) = q\,\phi(\vec{x},t) - \frac{q}{c}\,A(\vec{x},t)\cdot\dot{\vec{x}}$

Kinetic energy : $T(\dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2$ as usual

EM potentials : scalar $\phi(\vec{x},t)$ and vector $\vec{A}(\vec{x},t)$

EM fields :

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\ , \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Thus the Lagrangian is

$$L(\vec{x},\dot{\vec{x}},t) = \frac{1}{2}m\dot{\vec{x}}^2 - q\,\phi(\vec{x},t) + \frac{q}{c}\vec{A}(\vec{x},t)\cdot\dot{\vec{x}}$$

Canonical momentum : $\vec{p} = \dfrac{\partial L}{\partial\dot{\vec{x}}} = m\dot{\vec{x}} + \dfrac{q}{c}\vec{A}(\vec{x},t)$

NB : the <u>dynamical momentum</u> is $m\dot{\vec{x}} = \vec{p} - \frac{q}{c}\vec{A}$

Let's find the Hamiltonian $H(\vec{x},\vec{p},t)$:

$$\begin{aligned}
H(\vec{x},\vec{p},t) &= \vec{p}\cdot\dot{\vec{x}} - L \\
&= \left(m\dot{\vec{x}}^2 + \frac{q}{c}\vec{A}\cdot\dot{\vec{x}}\right) - \left(\frac{1}{2}m\dot{\vec{x}}^2 - q\phi + \frac{q}{c}\vec{A}\cdot\dot{\vec{x}}\right) \\
&= \frac{1}{2}m\dot{\vec{x}}^2 + q\phi
\end{aligned}$$

Thus, $\quad H(\vec{x},\vec{p},t) = \dfrac{1}{2m}\left(\vec{p} - \dfrac{q}{c}\vec{A}(\vec{x},t)\right)^2 + q\,\phi(\vec{x},t)$

If $\dfrac{\partial\phi}{\partial t} = 0$ and $\dfrac{\partial A}{\partial t} = 0$ then $\dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t} = 0$

and $H(\vec{x}(t),\vec{p}(t))$ is a constant of the motion.

Equations of motion: recall $L = \frac{1}{2} m \dot{\vec{x}}^2 - q\phi + \frac{q}{c} \vec{A} \cdot \dot{\vec{x}}$

EL eqns: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}^\alpha}\right) = \dfrac{\partial L}{\partial x^\alpha}$

$$\frac{d}{dt}\left(m\dot{x}^\alpha + \frac{q}{c}A^\alpha\right) = m\ddot{x}^\alpha + \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{q}{c}\frac{\partial A^\alpha}{\partial t}$$

$$\frac{\partial L}{\partial x^\alpha} = -q\frac{\partial \phi}{\partial x^\alpha} + \frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\dot{x}^\beta$$

Thus,

$$m\ddot{x}^\alpha + \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{q}{c}\frac{\partial A^\alpha}{\partial t} = -q\frac{\partial \phi}{\partial x^\alpha} + \frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\dot{x}^\beta$$

$$m\ddot{x}^\alpha = -q\frac{\partial \phi}{\partial x^\alpha} - \frac{q}{c}\frac{\partial A^\alpha}{\partial t} - \frac{q}{c}\left(\frac{\partial A^\alpha}{\partial x^\beta} - \frac{\partial A^\beta}{\partial x^\alpha}\right)\dot{x}^\beta$$

Now $B^\gamma = \epsilon_{\mu\nu\gamma}\,\partial_\mu A^\nu$, so

$$\begin{aligned} \epsilon_{\alpha\beta\gamma} B^\gamma &= \epsilon_{\alpha\beta\gamma}\,\epsilon_{\mu\nu\gamma}\,\partial_\mu A^\nu \\ &= \left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right)\partial_\mu A^\nu \\ &= \frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta} \end{aligned}$$

and we have

$$m\ddot{x}^\alpha = -q\frac{\partial \phi}{\partial x^\alpha} - \frac{q}{c}\frac{\partial A^\alpha}{\partial t} + \frac{q}{c}\epsilon_{\alpha\beta\gamma}\dot{x}^\beta B^\gamma$$

or in vector form,

$$\begin{aligned} m\ddot{\vec{x}} &= -q\vec{\nabla}\phi - \frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\dot{\vec{x}} \times \vec{B} \\ &= q\vec{E} + \frac{q}{c}\dot{\vec{x}} \times \vec{B} \qquad \text{(Lorentz force law)} \end{aligned}$$

Hamilton's equations of motion:

$$H(\vec{x}, \vec{p}, t) = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

- $$\dot{x}^\alpha = +\frac{\partial H}{\partial p^\alpha} = \frac{1}{m}\left(p^\alpha - \frac{q}{c}A^\alpha\right)$$
- $$\dot{p}^\alpha = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{m}\left(p^\beta - \frac{q}{c}A^\beta\right)\left(-\frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\right) - q\frac{\partial \phi}{\partial x^\alpha}$$

Thus,

$$m\dot{x}^\alpha = p^\alpha - \frac{q}{c}A^\alpha$$

$$\dot{p}^\alpha = \frac{q}{mc}\overbrace{\left(p^\beta - \frac{q}{c}A^\beta\right)}^{m\dot{x}^\beta}\frac{\partial A^\beta}{\partial x^\alpha} - q\frac{\partial \phi}{\partial x^\alpha}$$

Take the time derivative of the first equation:

$$m\ddot{x}^\alpha = \dot{p}^\alpha - \frac{q}{c}\frac{dA^\alpha}{dt}$$

$$= \left(\frac{q}{c}\dot{x}^\beta\frac{\partial A^\beta}{\partial x^\alpha} - q\frac{\partial \phi}{\partial x^\alpha}\right) - \left(\frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta - \frac{q}{c}\frac{\partial A^\alpha}{\partial t}\right)$$

$$= q\left(-\frac{\partial \phi}{\partial x^\alpha} - \frac{1}{c}\frac{\partial A^\alpha}{\partial t}\right) + \frac{q}{c}\dot{x}^\beta\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\right)$$

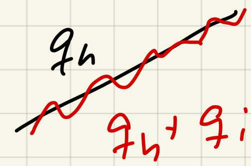
$$\Rightarrow \quad m\ddot{\vec{x}} = q\vec{E} + \frac{q}{c}\dot{\vec{x}} \times \vec{B}$$

Again, we obtain the Lorentz force law.

Fast Perturbations : Rapidly Oscillating Fields

Consider an oscillating force $F(t) = F_0 \sin \omega t$. Newton's 2^{nd} law then says $m\ddot{q} = F \sin \omega t$, the solution of which is

$$q(t) = \underbrace{a + bt}_{\substack{q_h(t) \\ \text{(homogeneous)}}} - \underbrace{\omega^{-2} F_0 \sin \omega t}_{\substack{q_i(t) \\ \text{(inhomogeneous)}}}$$



Note that $q_i(t) \propto \omega^{-2}$ is very small as $\omega \to \infty$.
Now consider the time-dependent Hamiltonian

$$H(q, p, t) = H^0(q, p, t) + \tilde{V}(q) \cos(\omega t)$$

The external force is then $F(q,t) = -\tilde{V}'(q) \cos(\omega t)$.
We now separate the motion $\{q(t), p(t)\}$ into slow components $\{Q(t), P(t)\}$ and fast components $\{\xi(t), \pi(t)\}$:

$$q(t) = Q(t) + \xi(t)$$
$$p(t) = P(t) + \pi(t)$$

$$H = H^0(Q + \xi, P + \pi) + \tilde{V}(Q + \xi) \cos(\omega t)$$

We further assume that ξ and π are small, and we expand in these quantities:

$$\dot{Q} + \dot{\xi} = \frac{\partial H}{\partial P} = \frac{\partial H^0}{\partial P} + \left(\frac{\partial^2 H^0}{\partial P^2} \pi + \frac{\partial^2 H^0}{\partial Q \partial P} \xi \right) + \frac{1}{2} \left(\frac{\partial^3 H^0}{\partial P^3} \pi^2 + 2 \frac{\partial^3 H^0}{\partial Q \partial P^2} \xi \pi + \frac{\partial^3 H^0}{\partial Q^2 \partial P} \xi^2 \right) + \ldots$$

$$H^0(Q + \xi, P + \pi) = H^0 + \frac{\partial H^0}{\partial Q} \xi + \frac{\partial H^0}{\partial P} \pi + \frac{1}{2} \frac{\partial^2 H^0}{\partial Q^2} \xi^2 + \ldots$$

$$\dot{P} + \dot{\pi} = -\frac{\partial H}{\partial Q} = -\frac{\partial H^0}{\partial Q} - \left(\frac{\partial^2 H^0}{\partial Q^2}\zeta + \frac{\partial^2 H^0}{\partial Q \partial P}\pi\right) - \frac{1}{2}\left(\frac{\partial^3 H^0}{\partial Q^3}\zeta^2 + 2\frac{\partial^3 H^0}{\partial Q^2 \partial P}\zeta\pi + \frac{\partial^3 H^0}{\partial Q \partial P^2}\pi^2\right) - \frac{\partial \tilde{V}}{\partial Q}\cos(\omega t) - \frac{\partial^2 \tilde{V}}{\partial Q}\zeta\cos(\omega t) + \dots$$

We can pick out from these equations the fast dynamics:

$$\dot{\zeta} = H^0_{QP}\zeta + H^0_{PP}\pi + \dots$$

$$\dot{\pi} = -H^0_{QQ}\zeta - H^0_{QP}\pi - \tilde{V}_Q\cos(\omega t) + \dots$$

where $H^0_{QQ} \equiv \dfrac{\partial^2 H^0}{\partial Q^2}$, $H^0_{QP} \equiv \dfrac{\partial^2 H^0}{\partial Q \partial P}$, etc.

We have ignored terms oscillating with frequencies near 0, 2ω, 3ω, etc. The slow dynamics are obtained by <u>averaging</u> over the fast dynamics, viz.

$$\dot{Q} = H^0_P + \frac{1}{2}H^0_{QQP}\langle\zeta^2\rangle + H^0_{QPP}\langle\zeta\pi\rangle + \frac{1}{2}H^0_{PPP}\langle\pi^2\rangle + \dots$$

$$\dot{P} = -H^0_Q - \frac{1}{2}H^0_{QQQ}\langle\zeta^2\rangle - H^0_{QQP}\langle\zeta\pi\rangle - \frac{1}{2}H^0_{QPP}\langle\pi^2\rangle - \tilde{V}_{QQ}\langle\zeta\cos(\omega t)\rangle + \dots$$

We solve the fast dynamics by writing $\tilde{V}_Q\cos(\omega t) = \mathrm{Re}\,\tilde{V}_Q e^{-i\omega t}$, $\zeta(t) = \mathrm{Re}\,\zeta_0 e^{-i\omega t}$, $\pi(t) = \mathrm{Re}\,\pi_0 e^{-i\omega t}$ and inverting

$$\begin{pmatrix} H^0_{QP} + i\omega & H^0_{PP} \\ -H^0_{QQ} & -H^0_{QP} + i\omega \end{pmatrix}\begin{pmatrix}\zeta_0 \\ \pi_0\end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{V}_Q \end{pmatrix}$$

We obtain

$$\xi(t) = \omega^{-2} H^0_{PP} \tilde{V}_Q \cos\omega t + O(\omega^{-4})$$

$$\pi(t) = -\omega^{-2} H^0_{QP} \tilde{V}_Q \cos\omega t - \omega^{-1} \tilde{V}_Q \sin\omega t + O(\omega^{-3})$$

Now we average, using $\langle \cos^2\omega t \rangle = \langle \sin^2\omega t \rangle = \frac{1}{2}$ and $\langle \cos\omega t \sin\omega t \rangle = 0$. We obtain

$$\langle \xi^2(t) \rangle = \frac{1}{2}\omega^{-4} \left(H^0_{PP} \tilde{V}_Q\right)^2 + \dots \quad \langle \xi(t)\pi(t) \rangle = \frac{1}{2}\omega^{-4} H^0_{PP} H^0_{QQ} \tilde{V}_Q^2 + \dots$$

$$\langle \pi^2(t) \rangle = \frac{1}{2}\omega^{-2} \tilde{V}_Q^2 + \frac{1}{2}\omega^{-4} \left(H^0_{QP} \tilde{V}_Q\right)^2 + \dots$$

$$\langle \xi(t) \cos\omega t \rangle = \frac{1}{2}\omega^{-2} H^0_{PP} \tilde{V}_Q + \dots$$

Plugging into the slow equations for $\dot{Q}$ and $\dot{P}$, we have

$$\dot{Q} = H^0_P + \frac{1}{4}\omega^{-2} H^0_{PPP} \tilde{V}_Q^2 + \dots$$

$$\dot{P} = -H^0_Q - \frac{1}{4}\omega^{-2} H^0_{QPP} \tilde{V}_Q^2 - \frac{1}{2}\omega^{-2} H^0_{PP} \tilde{V}_Q \tilde{V}_{QQ} + \dots$$

which may be written as

$$\dot{Q} = \frac{\partial K}{\partial P} \quad , \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

where the effective Hamiltonian is

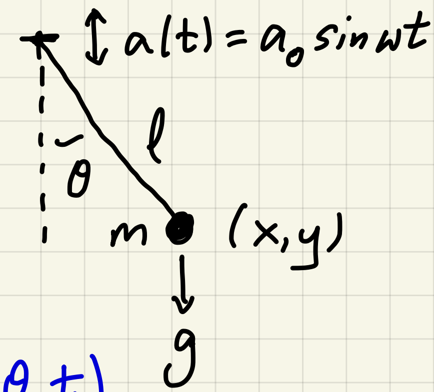
$$K(Q,P) = H^0(Q,P) + \frac{1}{4\omega^2} \frac{\partial^2 H^0}{\partial P^2} \left(\frac{\partial \tilde{V}}{\partial Q}\right)^2 + O(\omega^{-4})$$

Example: pendulum with oscillating support

Coordinates of mass m:

$$x = \ell \sin\theta$$

$$y = a(t) - \ell\cos\theta$$



The Lagrangian is

$$L = \frac{1}{2} m \ell^2 \dot{\theta}^2 + m(g+\ddot{a})\,\ell\cos\theta + \frac{d}{dt} G(\theta,t)$$

we may ignore this

From this, we obtain the Hamiltonian

$$H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta - m\ell\ddot{a}\cos\theta$$

With $a(t) = a_0 \sin\omega t$, the perturbing potential is

$$\tilde{V}(\theta) = m\ell a_0 \omega^2 \cos\theta$$

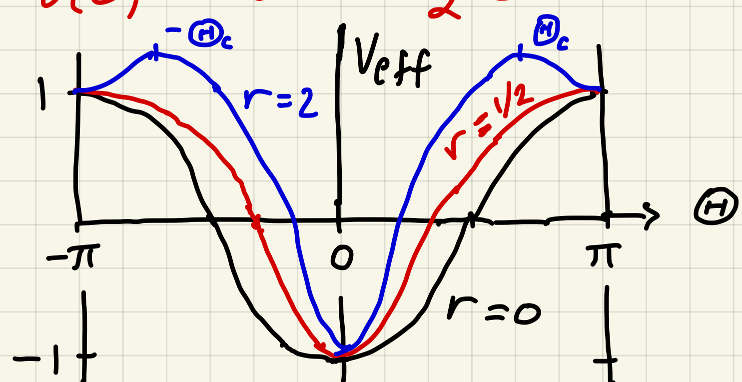
We write $\theta = \Theta + \zeta$, $p_\theta = L + \pi$ and compute $K(\Theta, L)$:

$$K(\Theta, L) = \frac{L^2}{2m\ell^2} - mg\ell\cos\Theta + \frac{1}{4} m a_0^2 \omega^2 \sin^2\Theta$$

Thus, the effective potential is

$$V_{eff}(\Theta) = mg\ell\, v(\Theta) \quad , \quad v(\Theta) = -\cos\Theta + \frac{r}{2}\sin^2\Theta$$

with $r = \omega^2 a_0^2 / 2g\ell$.

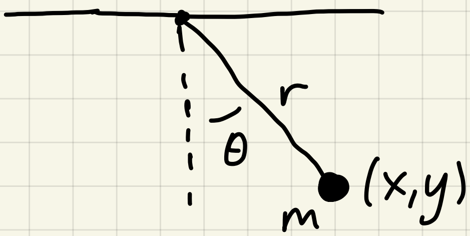


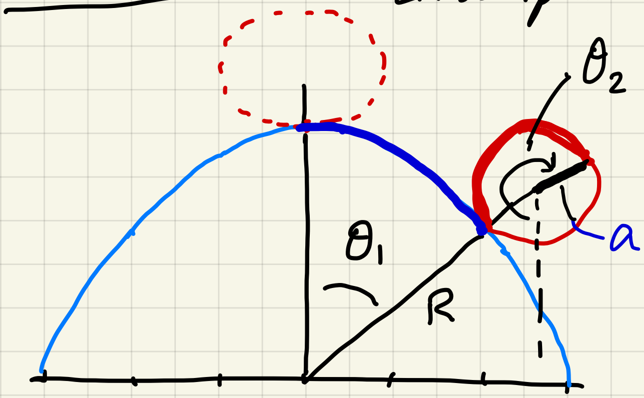
$r < 1$: $\Theta = 0$ stable, $\Theta = \pi$ unstable

$r > 1$: $\Theta = 0, \pi$ stable, $\pm\Theta_c$ unstable

Lecture 4 (Oct. 14)

Today's lecture is about constraints. Examples:





constraint: $r = \ell$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$= \frac{1}{2}m\ell^2\dot{\theta}^2$$

"no slip" condition: $R\theta_1 = a(\theta_2 - \theta_1)$

$$\Rightarrow \theta_2 = \left(1 + \frac{R}{a}\right)\theta_1$$

In these cases the constraint equations may easily be solved exactly and the number of generalized coordinates thereby reduced: $\{r, \theta\} \to \{\theta\}$, $\{\theta_1, \theta_2\} \to \{\theta_1\}$

In other cases the constraint equations are nonlinear or differential and they can't by solved to eliminate redundant degrees of freedom.

Constrained extremization of functions: Lagrange multipliers

Task: extremize $F(x_1, \ldots, x_n)$ subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ with $j \in \{1, \ldots, k\}$. We want to find solutions $\vec{x}^*$ such that $\vec{\nabla}F(\vec{x}^*)$ is linearly dependent on the k vectors $\{\vec{\nabla}G_j(\vec{x}^*)\}$.

That is,

$$\text{①} \qquad \vec{\nabla} F + \sum_{j=1}^{k} \lambda_j \vec{\nabla} G_j = 0 \qquad \text{(n equations)}$$

where the $\{\lambda_j\}$ are all real. This means that any displacement $d\vec{x}$ relative to $\vec{x}^*$ would result in a violation of one or more of the constraint equations. Eqn. ① provides n equations for the $(n+k)$ quantities $\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\}$. The remaining k equations are the constraints $G_j(x_1, \ldots, x_n) = 0$.

Equivalently, construct the function

$$F^*(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j G_j(x_1, \ldots, x_n)$$

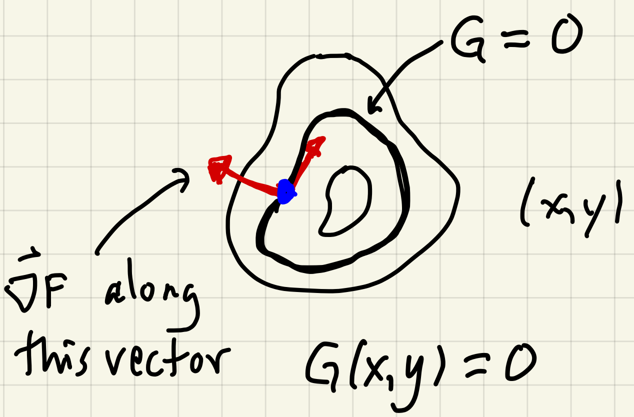
and freely extremize F^* over all its variables:

$$dF^* = \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j \equiv 0$$

This results in the $(n+k)$ equations

$$\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, n)$$

$$G_j = 0 \qquad (j = 1, \ldots, k)$$



usually we set $\vec{\nabla} F = 0 \Rightarrow$ n eqns in n unknowns $\{x_1, \ldots, x_n\}$ but in general these sol$^{\underline{ns}}$ will not satisfy $G_j(\vec{x}) = 0 \;\; \forall j$

Example

Extremize the volume of a cylinder of height h and radius a subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \qquad (b, \ell \text{ fixed})$$

Thus, we define

$$V^*(a,h;\lambda) = \pi a^2 h + \lambda\left(2\pi a + \frac{h^2}{b} - \ell\right)$$

$V = \pi a^2 h$

$$① \quad \frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0$$

$$② \quad \frac{\partial V^*}{\partial h} = \pi a^2 + \frac{2}{b}\lambda h = 0$$

$$③ \quad \frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0$$

Thus ① gives $\lambda = -ah$, whence ② yields

$$\pi a^2 - \frac{2}{b} a h^2 = 0 \Rightarrow a = \frac{2}{\pi b} h^2$$

Finally, ③ gives

$$\frac{4}{b} h^2 + \frac{h^2}{b} = \ell \Rightarrow h = \sqrt{\frac{b\ell}{5}}$$

and therefore $a = \dfrac{2\ell}{5\pi}$ and $\lambda = -\dfrac{2}{5^{3/2}\pi} b^{1/2} \ell^{3/2}$

Thus, the extremal volume is

$$V^* = \pi a^2 h = \frac{4}{5^{5/2}\pi} b^{1/2} \ell^{5/2}$$

Constraints and variational calculus

Consider the following class of functionals:

$$F[\vec{y}(x)] = \int_{x_L}^{x_R} dx \; L(\vec{y}, \vec{y}', x)$$

Here $\vec{y}(x)$ may stand for a vector of functions $\{y_\sigma(x)\}$. We consider two classes of constraints:

① Integral constraints: these are of the form

$$\int_{x_L}^{x_R} dx \; N_j(\vec{y}, \vec{y}', x) = C_j \quad , \quad j \in \{1, \ldots, k\}$$

② Holonomic constraints: these take the form

$$G_j(\vec{y}, x) = 0 \quad \text{on} \quad x \in [x_L, x_R]$$

Integral constraints

Here we introduce a separate multiplier λ_j for each integral constraint. That is, we extremize the extended functional

$$F^*[\vec{y}(x); \vec{\lambda}] = \int_{x_L}^{x_R} dx \; L(\vec{y}, \vec{y}', x) + \sum_{j=1}^{k} \lambda_j \int_{x_L}^{x_R} dx \; N_j(\vec{y}, \vec{y}', x)$$

$$\equiv \int_{x_L}^{x_R} dx \; L^*(\vec{y}, \vec{y}', x; \vec{\lambda})$$

$$L^*(\vec{y}, \vec{y}', x; \vec{\lambda}) \equiv L(\vec{y}, \vec{y}', x) + \sum_j \lambda_j N_j(\vec{y}, \vec{y}', x)$$

This results in the following set of equations:

$$\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{j=1}^{k} \lambda_j \left\{\frac{\partial N_j}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial N_j}{\partial y'_\sigma}\right)\right\} = 0 \qquad \sigma \in \{1, \dots, n\}$$

$$\int_{x_L}^{x_R} dx\, N_j(\vec{y}, \vec{y}\,', x) = C_j \qquad j \in \{1, \dots, k\}$$

Note that n of these are second order ODEs.
We have assumed that $\vec{y}(x_L)$ and $\vec{y}(x_R)$ are fixed.

Holonomic constraints

Now extremize

$$F[\vec{y}(x)] = \int_{x_L}^{x_R} dx\; L(\vec{y}, \vec{y}\,', x) \quad , \quad \vec{y}(x) = \{y_1(x), \dots, y_n(x)\}$$

subject to the k conditions

$$G_j(\vec{y}(x), x) = 0 \quad , \quad j \in \{1, \dots, k\}$$

Again, construct the extended functional

$$F^*[\vec{y}(x), \vec{\lambda}(x)] = \int_{x_L}^{x_R} dx\, \Big\{ \underbrace{L(\vec{y}, \vec{y}\,', x) + \sum_{j=1}^{k} \lambda_j\, G_j(\vec{y}, x)}_{L^*(\vec{y}, \vec{y}\,', x;\, \vec{\lambda})} \Big\}$$

and freely extremize wrt the $(n+k)$ functions

$$\{y_1(x), \dots, y_n(x);\ \lambda_1(x), \dots, \lambda_k(x)\}$$

This results in n second order ODEs plus k algebraic constraints:

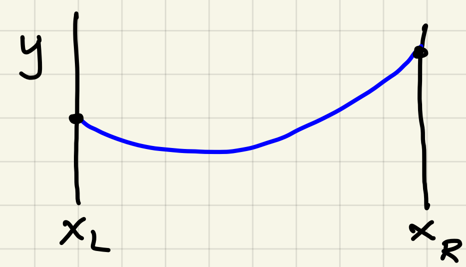
$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial y_\sigma} \quad , \quad \sigma \in \{1, \dots, n\}$$

$$G_j = 0 \quad , \quad j \in \{1, \dots, k\}$$

Each of these equations holds for all $x \in [x_L, x_R]$.

Examples

① hanging rope of fixed length



The potential energy functional is

$$U[y(x)] = \rho g \int_{x_L}^{x_R} ds \, y \quad ; \quad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}\, dx$$

The length is

$$C[y(x)] = \int_{x_L}^{x_R} ds = \int_{x_L}^{x_R} dx \sqrt{1 + (y')^2}$$

Thus we form

$$U^*[y(x), \lambda] = \int_{x_L}^{x_R} dx \, \overbrace{(\rho g y + \lambda) \sqrt{1 + (y')^2}}^{L^*(y, y', x; \lambda)}$$

Since $\partial L^* / \partial x = 0$, the "Hamiltonian" is conserved:

$$H = y' \frac{\partial L^*}{\partial y'} - L^* = - \frac{\rho g y + \lambda}{\sqrt{1 + (y')^2}} = \text{constant}$$

Thus,

$$\frac{dy}{dx} = \pm \frac{1}{H}\sqrt{(\rho g y+\lambda)^2 - H^2}$$

Integrate to get

$$y(x) = -\frac{\lambda}{\rho g} + \frac{H}{\rho g}\cosh\left(\frac{\rho g}{H}(x-a)\right)$$

where a is a constant of integration.
The constants λ, H, and a are fixed by the conditions $y(x_L) = y_L$, $y(x_R) = y_R$, and by the fixed length constraint $\int_{x_L}^{x_R} dx\,\sqrt{1+(y')^2} = C$.

Constraints in Lagrangian Mechanics

We write our system of constraints in differential form:

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\,dq_\sigma + h_j(q,t)\,dt = 0 \quad , \quad \begin{array}{l}\sigma\in\{1,\ldots,n\}\\ j\in\{1,\ldots,k\}\end{array}$$

where $q = \{q_1,\ldots,q_n\}$. If the partial derivatives satisfy the conditions

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma}$$

then the k differentials may be integrated to yield k holonomic constraints $G_j(q,t)=0$, with

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad \text{and} \quad h_j = \frac{\partial G_j}{\partial t}$$

One *may* then be able to eliminate redundant degrees of freedom directly.

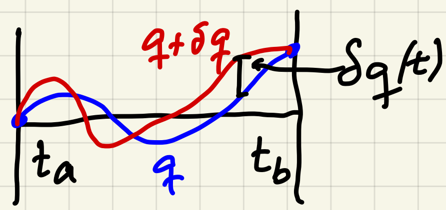
The action functional is

$$S[q(t)] = \int_{t_a}^{t_b} dt\, L(q, \dot{q}, t) \quad ; \quad \delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$$

Its variation is

$$\delta S = \int_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \right\} \delta q_\sigma(t)$$

Since the $\{\delta q_\sigma(t)\}$ are no longer all independent, we cannot infer that the term in curly brackets vanishes for each σ. What are the constraints on the $\{\delta q_\sigma(t)\}$? Since they occur in *zero time* we call them "virtual displacements", and setting $\delta t = 0$ we have the conditions



$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \delta q_\sigma(t) = 0$$

Now we may relax the constraint by introducing k Lagrange multipliers $\lambda_j(t)$ at each time, and write

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0$$

We may set each of the bracketed terms to zero.

Thus, we obtain a set of $(n+k)$ equations:

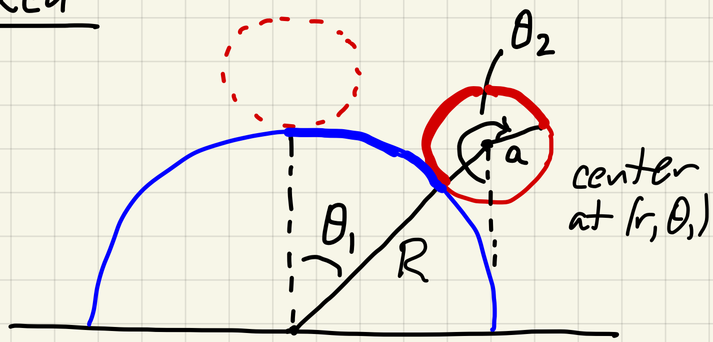
$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}_{\dot{p}_\sigma} - \underbrace{\frac{\partial L}{\partial q_\sigma}}_{F_\sigma} = \underbrace{\sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t)}_{Q_\sigma \,=\, \text{force of constraint}} \quad , \quad \sigma \in \{1, \ldots, n\}$$

and

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \dot{q}_\sigma + h_j(q,t) = 0 \quad , \quad j \in \{1, \ldots, k\}$$

- Please read § 3.16.8 on constraints and conservation laws!

Example: Two cylinders, one fixed



Constraints:

1) contact : $r = R + a$
2) no slip : $R\theta_1 = a(\theta_2 - \theta_1)$

$$g_{j\sigma} = \begin{array}{c} \\ \\ \\ \end{array}\!\!\begin{array}{l} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & R+a & -a \end{array}\right) \begin{array}{l} \leftarrow \text{contact } (j=1) \\ \leftarrow \text{no slip } (j=2) \end{array} \\ \begin{array}{ccc} \quad\;\uparrow & \;\uparrow & \;\;\uparrow \\ \quad\; r & \;\theta_1 & \;\;\theta_2 \end{array} \end{array} \quad , \quad h_j = 0$$

Lagrangian:

$$L = T - U = \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\theta}_1^2) + \frac{1}{2} I \dot{\theta}_2^2 - Mgr\cos\theta_1$$

M: mass of rolling cylinder

I: rotational inertia of rolling cylinder

$$\underset{\substack{\parallel \\ 1}}{g_{1r}}\, \dot{r} + \underbrace{g_{1\theta_1}\dot{\theta}_1 + g_{1\theta_2}\dot{\theta}_2 + h_1}_{\text{all vanish}} = 0$$

i.e. $\dot{r} = 0 \longrightarrow r = R + a$

$n = 3$ equations of motion:

$$r: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = M\ddot r - Mr\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 = Q_r$$

$$\theta_1: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = Mr^2\ddot\theta_1 + 2Mr\dot r\dot\theta_1 - Mgr\sin\theta_1 = \lambda_2(R+a) = Q_{\theta_1}$$

$$\theta_2: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = I\ddot\theta_2 = -\lambda_2 a = Q_{\theta_2}$$

$$\cancel{\lambda_1 g_{1\theta_1}} + \underbrace{\lambda_2 g_{2\theta_1}}_{R+a}$$

$k=2$ equations of constraint:

$$\left.\begin{aligned} &\text{contact}: \quad \dot r = 0 \\ &\text{no slip}: \quad R\dot\theta_1 - a(\dot\theta_2 - \dot\theta_1) = 0 \end{aligned}\right\} \xrightarrow[\text{integrate}]{} \left\{\begin{aligned} &r = R+a \\ &\theta_2 = \left(1+\frac{R}{a}\right)\theta_1 \end{aligned}\right.$$

Now we have 5 equations in 5 unknowns $\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\}$. We've already integrated the constraints so we may eliminate r and θ_2, yielding

$$-M(R+a)\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1$$
$$M(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = \lambda_2(R+a)$$
$$I\left(1+\frac{R}{a}\right)\ddot\theta_1 = -\lambda_2 a$$

We can ~~read~~ now read off the result $\lambda_2 = -\frac{I}{a^2}(R+a)\ddot\theta_1$

Substituting this into the second of these equations gives

$$\left(M + \frac{I}{a^2}\right)(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = 0$$

Multiply this by $\dot\theta_1$ and then integrate to obtain...

$$\dot\theta_1\ddot\theta_1 = \frac{d}{dt}\left(\frac{1}{2}\dot\theta_1^2\right), \quad \dot\theta_1\sin\theta_1 = \frac{d}{dt}(-\cos\theta_1)$$

$$\frac{1}{2} M\left(1+\frac{I}{Ma^2}\right)\dot{\theta}_1^2 + \frac{Mg}{R+a}\cos\theta_1 = \frac{Mg}{R+a}\cos\theta_1^\circ$$

where we assume the upper cylinder is released from rest (i.e. $\dot{\theta}_1^\circ = 0$) at $\theta_1 = \theta_1^\circ$. Finally, we may use this to express $\dot{\theta}_1^2$ in terms of θ_1, and stick the result into the first equation, resulting in

$$Q_r = \frac{Mg}{1+\alpha}\left\{(3+\alpha)\cos\theta_1 - 2\cos\theta_1^\circ\right\}$$

where $\alpha = I/Ma^2$ is dimensionless, with $\alpha \in [0,1]$

$\alpha = 0$: all mass of rolling cylinder at its center

$\alpha = 1$: all mass of rolling cylinder at its edge

When Q_r vanishes, the cylinders lose contact (the normal force of the bottom cylinder on the top one can only be positive). This happens for

$$\theta_1^* = \cos^{-1}\left(\frac{2\cos\theta_1^\circ}{3+\alpha}\right) = \text{detachment angle}$$

Note θ_1^* is an increasing function of α, i.e. larger rotational inertia I delays detachment. Physics here is that kinetic energy gain is split between translational and rotational motions.

Note also : $\dot{\theta}_1 = \left(\frac{2g}{R+a}\right)^{1/2}(\cos\theta_1^\circ - \cos\theta_1)$

$$dt = \left(\frac{R+a}{2g}\right)^{1/2}\frac{d\theta_1}{\sqrt{\cos\theta_1^\circ - \cos\theta_1}} \rightarrow \text{integrate for } \theta_1(t)$$

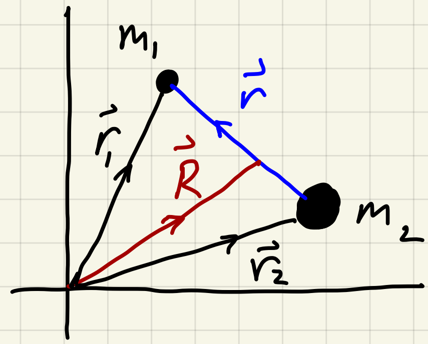
Lecture 5 (Oct. 19)

Two body central force problem:

$$L = T - U = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

① Change to CM and relative coordinates:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad , \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$



Invert to obtain:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r} \quad , \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2} \vec{r}$$

Substitute in $L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$:

$$L(\vec{R}, \dot{\vec{R}}, \vec{r}, \dot{\vec{r}}) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - U(r)$$

{ decoupled CM and relative motion!

where $M = m_1 + m_2$ (total mass)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{(reduced mass)}$$

NB: $m_1 \ll m_2 \Rightarrow \mu = m_1 - \frac{m_1^2}{m_2} + \cdots$

$m_1 = m_2 = m \Rightarrow \mu = \frac{1}{2} m$

② Integrate CM eqns of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{R}}} = \frac{\partial L}{\partial \vec{R}} \Rightarrow M\ddot{\vec{R}} = 0 \quad , \quad \vec{P} = \frac{\partial L}{\partial \dot{\vec{R}}} = M\dot{\vec{R}} = \text{const}$$

$$\vec{R}(t) = \vec{R}(0) + \dot{\vec{R}}(0)\, t$$

③ Relative coordinate problem

$$L_{rel} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$$

Continuous rotational symmetry $\Rightarrow$

$$\vec{l} = \vec{r}\times\vec{p} = \mu\vec{r}\times\dot{\vec{r}} \quad \text{conserved}$$

Since $\vec{r}\cdot\vec{l} = 0$, all motion $\vec{r}(t)$ is confined to the plane perpendicular to $\vec{l}$. Choose 2D polar coordinates (r,ϕ) in this plane. The relative coordinate Lagrangian is then

$$L_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Since the coordinate ϕ is cyclic, the angular momentum $l = \mu r^2\dot{\phi}$ is conserved. And since $\partial L/\partial t = 0$, $H = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L$ is conserved.
Find

$$H = E = T + U = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$
$$= \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

where

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

We can now solve to obtain radial motion $r(t)$, and then obtain ϕ by integrating $\dot{\phi} = l/\mu r^2(t)$.

Specifically, from $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$, we have

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)} \Rightarrow$$

$$\left.\begin{array}{l} + \text{ for } dr>0 \\ - \text{ for } dr<0 \end{array}\right\} \quad dt = \pm\sqrt{\frac{\mu}{2}}\,\frac{dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}}$$

Integrate to get $t(r)$. In principle this is possible. This introduces a constant of integration $r_0 = r(t=0)$

Next, with $r(t)$ in hand, integrate

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\ell^2}{2\mu r^2} \Rightarrow d\phi = \frac{\ell}{\mu}\frac{dt}{r^2(t)}$$

to get $\phi(t)$. This introduces a second constant, $\phi_0 = \phi(t=0)$.

Now we have the complete motion of the system, $\{r(t), \phi(t)\}$ with four constants of integration: E, ℓ, r_0, ϕ_0.

Recall that the three-dimensional motion is confined to a plane perpendicular to $\vec{\ell}$, so its direction $\hat{\ell}$ accounts for two additional constants of integration.

Overall, there are 12 such constants:

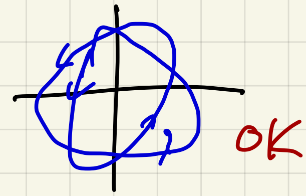
$$\vec{R}(0)\ (\times 3),\ \dot{\vec{R}}(0)\ (\times 3),\ E_{rel},\ \vec{\ell}\ (\times 3),\ r_0,\ \phi_0$$

which is expected given two coupled second order equations of motion for the six quantities $\vec{r}_1, \vec{r}_2$.

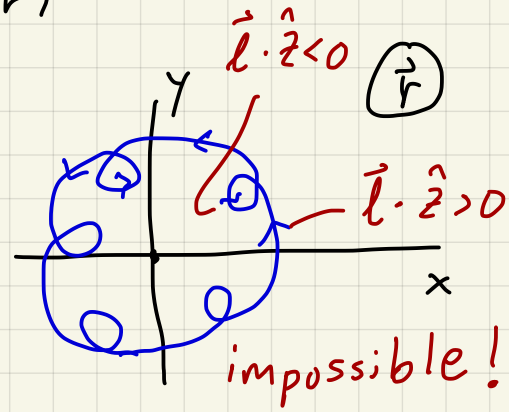
- Geometric equation of the orbit

The 2nd order ODE for $r(t)$ is

$$\mu \ddot{r} = -\frac{\partial U_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} - U'(r)$$



Since $\ell = \mu r^2 \frac{d\phi}{dt}$ is conserved,



$$\frac{d}{dt} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$$

Therefore

$$\mu\left(\frac{\ell}{\mu r^2}\frac{d}{d\phi}\right)\left(\frac{\ell}{\mu r^2}\frac{d}{d\phi}\right) r = \frac{\ell^2}{\mu r^3} - U'(r)$$

$$\frac{\ell^2}{\mu r^4}\frac{d^2 r}{d\phi^2} - \frac{2\ell^2}{\mu r^5}\left(\frac{dr}{d\phi}\right)^2 = \frac{\ell^2}{\mu r^3} - U'(r)$$

$$\Rightarrow \quad \frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = r + \frac{\mu r^4}{\ell^2} F(r)$$

where $F(r) = -U'(r)$ is the radial force. Using energy conservation, we can write

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$$

$$= \frac{\ell^2}{2\mu r^2}\left(\frac{dr}{d\phi}\right)^2 + U_{eff}(r)$$

to obtain

$$d\phi = \pm \frac{\ell}{\sqrt{2\mu}} \frac{dr}{r^2\sqrt{E - U_{eff}(r)}}$$

It is sometimes convenient to write the equation

$$r'' - \frac{2}{r}(r')^2 = \frac{\mu r^4}{\ell^2} F(r) + r \qquad \left(r' = \frac{dr}{d\phi} \text{ etc.}\right)$$

in terms of the variable $s \equiv 1/r$. Then

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} F(s^{-1})$$

Suppose for example that $r(\phi) = r_0 e^{\kappa\phi}$, i.e. a logarithmic spiral. Then $s(\phi) = s_0 e^{-\kappa\phi}$, and

$$(\kappa^2+1)\, s = -\frac{\mu}{\ell^2 s^2} F(s^{-1})$$

$$F(s^{-1}) = -\frac{\ell^2}{\mu}(\kappa^2+1) s^3 \iff F(r) = -\frac{\ell^2}{\mu}(\kappa^2+1)\frac{1}{r^3}$$

This corresponds to a potential $U(r) = -\frac{C}{r^3}$ $(C>0)$ with

$$\kappa = \left(\frac{\mu C}{\ell^2} - 1\right)^{1/2}$$

Thus, the general shape of the orbit for $\ell^2 > \mu C > 0$ is

$a, b \in \mathbb{R}$
2 real const.

$$r(\phi) = \frac{1}{a e^{\kappa\phi} + b e^{-\kappa\phi}}$$

spiral orbit for $a = 0$ or $b = 0$

When $\mu C > \ell^2 > 0$, let $\bar{\kappa} = \left(1 - \frac{\mu C}{\ell^2}\right)^{1/2}$, in which case

$A \in \mathbb{C}$
1 complex const.

$$r(\phi) = \frac{1}{A e^{i\bar{\kappa}\phi} + A^* e^{-i\bar{\kappa}\phi}}$$

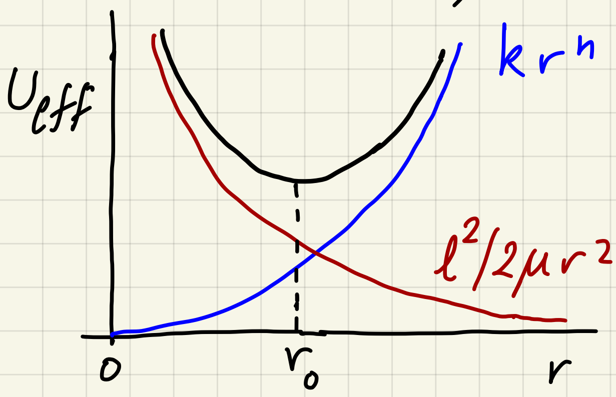
orbit is unbound, with $r(\phi) = \infty$ when

$\bar{\kappa}\phi = (n+\frac{1}{2})\pi - \arg A$

- Almost circular orbits

A circular orbit $r(t) = r_0$ requires $U'_{eff}(r_0) = 0$.

For a homogeneous attractive potential $U(r) = kr^n$ with $k>0$, $n>0$, we have:

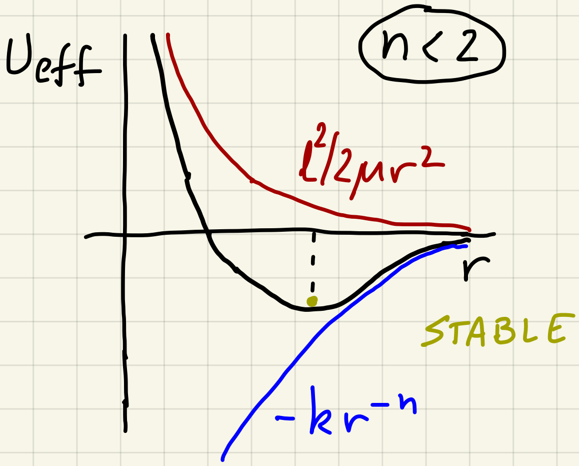


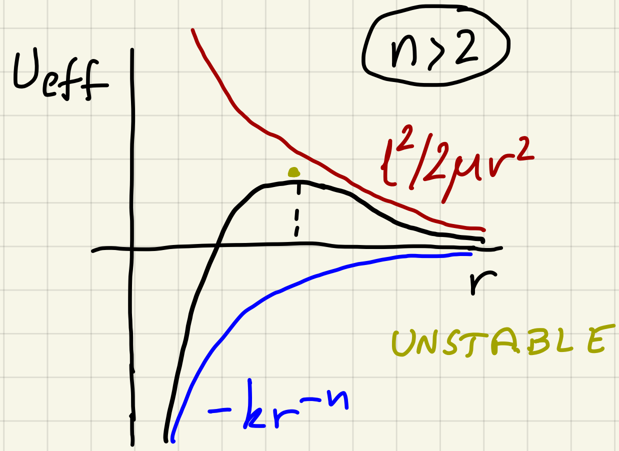
$$U_{eff} = \frac{\ell^2}{2\mu r^2} + kr^n$$

$$U'_{eff} = -\frac{\ell^2}{\mu r^3} + nkr^{n-1} \equiv 0$$

$$r_0 = \left(\ell^2/n\mu k\right)^{1/(n+2)}$$

For $U(r) = -kr^{-n}$ with $k>0$, $n>0$, we have





$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r^n}, \quad U'_{eff} = -\frac{\ell^2}{\mu r^3} + \frac{nk}{r^{n+1}}$$

$$r_0 = \left(\frac{n\mu k}{\ell^2}\right)^{1/(n-2)}$$

If we write $r = r_0 + \eta$ with $|\eta| \ll r_0$, then

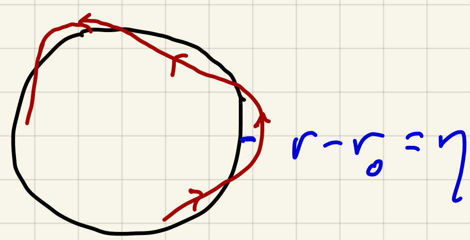
$$\mu\ddot{\eta} = -U''_{eff}(r_0)\,\eta \Rightarrow \ddot{\eta} = -\omega^2\eta \text{ with } \omega^2 = \frac{U''_{eff}(r_0)}{\mu}$$

We can also use

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2} F(r) + r$$

and linearize in η with $r = r_0 + \eta$. This yields

$$\eta'' = \underbrace{\left[\frac{\mu r_0^4}{\ell^2} F(r_0) + r_0\right]}_{= -\frac{\mu r_0^4}{\ell^2} U'_{eff}(r_0) = 0} + \left(\underbrace{\frac{4\mu r_0^3}{\ell^2} F(r_0)}_{= 4} + \frac{\mu r_0^4}{\ell^2} F'(r_0) - 1\right)\eta + \mathcal{O}(\eta^2)$$



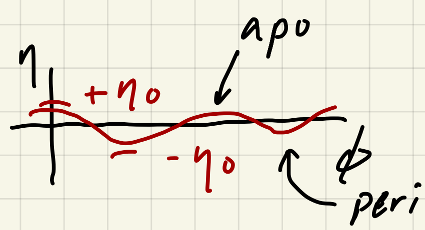
and hence

$$\eta''(\phi) = -\beta^2 \eta(\phi)$$

with
$$\beta^2 = 3 - \frac{\mu r_0^4}{\ell^2} F'(r_0) = 3 - \left.\frac{d\ln F}{d\ln r}\right|_{r_0}$$

The solution is

$$\eta(\phi) = \eta_0 \cos\left[\beta(\phi - \delta_0)\right]$$



where η_0 and ϕ_0 set the initial conditions. Note that $\eta(\phi) = +\eta_0$ for $\phi = \phi_n \equiv 2\pi\beta^{-1}n + \delta_0$. This is called apoapsis (farthest point). The condition for periapsis (closest point) occurs for $\phi = \phi_n + \pi\beta^{-1}$. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi(\beta^{-1} - 1)$$

is the angle by which the apsides (i.e. periapsis and apoapsis) precess during each cycle. If $\beta > 1$, the apsides advance, (come sooner) while if $\beta < 1$ the apsides recede (later).

If $\beta = \frac{p}{q} \in \mathbb{Q}$ is a rational number, then the orbit is <u>closed</u> and will retrace itself every q revolutions.

- Example: $U(r) = -k r^{-\alpha}$ with $k>0$, $n>0$. Then

$$U'_{eff}(r) = -\frac{\ell^2}{\mu r^3} + \frac{\alpha k}{r^{\alpha+1}} \Rightarrow r_0 = \left(\frac{\ell^2}{\alpha \mu k}\right)^{1/(2-\alpha)}$$

We then have $\beta^2 = 3 - \left.\frac{d\ln F}{d\ln r}\right|_{r_0} = 2-\alpha$. These orbits are stable only for $\alpha<2$. For $\alpha>2$ the circular orbit is unstable and $r(t)$ either falls to the force center or escapes to infinity. In either case, for $\alpha>2$ the orbit is <u>unbound</u>. ($r\to\infty$ or $r\to 0$ whence $p_r\to\infty$). In order that small perturbations about a stable orbit be <u>closed</u>, we must have $\alpha = 2-(p/q)^2$.

- Fun fact: If we consider <u>nonlinear</u> perturbations of a circular orbit, the <u>only</u> values of β which yield a closed orbit are $\beta^2=1$ (Kepler problem, $\alpha=1$) and $\beta^2=4$ (harmonic oscillator, $\alpha=-2$). See §14.7.1.

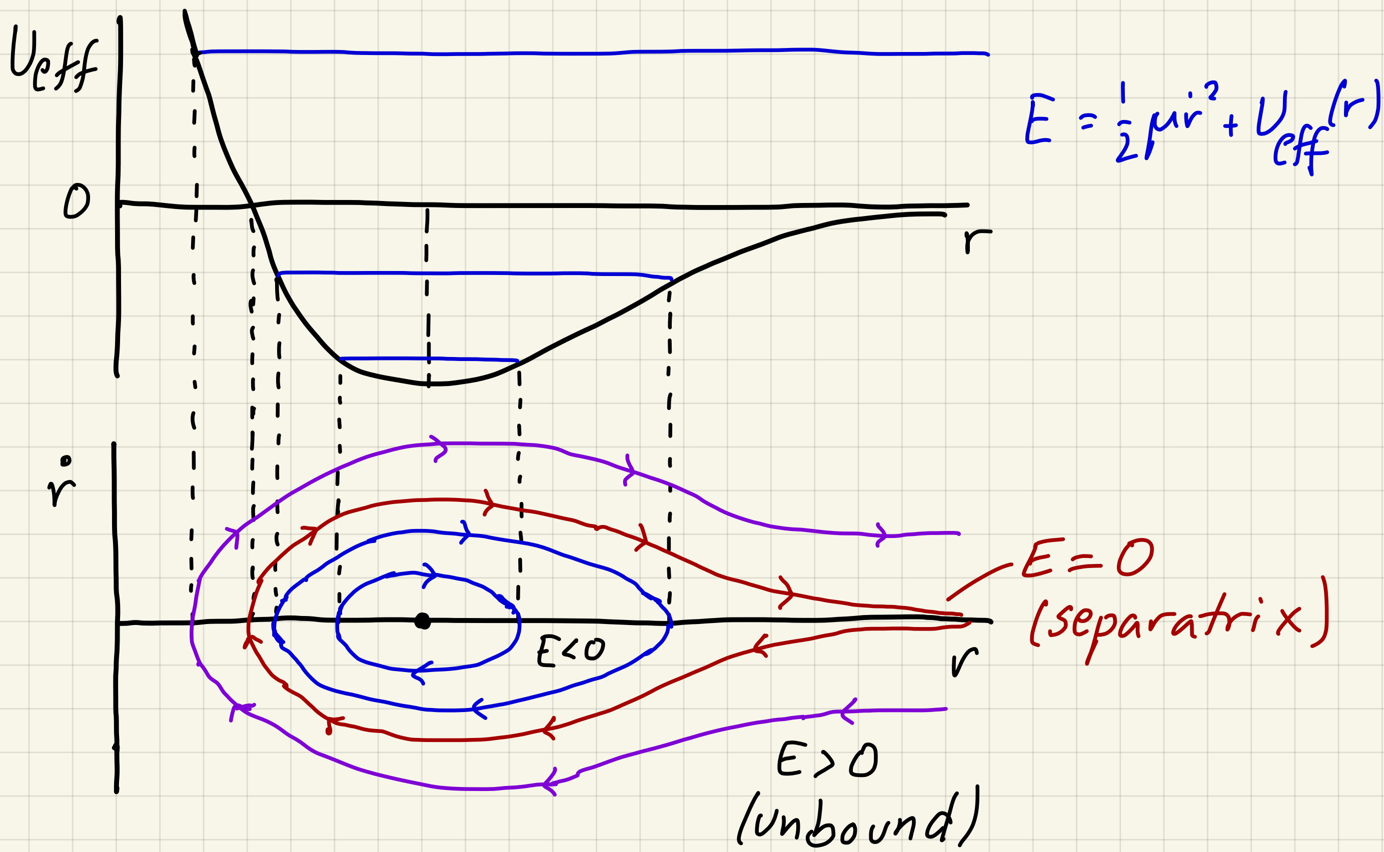
- Read §4.3: "Precession in a Soluble Model"

$$F = -\frac{k}{r} + \frac{C}{r^2} \Rightarrow r(\phi) = \frac{r_0}{1-\epsilon\cos\beta\phi}, \quad \beta = \left(1+\frac{\mu C}{\ell^2}\right)^{1/2}$$

$\epsilon^2 = 1 + \frac{2E(\ell^2+\mu C)}{\mu k^2}$ = eccentricity, E = energy (see Fig 4.3)

- The Kepler Problem : $U(r) = -\frac{k}{r}$, $k = Gm_1m_2 = GM\mu$

Effective potential and phase curves :



From $F(r) = -kr^{-2}$, we have, with $s = 1/r$,

$$s''(\phi) + s = -\frac{\mu}{\ell^2 s^2} F(s^{-1}) = \frac{\mu k}{\ell^2} = \text{const.}$$

Thus, $s(\phi) = \frac{\mu k}{\ell^2} - C\cos(\phi - \phi_0)$, i.e.

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos(\phi - \phi_0)}$$

with $r_0 = \frac{\ell^2}{\mu k}$ and $\epsilon \equiv C r_0$. Since $r(\phi) = r(\phi + 2\pi n)$, the bound Kepler orbits (circles, ellipses) are closed.

- Laplace - Runge - Lenz vector

Define $\vec{A} \equiv \vec{p} \times \vec{\ell} - \mu k \hat{r}$ $\quad \left(\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \text{unit vector} \right)$

Then :

$$\frac{d\vec{A}}{dt} = \dot{\vec{p}} \times \vec{\ell} + \vec{p} \times \overbrace{\dot{\vec{\ell}}}^{=0} - \mu k \frac{\dot{\vec{r}}}{r} + \mu k \frac{\dot{r}\vec{r}}{r^2}$$

$$= -\frac{k\vec{r}}{r^3} \times (\mu \vec{r} \times \dot{\vec{r}}) - \mu k \frac{\dot{\vec{r}}}{r} + \mu k \frac{\dot{r}\vec{r}}{r^2}$$

interlude : $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{c} \cdot \vec{a})\vec{b}$

$$\frac{d\vec{A}}{dt} = -\frac{\mu k}{r^3} \left[\cancel{\vec{r} (\underbrace{\vec{r} \cdot \dot{\vec{r}}}_{r\dot{r}})} - \cancel{\dot{\vec{r}} (\underbrace{\vec{r} \cdot \vec{r}}_{r^2})} \right] - \cancel{\mu k \frac{\dot{\vec{r}}}{r}} + \cancel{\mu k \frac{\dot{r}\vec{r}}{r^2}} = 0$$

Thus, $\vec{A}$ is a conserved vector lying in the plane of the motion. If we assume apoapsis occurs at $\phi = \phi_0$,

$$\vec{A} \cdot \vec{r} = -Ar\cos(\phi - \phi_0) = \ell^2 - \mu k r$$

and

$$r(\phi) = \frac{\ell^2}{\mu k - A\cos(\phi - \phi_0)} = \frac{a(1-\epsilon^2)}{1 - \epsilon\cos(\phi - \phi_0)}$$

where

$$\epsilon = \frac{A}{\mu k} \quad , \quad a(1-\epsilon^2) = \frac{\ell^2}{\mu k}$$

From $\vec{A}^2 = 2\mu\ell^2 \left(E + \frac{\mu k^2}{2\ell^2} \right)$, we find

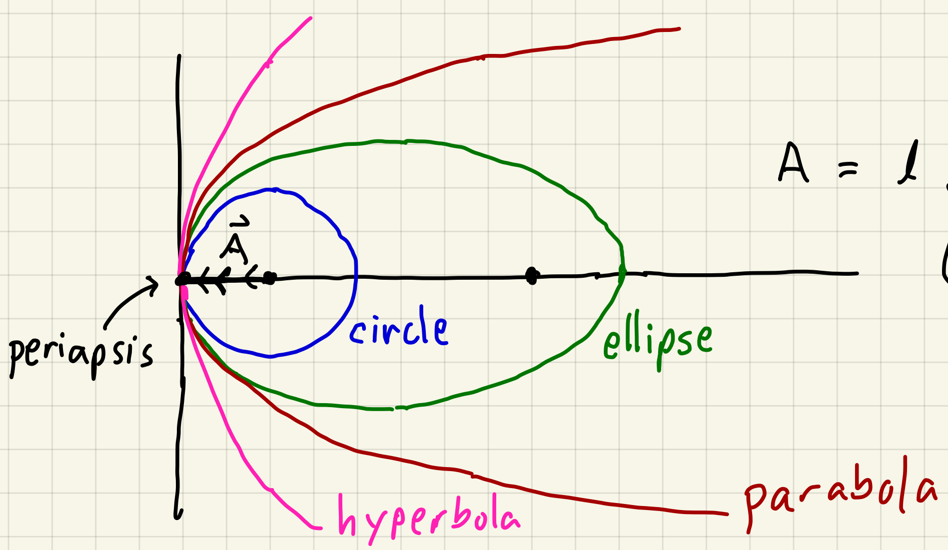
$$a = -\frac{k}{2E} \quad , \quad \epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$$

One can now show (§ 4.4.3) that Keplerian orbits are <u>conic sections</u> :

$$r(\phi) = \frac{a(1-\epsilon^2)}{1-\epsilon\cos(\phi-\phi_0)} \quad , \quad a = -\frac{k}{2E} \quad , \quad \epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$$

Note $\epsilon^2 > 0$ since $E_0 = -\frac{\mu k^2}{2\ell^2}$ is the energy of the (stable) circular orbit.

- circle : $E = -\frac{\mu k^2}{2\ell^2}$, $\epsilon = 0$, $a = \frac{\ell^2}{\mu k} = r_0$
- ellipse : $-\frac{\mu k^2}{2\ell^2} < E < 0$, $0 < \epsilon < 1$, semimajor axis length $a = -\frac{k}{2E}$, semiminor $b = a\sqrt{1-\epsilon^2}$
- parabola : $E = 0$, $\epsilon = 1$, $a(1-\epsilon^2) = \frac{\ell^2}{\mu k} = r_0$
  focus lies at force center
- hyperbola : $E > 0$, $\epsilon > 1$, $\phi = \phi_0 + \cos^{-1}(1/\epsilon) \Rightarrow r(\phi) = \infty$
  Force center is closest (attractive) or furthest (repulsive) focus.



$$A = \ell\sqrt{2\mu\left(E + \frac{\mu k^2}{2\ell^2}\right)} = \mu k \epsilon$$

($A = 0$ for circles)

- Period of bound Kepler orbits (circles, ellipses)

Since $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\Sigma}$, where $d\Sigma = \frac{1}{2} r^2 d\phi$ is the differential area enclosed, the period is

$$\tau = \frac{2\mu}{\ell} \Sigma = \frac{2\mu}{\ell} \underbrace{\pi a^2 \sqrt{1-\epsilon^2}}_{\text{area of ellipse/circle}}$$

Now $\epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$ and $a = -\frac{k}{2E}$, so eliminating $E \Rightarrow$

$$E = -\frac{k}{2a} \quad \Rightarrow \quad 1-\epsilon^2 = \frac{\ell^2}{\mu k a}$$

and we conclude $\tau = 2\pi (\mu a^3/k)^{1/2} = 2\pi (a^3/GM)^{1/2}$

since $k = Gm_1 m_2 = GM\mu$. Equivalently,

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} = \text{Const.}$$

For planets orbiting the sun, $\frac{a^3}{\tau^2} = \left(1 + \frac{m_p}{M_\odot}\right) \frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2}$

Note $m_p/M_\odot \lesssim 10^{-3}$ even for Jupiter.

- Escape velocity : threshold for energy is $E = 0$

$$E = 0 = \frac{1}{2} \mu v_{esc}^2(r) - \frac{Gm_1 m_2}{r}$$

$$\Rightarrow v_{esc}(r) = \sqrt{\frac{2GM}{r}}$$

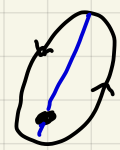
On earth's surface, $g = \frac{GM_E}{R_E^2} \Rightarrow v_{esc,E} = \sqrt{2gR_E}$
$= 11.2\,\text{km/s}$

- Satellites and spacecraft

Recall: $\tau = \frac{2\pi}{\sqrt{GM_E}} (R_E + h)^{3/2}$ $\quad (m_s << M_E)$

LEO = "Low Earth Orbit" $(h << R_E = 6.37 \times 10^6 m)$

So find $\tau_{LEO} = 1.4$ hr.



Problem: $h_p = 200$ km, $h_a = 7200$ km

$a = \frac{1}{2}(R_E + h_p + R_E + h_a) = 10071$ km

$\tau_{sat} = (a/R_E)^{3/2} \cdot \tau_{LEO} \simeq 2.65$ hr

- Read §§ 4.5 and 4.6

Lecture 6 (Oct. 21)

- A **rigid body** is a collection of point particles whose separations $|\vec{r}_i - \vec{r}_j|$ are all fixed in magnitude. Six independent coordinates are required to specify completely the position and orientation of a rigid body. For example, the location of the first particle (i) is specified by $\vec{r}_i$, which is three coordinates. The second (j) is then specified by a direction unit vector $\hat{n}_{ij}$, which requires two additional coordinates (polar and azimuthal angle). Finally, a third particle, k, is then fixed by its angle relative to the $\hat{n}_{ij}$ axis. Thus, six generalized coordinates in all are required.

Usually, one specifies three CM coordinates $\vec{R}$, and three orientational coordinates (e.g. the Euler angles). The equations of motion are then

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i \quad , \quad \dot{\vec{P}} = \vec{F}^{ext} \quad \text{(external force)}$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \quad , \quad \dot{\vec{L}} = \vec{N}^{ext} \quad \text{(external torque)}$$

- Inertia tensor

Suppose a point within a rigid body is fixed. This eliminates the translational motion. If we measure distances relative to this fixed point, then in an inertial frame,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \quad ; \quad \vec{\omega} = \text{angular velocity}$$

The kinetic energy is then

$$T = \frac{1}{2} \sum_i m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 = \frac{1}{2} \sum_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} \sum_i m_i \left[\omega^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2\right] \equiv \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$$

where $I_{\alpha\beta}$ is the **inertia tensor**,

3x3 real symmetric matric ⇒ 6 DOF

$$I_{\alpha\beta} = \sum_i m_i \left[\vec{r}_i^{\,2} \delta^{\alpha\beta} - r_i^\alpha r_i^\beta\right] \quad \text{(discrete)}$$

$$= \int d^d r \, \rho(\vec{r}) \left[\vec{r}^{\,2} \delta^{\alpha\beta} - r^\alpha r^\beta\right] \quad \text{(continuous)}$$

Diagonal elements of $I_{\alpha\beta}$ are <u>moments</u> of inertia, while off-diagonal elements are <u>products</u> of inertia.

- Satellites and spacecraft

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Usually, one specifies three CM coordinates $\vec{R}$, and three orientational coordinates (e.g. the Euler angles). The equations of motion are then

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i \quad , \quad \dot{\vec{P}} = \vec{F}^{ext} \quad \text{(external force)}$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \quad , \quad \dot{\vec{L}} = \vec{N}^{ext} \quad \text{(external torque)}$$

- Inertia tensor

Suppose a point within a rigid body is fixed. This eliminates the translational motion. If we measure distances relative to this fixed point, then in an inertial frame,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \quad ; \quad \vec{\omega} = \text{angular velocity}$$

The kinetic energy is then

$$T = \frac{1}{2} \sum_i m_i \left(\frac{d\vec{r}_i}{dt} \right)^2 = \frac{1}{2} \sum_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} \sum_i m_i \left[\omega^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2 \right] \equiv \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$$

where $I_{\alpha\beta}$ is the **inertia tensor**,

$$I_{\alpha\beta} = \sum_i m_i \left[\vec{r}_i^{\,2} \delta^{\alpha\beta} - r_i^\alpha r_i^\beta \right] \quad \text{(discrete)}$$

$$= \int d^d r \, \rho(\vec{r}) \left[\vec{r}^{\,2} \delta^{\alpha\beta} - r^\alpha r^\beta \right] \quad \text{(continuous)}$$

3x3 real symmetric matrix ⇒ 6 DOF

Diagonal elements of $I_{\alpha\beta}$ are <u>moments</u> of inertia, while off-diagonal elements are <u>products</u> of inertia.

- coordinate transformations

$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ = orthonormal basis ; $\hat{e}_\alpha \cdot \hat{e}_\beta = \delta_{\alpha\beta}$

Orthogonal basis transformation :

$$\hat{e}'_\alpha = R_{\alpha\mu}\hat{e}_\mu \quad ; \quad \hat{e}'_\alpha \cdot \hat{e}'_\beta = R_{\alpha\mu}R_{\beta\nu}\hat{e}_\mu \cdot \hat{e}_\nu = (R^T R)_{\alpha\beta} = \delta_{\alpha\beta}$$

Let $\vec{A} = A^\mu \hat{e}_\mu$ be a vector with A^α the components. Then

$$\vec{A} = A^\mu \hat{e}_\mu = A^\mu R_{\alpha\mu}\hat{e}'_\alpha \Rightarrow \underbrace{A'^\alpha = R_{\alpha\mu}A^\mu}_{\text{coordinate transformation}}$$

How does the inertia tensor transform?

$$\begin{aligned} I'_{\alpha\beta} &= \int d^3r' \rho'(r')\left[\vec{r}'^2 \delta^{\alpha\beta} - r'^\alpha r'^\beta\right] \\ &= \int d^3r \rho(\vec{r})\left[\vec{r}^2\delta^{\alpha\beta} - R_{\alpha\mu}r^\mu R_{\beta\nu}r^\nu\right] \\ &= R_{\alpha\mu} I_{\mu\nu} R^T_{\nu\beta} \quad , \quad \text{since } \rho'(\vec{r}') = \rho(\vec{r}) \end{aligned}$$

i.e. $\vec{v}' = R\vec{v}$ is the transformation rule for vectors, and $I' = RIR^T$ the rule for tensors. For scalars, $s' = s$. Note $\vec{\omega}$ is a vector, as is $\vec{L}$, but

$$T = \frac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta \text{ is a scalar}$$

Note : $T' = \frac{1}{2}R^T_{\alpha\mu}\omega'_\mu I_{\alpha\beta}R^T_{\beta\nu}\omega'_\nu = \frac{1}{2}\omega'_\mu(R_{\mu\alpha}I_{\alpha\beta}R^T_{\beta\nu})\omega'_\nu$

$$= \frac{1}{2}\omega'_\mu I'_{\mu\nu}\omega'_\nu$$

- The case of no fixed point

If there is no fixed point, choose CM as instantaneous origin for the body-fixed frame:

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = \frac{1}{M}\int d^3r\, \rho(\vec{r})\, \vec{r}$$

$$M = \sum_i m_i = \int d^3r\, \rho(\vec{r}) = \text{total mass}$$

Then

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}I_{\alpha\beta}\omega^\alpha\omega^\beta$$

$$L_\alpha = \epsilon_{\alpha\beta\gamma} M R^\beta \dot{R}^\gamma + I_{\alpha\beta}\omega^\beta$$

- Parallel axis theorem

Suppose we have $I_{\alpha\beta}$ in a body-fixed frame. Now shift the origin from O to $\vec{d}$. A mass at position $\vec{r}_i$ is located at $\vec{r}_i - \vec{d}$ as a result. Thus,

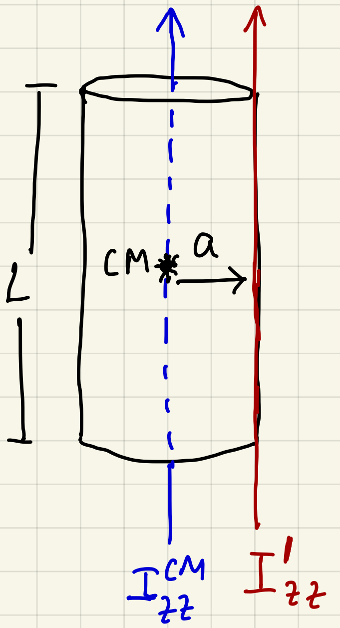
$$I_{\alpha\beta}(\vec{d}) = \sum_i m_i \left[(\vec{r}_i^{\,2} - 2\vec{d}\cdot\vec{r}_i + \vec{d}^{\,2})\delta^{\alpha\beta} - (r_i^\alpha - d^\alpha)(r_i^\beta - d^\beta) \right]$$

If $\vec{r}_i$ in the original frame is wrt the CM, then $\sum_i m_i \vec{r}_i = 0$, and we have

$$I_{\alpha\beta}(d) = I_{\alpha\beta}^{CM} + M(\vec{d}^{\,2}\delta^{\alpha\beta} - d^\alpha d^\beta)$$

Since we are only translating the origin, the coordinate axes remain parallel. Hence this result is known as the parallel axis theorem.

Example : uniform cylinder of radius a, height L



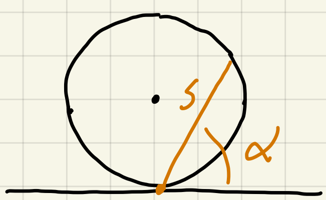
With origin at CM,

$$I^{CM}_{zz} = \int d^3r\, \rho(\vec{r})\,(\underbrace{x^2+y^2}_{\vec{r}^{\,2}-z^2})$$

$$= 2\pi\rho L \int_0^a dr_\perp\, r_\perp^3 = \frac{\pi}{2}\rho L a^4$$

$$= \frac{1}{2} M a^2 \quad \text{since} \quad M = \pi a^2 L \rho$$

Displace origin to surface : $\vec{d} = a\hat{\rho}$



Distance s ranges from 0 to s_0, with

$$a^2 = (s_0\cos\alpha)^2 + (s_0\sin\alpha - a)^2$$

$$= s_0^2 + a^2 - 2as_0\sin\alpha \Rightarrow s_0 = 2a\sin\alpha$$

Thus, $$I'_{zz} = \rho L \int_0^\pi d\alpha \int_0^{2a\sin\alpha} ds\, s^3 = \frac{M}{\pi a^2}\cdot 4a^4 \cdot \underbrace{\int_0^\pi d\alpha\, \sin^4\alpha}_{3\pi/8}$$

$$I'_{zz} = \frac{3}{2} M a^2$$

Using parallel axis theorem : $\vec{d} = a\hat{x}$

$$I'_{zz} = I^{CM}_{zz} + M(\vec{d}^{\,2}\delta^{zz} - d^z d^z)$$

$$= \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2 \;\checkmark$$

No need for trigonometry or integration!

- Read § 8.3.1 (inertia tensor for right triangle)

- Planar mass distributions:

If $\rho(x,y,z) = \sigma(x,y)\,\delta(z)$, then $I_{xz} = I_{yz} = 0$

Furthermore,

$$I_{xx} = \int dx \int dy\, \sigma(x,y)\, y^2$$

$$I_{yy} = \int dx \int dy\, \sigma(x,y)\, x^2 \qquad I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{xx}+I_{yy} \end{pmatrix}$$

$$I_{xy} = -\int dx \int dy\, \sigma(x,y)\, xy$$

and $I_{zz} = I_{xx} + I_{yy}$. Only 3 parameters.

- Principal axes of inertia

In general, if you have a symmetric matrix and you diagonalize it, good things will happen.

Recall that basis transformation $\hat{e}'_\alpha = R_{\alpha\mu}\hat{e}_\mu$ entails the transformation rules for vectors and tensors,

$$\vec{A}' = R\vec{A} \quad , \quad I' = RIR^T$$

i.e. $\quad A'^\alpha = R_{\alpha\mu} A^\mu \quad , \quad I'_{\alpha\beta} = R_{\alpha\mu} I_{\mu\nu} R^T_{\nu\beta}$

Since $I = I^T$ is symmetric, we can find a new orthonormal basis $\{\hat{e}'_\mu\}$ with respect to which I' is diagonal. Dropping the primes, we have that in a diagonal basis,

$$I = \mathrm{diag}(I_1, I_2, I_3) \quad , \quad \vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$T = \tfrac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta = \tfrac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

How to diagonalize $I_{\alpha\beta}$ (or any real symmetric matrix):

1) Find the diagonal elements of I', which are the eigenvalues of I, by solving $P(\lambda) \equiv \det(\lambda \cdot \mathbb{1} - I) = 0$. If $I_{\alpha\beta}$ is of rank n, $P(\lambda)$ is a polynomial in λ of order n.

2) For each eigenvalue λ_a $(a = 1, \ldots, n)$, solve the n equations

$$\sum_{\nu=1}^{n} I_{\mu\nu}\, \psi^a_\nu = \lambda_a\, \psi^a_\mu$$

where ψ^a_μ is the μ^{th} component of the a^{th} eigenvector $\vec{\psi}^a$. Since $(\lambda_a \cdot \mathbb{1} - I)$ is degenerate, the above equations are linearly dependent, and we may solve for the $(n-1)$ ratios $\{\psi^a_2/\psi^a_1, \ldots, \psi^a_n/\psi^a_1\}$.

3) Since $I_{\alpha\beta}$ is real and symmetric, its eigenfunctions corresponding to distinct eigenvalues are necessarily orthogonal. Eigenvectors corresponding to degenerate eigenvalues may be chosen to be orthogonal via the Gram–Schmidt procedure. Finally, the eigenvectors are normalized, thus

$$\langle \vec{\psi}^a | \vec{\psi}^b \rangle = \sum_{\mu=1}^{n} \psi^a_\mu\, \psi^b_\mu = \delta^{ab}$$

4) The matrix elements of R are then given by $R_{a\mu} = \psi^a_\mu$, i.e. the a^{th} row of R is the eigenvector ψ^a_μ, which is the a^{th} column of R^T.

5) The eigenvectors are complete and orthonormal.

$$\text{completeness}: \sum_a \psi^a_\mu \psi^a_\nu = R_{a\mu} R_{a\nu} = (R^T R)_{\mu\nu} = \delta_{\mu\nu}$$

$$\text{orthogonality}: \sum_\mu \psi^a_\mu \psi^b_\mu = R_{a\mu} R_{b\mu} = (R R^T)_{ab} = \delta_{ab}$$

See § 8.4 Eqns. 8.32 – 8.38 for an example

- Euler's equations

We choose our coordinate axes such that $I_{\alpha\beta}$ is diagonal. Such a choice $\{\hat{e}_\alpha\}$ are called **principal axes of inertia.** We further choose the origin to be located at the CM. Thus

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \vec{L} = I\vec{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

The equations of motion are then

in body-fixed frame

$$\vec{N}^{ext} = \left(\frac{d\vec{L}}{dt}\right)_{inertial} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

in inertial frame

$$= I\dot{\vec{\omega}} + \vec{\omega} \times (I\vec{\omega})$$

Here we have used the important relation

$$\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \vec{\omega} \times \vec{A},$$

valid for any vector $\vec{A}$. Let's derive this important result.

- Interlude : accelerated coordinate systems (§ 7.1)

Consider an inertial frame with fixed coordinate axes $\hat{e}_\mu$, and a rotating frame with axes $\hat{e}'_\mu$, where $\mu \in \{1, \ldots, d\}$. The two frames share a common origin which is fixed within the body.

Any vector $\vec{A}$ may be written as

$$\vec{A} = \sum_\mu A_\mu \hat{e}_\mu = \sum_\mu A'_\mu \hat{e}'_\mu$$

Thus in the inertial frame

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \sum_\mu \frac{dA_\mu}{dt} \hat{e}_\mu$$

$$= \underbrace{\sum_\mu \frac{dA'_\mu}{dt} \hat{e}'_\mu}_{\text{this is } (d\vec{A}/dt)_{\text{body}}} + \sum_\mu A'_\mu \frac{d\hat{e}'_\mu}{dt}$$

What is $d\hat{e}'_\mu/dt$? Since the basis $\{\hat{e}'_\nu\}$ is complete, we may expand

$$d\hat{e}'_\mu = \sum_\nu d\Omega_{\mu\nu} \hat{e}'_\nu \iff d\Omega_{\mu\nu} = d\hat{e}'_\mu \cdot \hat{e}_\nu$$

But $d(\underbrace{\hat{e}'_\mu \cdot \hat{e}'_\nu}_{\delta_{\mu\nu}}) = d\hat{e}'_\mu \cdot \hat{e}'_\nu + \hat{e}'_\mu \cdot d\hat{e}'_\nu = d\Omega_{\mu\nu} + d\Omega_{\nu\mu} = 0$

Thus, $d\Omega_{\mu\nu}$ is a real, antisymmetric, infinitesimal $d \times d$ matrix.

A $d\times d$ real antisymmetric matrix has $\frac{1}{2}d(d-1)$ independent entries. For $d=3$, we may write

$$d\Omega_{\mu\nu} = \sum_{\sigma} \epsilon_{\mu\nu\sigma}\, d\Omega_{\sigma}$$

and we define $\omega_\sigma \equiv d\Omega_\sigma / dt$. This yields

$$\frac{d\hat{e}'_\mu}{dt} = \vec{\omega} \times \hat{e}'_\mu$$

and we have

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{A}$$

is valid for any vector $\vec{A}$. We may then write

$$\frac{d}{dt}\bigg|_{\text{inertial}} = \frac{d}{dt}\bigg|_{\text{body}} + \vec{\omega}\,\times$$

so long as we apply this to vectors only. Applied to the vector $\vec{\omega}$ itself, this yields $\dot{\vec{\omega}}_{\text{inertial}} = \dot{\vec{\omega}}_{\text{body}}$. Applied twice,

$$\frac{d^2\vec{A}}{dt}\bigg|_{\text{inertial}} = \frac{d^2\vec{A}}{dt}\bigg|_{\text{body}} + \frac{d\vec{\omega}}{dt}\times\vec{A} + 2\vec{\omega}\times\frac{d\vec{A}}{dt}\bigg|_{\text{body}} + \vec{\omega}\times(\vec{\omega}\times\vec{A})$$

This formula contains the description of centrifugal and Coriolis forces, which you can read about in chapter 7 of the notes. But for now, back to rigid body dynamics...

Euler's equations along body-fixed principal axes:

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{L} = I\dot{\vec{\omega}} + \vec{\omega}\times(I\vec{\omega}) = \vec{N}^{ext}$$

Component by component,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_2\omega_3 + N_1^{ext}$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\,\omega_3\omega_1 + N_2^{ext}$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\,\omega_1\omega_2 + N_3^{ext}$$

These three equations are coupled and nonlinear. The components N_α^{ext} must be evaluated along the body-fixed principal axes. The simplest case is when there is no net external torque, which is the case when a body moves in free space, but also in a uniform gravitational field:

$$\vec{N}^{ext} = \sum_i \vec{r}_i \times (m_i\vec{g}) = \left(\sum_i m_i\vec{r}_i\right)\times\vec{g}$$

In a body fixed frame with the origin at the CM, the term in parentheses vanishes, hence $\vec{N}^{ext} = 0$, and

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3\,,\quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1\,,\quad \dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2$$

- Torque-free symmetric tops:

Suppose $I_1 = I_2 \neq I_3$. Then $\dot{\omega}_3 = 0$, hence $\omega_3 = \text{const.}$

The remaining two equations are

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_3\omega_2 \quad , \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1$$

hence $\dot{\omega}_1 = -\Omega\omega_2$, $\dot{\omega}_2 = +\Omega\omega_1$, with $\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3$.

Thus,

$$\omega_1(t) = \omega_\perp \cos(\Omega t + \delta) \;,\; \omega_2(t) = \sin(\Omega t + \delta) \;,\; \omega_3(t) = \omega_3$$

where $\omega_\perp$ and δ are constants of integration.

Therefore, in the body-fixed frame, $\vec{\omega}_\perp(t)$ precesses about $\hat{e}_3$ $(\equiv \hat{e}_3^{\,body})$ with frequency Ω at an angle $\lambda = \tan^{-1}(\omega_\perp/\omega_3)$. For the earth, this is called the **Chandler wobble**, and $\lambda \simeq 6\times10^{-7}$ rad, meaning that the north pole moves by about four meters during the wobble. Again for earth, $(I_3 - I_1)/I_1 \approx \frac{1}{305}$, hence the precession period is predicted to be about 305 days. In fact, the period of the Chandler wobble is about 14 months, which is a substantial discrepancy, attributed to the mechanical properties of the earth (elasticity and fluidity): the earth isn't solid!

- Asymmetric tops

In principal, we may invoke energy and angular momentum conservation,

$$E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$\vec{L}^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2$$

and obtain ω_1 and ω_2 in terms of ω_3. Then

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3} \right) \omega_1 \omega_2$$

becomes a nonlinear first order ODE. Using Lagranges method and extremizing the energy at fixed L^2, we obtain the following:

conditions	energy E	extremum classification $I_i < I_j < I_k$					
		123	213	132	312	231	321
$\omega_2 = \omega_3 = 0$	$\frac{1}{2} I_1 \omega_1^2 = \frac{L^2}{2I_1}$	MAX	SP	MAX	SP	MIN	MIN
$\omega_1 = \omega_3 = 0$	$\frac{1}{2} I_2 \omega_2^2 = \frac{L^2}{2I_2}$	SP	MAX	MIN	MIN	MAX	SP
$\omega_1 = \omega_2 = 0$	$\frac{1}{2} I_3 \omega_3^2 = \frac{L^2}{2I_3}$	MIN	MIN	SP	MAX	SP	MAX

We can then analyze the nonlinear ODE $\dot{\omega}_3 = f(\omega_3)$. This is somewhat unpleasant.

We can however easily linearize the equations of motion about a known solution. For example, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ is a solution of Euler's equations. Let us then write $\vec{\omega} = \omega_0 \hat{e}_3 + \delta\vec{\omega}$. Then

$$\delta\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_0\,\delta\omega_2 + \mathcal{O}(\delta\omega_2\,\delta\omega_3)$$

$$\delta\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_0\,\delta\omega_1 + \mathcal{O}(\delta\omega_1\,\delta\omega_3)$$

$$\delta\dot{\omega}_3 = 0 + \mathcal{O}(\delta\omega_1\,\delta\omega_2)$$

Thus, we have $\delta\ddot{\omega}_1 = -\Omega^2\,\delta\omega_1$ and $\delta\ddot{\omega}_2 = -\Omega^2\,\delta\omega_2$ with

$$\Omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\,\omega_0^2$$

The solution is $\delta\omega_1(t) = \epsilon\cos(\Omega t + \eta)$, in which case

$$\delta\omega_2(t) = \omega_0^{-1}\,\frac{I_1}{I_2 - I_3}\,\delta\dot{\omega}_1 = \left(\frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\right)^{1/2}\epsilon\sin(\Omega t + \delta)$$

If $\Omega \in \mathbb{R}$, $\delta\omega_1(t)$ and $\delta\omega_2(t)$ are harmonic functions with period $2\pi/\Omega$. This is the case when $I_3 > I_{1,2}$ or $I_3 < I_{1,2}$. But if I_3 is in the middle, i.e. $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$, then $\Omega^2 < 0$, $\Omega \in i\mathbb{R}$, and the behavior is exponential, i.e. $\vec{\omega}(t) = \omega_0 \hat{e}_3$ is <u>unstable</u>.

- Read § 8.5.1 (example problem for Euler's equations)

- Euler's angles

  The dimension of the orthogonal group $O(n)$ is

$$\dim O(n) = \frac{1}{2} n(n-1)$$

Thus in dimension $n=2$, a rotation is specified by a single parameter, i.e. the planar angle. In $n=3$ dimensions, we require three parameters in order to specify a general rotation, i.e. a general orientation of an object with respect to some fiducial orientation. These three parameters are often taken to be **Euler's angles** $\{\phi, \theta, \psi\}$.

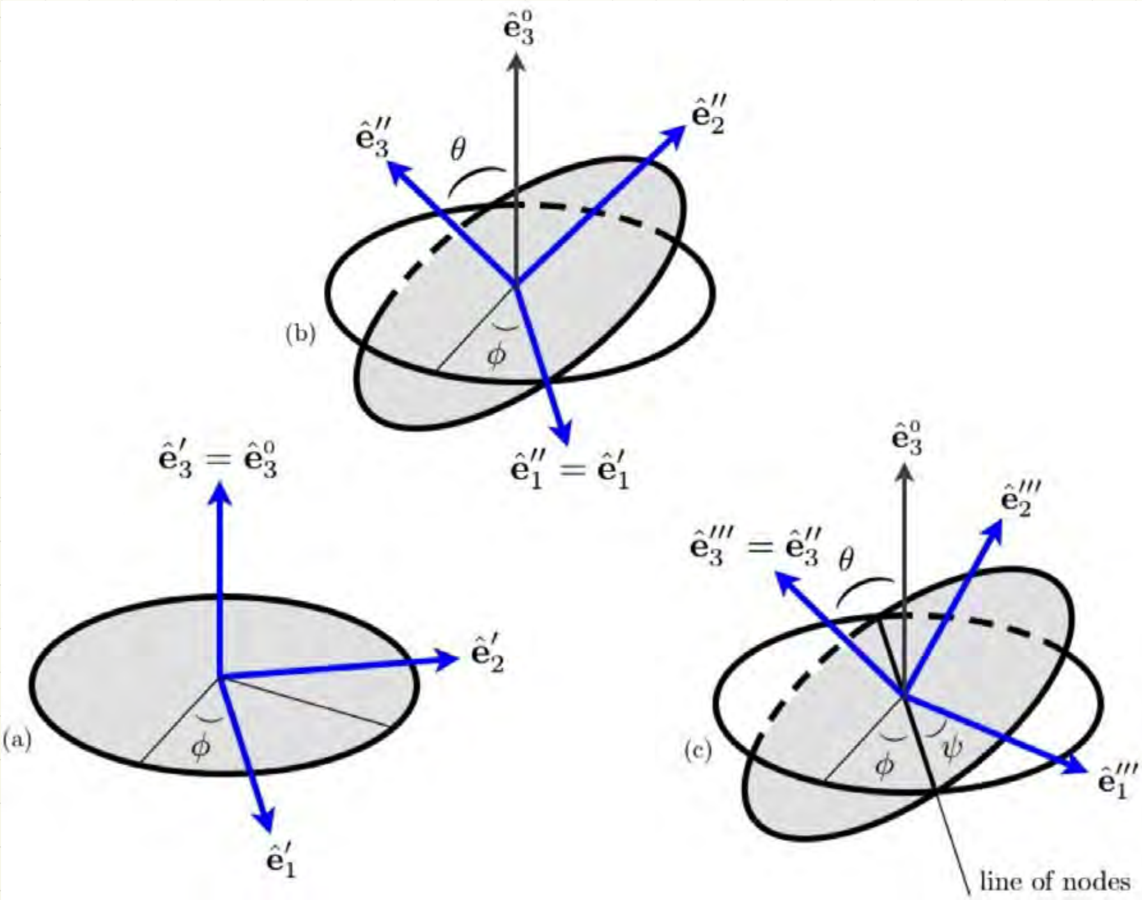
- General rotation matrix $R(\phi, \theta, \psi) \in SO(3)$:

  Start with an orthonormal triad $\{\hat{e}_\mu^0\}$. We first rotate by ϕ about the $\hat{e}_3^0$ axis:

$$\hat{e}_\mu' = R_{\mu\nu}(\phi, \hat{e}_3^0)\, \hat{e}_\nu^0 \; ; \; R(\phi, \hat{e}_3^0) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The next step is to rotate by θ about $\hat{e}_1'$:

$$\hat{e}_\mu'' = R_{\mu\nu}(\theta, \hat{e}_1')\, \hat{e}_\nu' \; ; \; R(\theta, \hat{e}_1') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$



Constructing a general rotation in SO(3) using Euler's angles $\{\phi, \theta, \psi\}$

Finally, rotate by ψ about $\hat{e}_3''$:

$$\hat{e}_\mu \equiv \hat{e}_\mu''' = R_{\mu\nu}(\psi, \hat{e}_3'')\,\hat{e}_\nu'' \quad ; \quad R(\psi, \hat{e}_3'') = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply the three matrices to get $\hat{e}_\mu = R_{\mu\nu}(\phi,\theta,\psi)\,\hat{e}_\nu^0$ with

$$R(\phi,\theta,\psi) = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\theta - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

See the figure at the top of this page.

Next we relate the components of $\vec{\omega}$ to the derivatives $\{\dot{\phi}, \dot{\theta}, \dot{\psi}\}$. This is accomplished by writing

$$\vec{\omega} = \dot{\phi}\,\hat{e}_\phi + \dot{\theta}\,\hat{e}_\theta + \dot{\psi}\,\hat{e}_\psi$$

where (consult previous figure)

$$\hat{e}_\phi = \sin\theta \sin\psi\, \hat{e}_1 + \sin\theta\cos\psi\,\hat{e}_2 + \cos\theta\,\hat{e}_3 = \hat{e}^{\,0}_3$$

$$\hat{e}_\theta = \cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2 \quad \text{("line of nodes")}$$

$$\hat{e}_\psi = \hat{e}_3$$

We may now read off

$$\omega_1 = \vec{\omega}\cdot\hat{e}_1 = \dot{\theta}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \vec{\omega}\cdot\hat{e}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \vec{\omega}\cdot\hat{e}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Note that:

$$\dot{\phi} \leftrightarrow \text{precession}, \quad \dot{\theta} \leftrightarrow \text{nutation}, \quad \dot{\psi} \leftrightarrow \text{axial rotation}$$

In spinning tops, axial rotation is sufficiently fast that it appears to us as a blur. We can, however, discern precession and nutation. The rotational kinetic energy is then

$$T_{rot} = \tfrac{1}{2} I_1 \left(\dot{\theta}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \tfrac{1}{2} I_2 \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \tfrac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

The canonical momenta are then

$$p_\phi = \frac{\partial T}{\partial \dot\phi}, \qquad p_\theta = \frac{\partial T}{\partial \dot\theta}, \qquad p_\psi = \frac{\partial T}{\partial \dot\psi}$$

and the angular momentum vector is

$$\vec{L} = p_\phi \hat{e}_\phi + p_\theta \hat{e}_\theta + p_\psi \hat{e}_\psi$$

Note that we don't need to specify the reference frame when writing $\vec{L}$ – only for time-derivatives of vectors must we specify inertial or body-fixed frame.

- Torque-free symmetric top: $\vec{N}^{ext} = 0$

Let $I_1 = I_2$. Then

$$T = \frac{1}{2} I_1 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) + \frac{1}{2} I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)^2$$

The potential is $U = 0$ so the Lagrangian is $L = T$.
Since ϕ and ψ are cyclic in L, their momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3 \cos\theta \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

Since $p_\psi = I_3 \omega_3$, we have $\omega_3 = \text{const.}$, as we have already derived from Euler's equations.

Let's solve for the motion. Note that $\vec{L}$ is conserved in the inertial frame, i.e. $(\dot{\vec{L}})_{\text{inertial}} = 0$. We choose $\hat{e}_3^\circ = \hat{e}_\phi = \vec{L}$. From $\hat{e}_\phi \cdot \hat{e}_\psi = \cos\theta$, we have $p_\psi = \vec{L}\cdot\hat{e}_\psi = L\cos\theta$ and conservation of p_ψ thus entails $\dot{\theta} = 0$. From

$$\dot{p}_\theta = I_1 \ddot{\theta} = \frac{\partial L}{\partial \theta} = (I_1 \cos\theta\,\dot{\phi} - p_\psi)\sin\theta\,\dot{\phi}$$

and $\dot{\theta} = 0$, we conclude $\dot{\phi} = p_\psi / I_1 \cos\theta$. Now, from the equation for p_ψ, we have

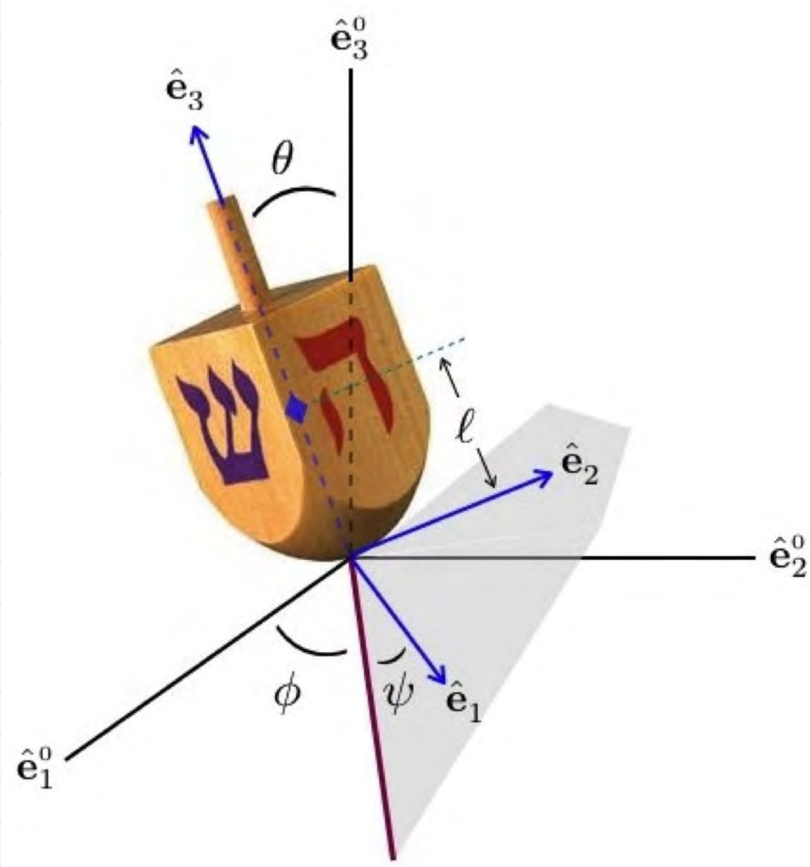
$$\dot{\psi} = \frac{p_\psi}{I_3} - \cos\theta\,\dot{\phi} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right)p_\psi = \left(\frac{I_3 - I_1}{I_3}\right)\omega_3$$

as we had derived from Euler's equations.

- Symmetric top with one point fixed:
  Now gravity exerts a torque. The Lagrangian is

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + \frac{1}{2} I_3 (\cos\theta\,\dot{\phi} + \dot{\psi})^2 - Mgl\cos\theta$$

where l is the distance from the fixed point to the CM. Let us now analyze the motion of this system.



The dreidl (Yid. דריידל, Heb. סביבון = spinner) is a symmetric top. Fourfold rotational symmetry is good enough to guarantee $I_1 = I_2$ and $I_{12} = 0$.

We have that ϕ and ψ are still cyclic, so

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3 \cos\theta \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

are again conserved. Thus,

$$\dot\phi = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} \quad , \quad \dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1 \sin^2\theta}$$

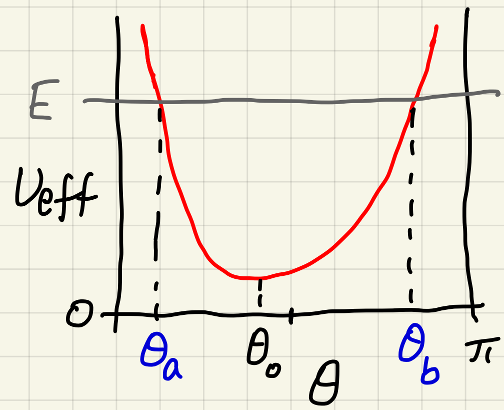
Energy $E = T + U$ is conserved:

$$E = \frac{1}{2} I_1 \dot\theta^2 + \underbrace{\frac{(p_\phi - p_\psi\cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} + Mg\ell\cos\theta}_{\text{effective potential } U_{\text{eff}}(\theta)}$$

Again:

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \overbrace{\frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} + Mgl\cos\theta}^{U_{eff}(\theta)}$$

Straightforward analysis (see lecture notes, ch. 8, p. 18) reveals that $U_{eff}(\theta)$ has a single minimum at θ_0 $[0, \pi]$, and that $U_{eff}(\theta)$ diverges as $\theta \to 0$ and $\theta \to \pi$. Thus, the equation of motion,

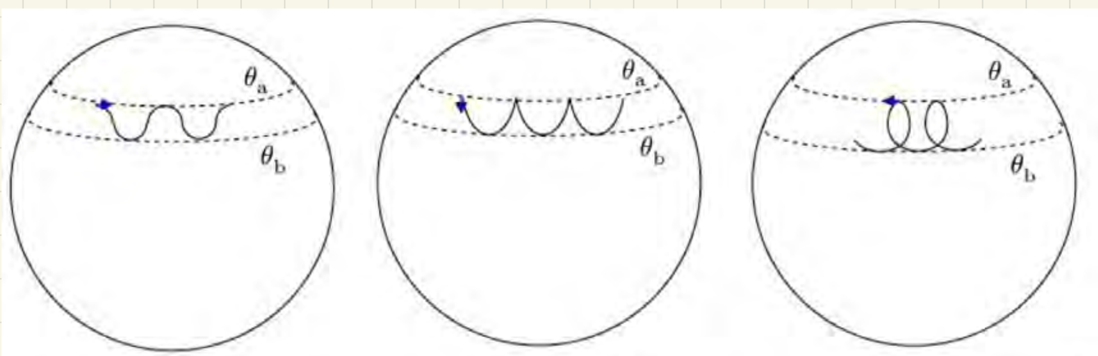


$$I_1 \ddot{\theta} = -U'_{eff}(\theta)$$

yields two **turning points**, which we label θ_a and θ_b, satisfying $E = U_{eff}(\theta_{a,b})$. Now we have already derived the result

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

Thus we conclude that if $p_\psi \cos\theta_b < p_\phi < p_\psi \cos\theta_a$ then $\dot{\phi}$ will change sign when θ reaches $\theta^* = \cos^{-1}(p_\phi / p_\psi)$. This leads to two types of motion, as shown below. Note that $\hat{e}_3 = \sin\theta \sin\phi \, \hat{e}_1^0 - \sin\theta\cos\phi \, \hat{e}_2^0 + \cos\theta \, \hat{e}_3^0$.



$p_\phi > p_\psi \cos\theta_a$ | $p_\phi = p_\psi \cos\theta_a$ | $p_\psi \cos\theta_b < p_\phi < p_\psi \cos\theta_a$

ϕ : precession

θ : nutation

ψ : axial angle

Lecture 7 (Oct. 26)

We now turn to the subject of small oscillations. We assume that the kinetic energy is homogeneous of degree two in the generalized velocities: $T = \frac{1}{2} T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma \dot{q}_{\sigma'}$, and that the potential $U(q_1, \ldots, q_n)$ is degree zero in the $\{\dot{q}_\sigma\}$. The equations of motion are then obtained as follows:

$$L = T - U \Rightarrow \begin{cases} p_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}(q)\, \dot{q}_{\sigma'} \\[2ex] F_\sigma = \dfrac{\partial L}{\partial q_\sigma} = \dfrac{1}{2} \dfrac{\partial T_{\sigma'\sigma''}(q)}{\partial q_\sigma} \dot{q}_{\sigma'} \dot{q}_{\sigma''} - \dfrac{\partial U(q)}{\partial q_\sigma} \end{cases}$$

Thus, $\dot{p}_\sigma = F_\sigma$ says

$$T_{\sigma\sigma'} \ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2} \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma} \right) \dot{q}_{\sigma'} \dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma}$$

This may be written as

$$T_{\sigma\alpha} \ddot{q}_\alpha + \frac{1}{2} \left(\frac{\partial T_{\sigma\mu}}{\partial q_\nu} + \frac{\partial T_{\sigma\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\sigma} \right) \dot{q}_\mu \dot{q}_\nu = -\frac{\partial U}{\partial q_\sigma}$$

multiply by $T^{-1}_{\lambda\sigma}$

$$\ddot{q}_\lambda + \Gamma^\lambda_{\mu\nu}\, \dot{q}_\mu \dot{q}_\nu = A_\lambda \quad , \quad \text{with}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} T^{-1}_{\lambda\sigma} \left(\frac{\partial T_{\sigma\mu}}{\partial q_\nu} + \frac{\partial T_{\sigma\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\sigma} \right) \leftarrow \text{Christoffel symbols}$$

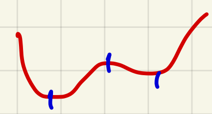
$$A_\lambda = -T^{-1}_{\lambda\sigma} \frac{\partial U}{\partial q_\sigma}$$

- Static equilibrium: $\dot{q}_\sigma = 0 \ \forall \sigma \in \{1, \dots, n\} \Rightarrow$

$\frac{\partial U}{\partial q_\sigma} = 0 \ \forall \sigma$; n equations in n unknowns $\{q_1, \dots, q_n\}$

Generically this has pointlike solutions, $\{\bar{q}_1, \dots, \bar{q}_n\}$.

Let's write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand the Lagrangian to quadratic order in the q_σ and $\dot{q}_\sigma$:



$$L = \tfrac{1}{2} T_{\sigma\sigma'} \dot{\eta}_\sigma \dot{\eta}_{\sigma'} - \tfrac{1}{2} V_{\sigma\sigma'} \eta_\sigma \eta_{\sigma'} + \dots$$

where

$$\left. \begin{aligned} T_{\sigma\sigma'} &= T_{\sigma\sigma'}(\bar{q}) = \left.\frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}}\right|_{\bar{q}} \\ V_{\sigma\sigma'} &= \left.\frac{\partial^2 U}{\partial q_\sigma \partial q_{\sigma'}}\right|_{\bar{q}} \end{aligned} \right\}$$

T and V are constant, real, symmetric, $n \times n$ matrices

So to quadratic order, $L = \frac{1}{2} \dot{\eta}^t T \dot{\eta} - \frac{1}{2} \eta^t V \eta$

- Method of small oscillations

The idea here is to express the η_σ in terms of normal modes, ξ_i, which diagonalize the equations of motion,

$$T_{\sigma\sigma'} \ddot{\eta}_{\sigma'} = - V_{\sigma\sigma'} \eta_\sigma$$

This being a linear problem, we write $\eta_\sigma = A_{\sigma i} \xi_i$ and demand

$A_{\sigma i}$: $n \times n$ real matrix

$$A^t T A = \mathbb{1}$$
$$A^t V A = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2)$$

The vector form of the linearized EL eqns is

$$T\ddot{\vec{\eta}} = -V\vec{\eta}$$

so

$$TA\ddot{\vec{\xi}} = -VA\vec{\xi}$$

Multiplying on the left by A^t, we then have

$$\underbrace{(A^tTA)}_{=\mathbb{1}}\ddot{\vec{\xi}} = -\underbrace{(A^tVA)}_{=\mathrm{diag}(\omega_1^2,\dots,\omega_n^2)}\vec{\xi}$$

Thus we have n decoupled second order ODEs:

$$\ddot{\xi}_i = -\omega_i^2\xi_i$$

with solutions

$$\xi_i(t) = C_i\cos(\omega_i t) + D_i\sin(\omega_i t)$$

with $2n$ constants of integration $\{C_i, D_i\}$ with $i\in\{1,\dots,n\}$.
Note $\vec{\eta} = A\vec{\xi}$ yields $\vec{\xi} = A^{-1}\vec{\eta} = A^tT\vec{\eta}$, thus

$$\eta_\sigma(t) = \sum_i A_{\sigma i}\left[C_i\cos(\omega_i t) + D_i\sin(\omega_i t)\right]$$

Multiplying on the left by A^tT, we obtain

$$C_i\cos(\omega_i t) + D_i\sin(\omega_i t) = A^t_{i\sigma}T_{\sigma\sigma'}\eta_{\sigma'}(t)$$

and thus

$$C_i = A^t_{i\sigma}T_{\sigma\sigma'}\eta_{\sigma'}(0)$$

$$D_i = \omega_i^{-1}A^t_{i\sigma}T_{\sigma\sigma'}\dot{\eta}_{\sigma'}(0) \qquad \text{(no sum on } i\text{)}$$

At this point, we have the complete solution to the problem for arbitrary initial conditions $\{\eta_\sigma(0), \dot{\eta}_\sigma(0)\}$. The matrix $A_{\sigma i}$ is called the modal matrix. If all the generalized coordinates have dimensions $[q_\sigma] = L$,

$$[T_{\sigma\sigma'}] = M \quad , \quad [V_{\sigma\sigma'}] = \frac{E}{L^2} = \frac{M}{T^2}$$

$$[A_{\sigma i}] = M^{-1/2} \quad , \quad [\xi_i] = M^{1/2} L$$

- Why can we demand $A^t T A = \mathbb{1}$ and $A^t V A = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2)$?

Proof by construction:

(i) Since $T_{\sigma\sigma'}$ is symmetric, there exists $O_1 \in O(n)$ such that $O_1^t T O_1 = T_d$, where T_d is diagonal. Additionally, the entries of T_d are all positive because the kinetic energy is in general positive (only zero if $\dot{q}_\sigma = 0 \ \forall \sigma$).

(ii) T_d being positive definite, we may construct its square root $T_d^{1/2}$ simply by taking the square root of each diagonal entry. Note then that

$$T_d^{-1/2} O_1^t T O_1 T_d^{-1/2} = T_d^{-1/2} T_d T_d^{-1/2} = \mathbb{1}$$

(iii) The matrix $T_d^{-1/2} O_1^t V O_1 T_d^{-1/2}$ is symmetric, and hence diagonalized by some $O_2 \in O(n)$. Thus, we have two matrices O_1 and O_2 such that

$$O_2^t T_d^{-1/2} O_1^t \, T \, O_1 T_d^{-1/2} O_2 = \mathbb{1}$$

$$O_2^t T_d^{-1/2} O_1^t \, V O_1 T_d^{-1/2} O_2 = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2)$$

Therefore the modal matrix is

$$A = O_1 T_d^{-1/2} O_2 \qquad \text{(NB: A not orthogonal!)}$$

We can see that it is in general not possible to simultaneously diagonal three symmetric matrices. Two is the limit!

- How to find the modal matrix

(i) Assume $\eta_\sigma(t) = \mathrm{Re}\, \psi_\sigma e^{-i\omega t}$. Then from the EL eqn $T\ddot{\vec{\eta}} = -V\vec{\eta}$ we have $(\omega^2 T - V)_{\sigma\sigma'} \psi_{\sigma'} = 0$. In order to have nontrivial solutions, we demand

$$\det(\omega^2 T - V) = 0$$

This yields an n^{th} order polynomial equation in ω^2. Its n roots are the n normal mode frequencies, ω_i^2.

(ii) Next, find the eigenvectors $\psi_\sigma^{(i)}$ by demanding

$$\sum_{\sigma'} (\omega_i^2 T_{\sigma\sigma'} - V_{\sigma\sigma'}) \psi_{\sigma'}^{(i)} = 0$$

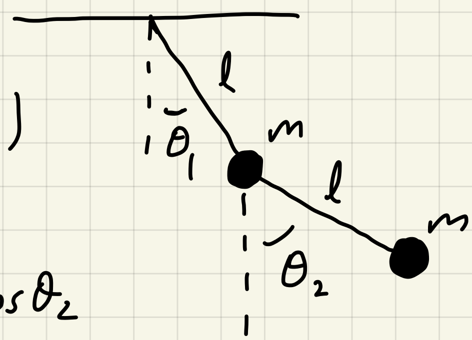
Since $\omega_i^2 T - V$ is defective, these equations are $(n-1)$ inhomogeneous linear equations for $\{\psi_2^{(i)}, \ldots, \psi_n^{(i)}\}$ yielding the ratios $\{\psi_2^{(i)}/\psi_1^{(i)}, \ldots, \psi_n^{(i)}/\psi_1^{(i)}\}$. It then follows (see §5.3.3) that $\psi_\sigma^{(i)} T_{\sigma\sigma'} \psi_{\sigma'}^{(j)} = 0$ if $i \neq j$. In fact, this is only guaranteed if $\omega_i^2 \neq \omega_j^2$, but for degenerate eigenvalues $\omega_i^2 = \omega_j^2$, we may still choose the eigenvectors to be orthogonal (wrt T) via the Gram–Schmidt process. Finally, we may choose to normalize each eigenvector, so that

$$\langle \psi^{(i)} | \psi^{(j)} \rangle \equiv \psi_\sigma^{(i)} T_{\sigma\sigma'} \psi_{\sigma'}^{(j)} = \delta_{ij}$$

(iii) The modal matrix is then given by $A_{\sigma i} = \psi_\sigma^{(i)}$.

(iv) Since $\vec{\eta} = A\vec{\xi}$ and $A^t T A = \mathbb{1}$, $A^{-1} = A^t T$ and $\vec{\xi} = A^t T \vec{\eta}$.

- Example: the double pendulum



(For simplicity, choose $\ell_1 = \ell_2 = \ell$, $m_1 = m_2 = m$)

$x_1 = \ell \sin\theta_1$, $y_1 = -\ell\cos\theta_1$

$x_2 = \ell\sin\theta_1 + \ell\sin\theta_2$, $y_2 = -\ell\cos\theta_1 - \ell\cos\theta_2$

$$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m\ell^2 \left(2\dot{\theta}_1^2 + 2\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$$

$V = -mg\ell(2\cos\theta_1 + \cos\theta_2)$; equilibrium @ $\theta_1 = \theta_2 = 0$

$$T = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix}, \quad V = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix}$$

Let $\omega_0^2 \equiv g/\ell$. Then

$$\omega^2 T - V = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix}$$

$$\det(\omega^2 T - V) = (m\ell^2)^2 \cdot \left\{ 2(\omega^2 - \omega_0^2)^2 - \omega^4 \right\}$$

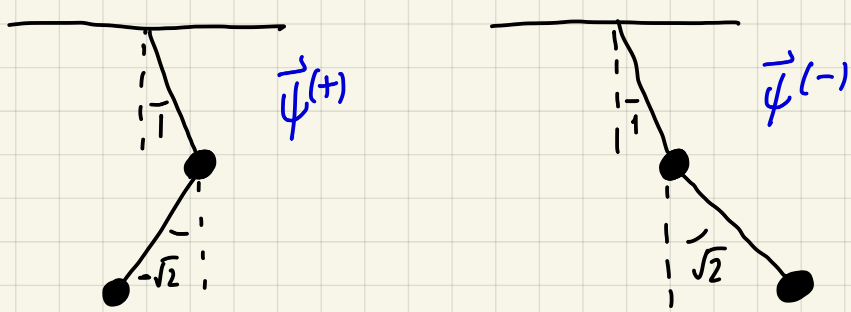
Setting $\det(\omega^2 T - V) = 0$ then yields $\omega_\pm^2 = (2 \pm \sqrt{2})\,\omega_0^2$.

Find

$$A = \begin{pmatrix} \psi_1^{(+)} & \psi_1^{(-)} \\ \psi_2^{(+)} & \psi_2^{(-)} \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{array}{c c} i=+ & i=- \\ \left(\begin{matrix} \sqrt{2+\sqrt{2}} \\ -\sqrt{2}\cdot\sqrt{2+\sqrt{2}} \end{matrix} \right. & \left. \begin{matrix} \sqrt{2-\sqrt{2}} \\ \sqrt{2}\cdot\sqrt{2-\sqrt{2}} \end{matrix} \right) \end{array} \begin{matrix} \sigma=1 \\ \sigma=2 \end{matrix}$$

Note that $\vec{\psi}^{(+)} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ and $\vec{\psi}^{(-)} \propto \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

Normal mode shapes:



In the low frequency normal mode, the two masses oscillate in phase, while in the high frequency normal mode, they are π out of phase.

- Zero modes

Recall that to each continuous one-parameter family of coordinate transformations

$$q_\sigma \to \tilde{q}_\sigma(q,\zeta) \quad , \quad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma$$

leaving L invariant corresponds a conserved "charge",

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \quad , \quad \frac{d\Lambda}{dt} = 0$$

Let us label the various one-parameter invariances with a label k. For small oscillations,

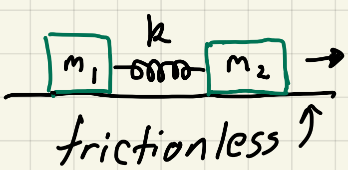
$$\frac{\partial L}{\partial \dot{q}_\sigma} = \frac{\partial L}{\partial \dot{\eta}_\sigma} = T_{\sigma\sigma'}\,\dot{\eta}_{\sigma'}$$

which says $C_{k\sigma} = \sum_{\sigma'} T_{\sigma\sigma'} \dfrac{\partial \tilde{q}_\sigma}{\partial \zeta_k}\bigg|_{\vec{\zeta}=0}$ so that

$$\xi_k = \sum_\sigma C_{k\sigma}\,\eta_\sigma$$

is a zero mode, satisfying $\ddot{\xi}_k = 0$. (As written it is unnormalized. Thus, in systems with continuous symmetries, associated with each such symmetry is a zero mode of the corresponding small oscillations problem.

Example 1 : $L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1 - a)^2$

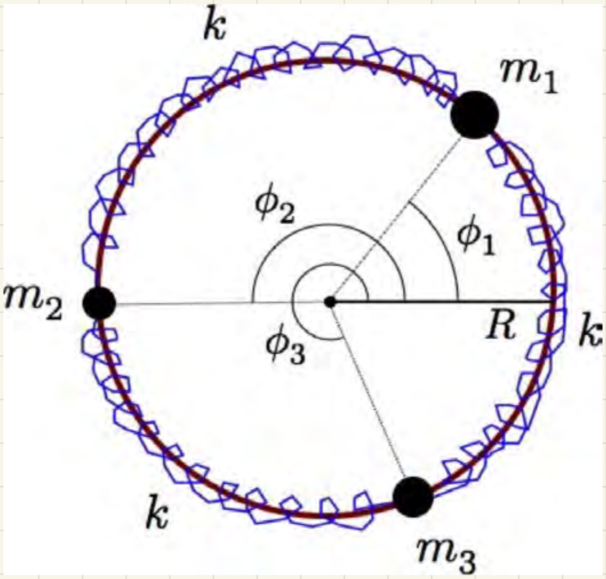


$$= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2}\mu \dot{x}^2 - \tfrac{1}{2} k (x-a)^2 \Rightarrow X \text{ (CM) is a ZM}$$

$$X = \tfrac{1}{2}(x_1 + x_2) \quad , \quad x = x_2 - x_1$$

Example 2

Consider the system to the right, for which



$$T = \frac{1}{2} R^2 \left(m_1 \dot{\phi}_1^2 + m_2 \dot{\phi}_2^2 + m_3 \dot{\phi}_3^2 \right)$$

and

$$U = \frac{1}{2} k R^2 \left[(\phi_2 - \phi_1 - \chi)^2 + (\phi_3 - \phi_2 - \chi)^2 + (2\pi + \phi_1 - \phi_3 - \chi)^2 \right]$$

where $\phi_3 - 2\pi < \phi_1 < \phi_2 < \phi_3 < \phi_1 + 2\pi$, and where $R\chi \equiv a$ is the unstretched length of each spring. The equilibrium configuration is

$$\phi_1^0 = \zeta \quad , \quad \phi_2^0 = \zeta + \frac{2\pi}{3} \quad , \quad \phi_3^0 = \zeta + \frac{4\pi}{3}$$

where ζ is an arbitrary continuous parameter, corresponding to the continuous translational symmetry that is present. Find

$$T = \begin{pmatrix} m_1 R^2 & 0 & 0 \\ 0 & m_2 R^2 & 0 \\ 0 & 0 & m_3 R^2 \end{pmatrix} \quad , \quad V = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix}$$

and

$$\omega^2 T - V = kR^2 \begin{pmatrix} \frac{\omega^2}{\nu_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\nu_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\nu_3^2} - 2 \end{pmatrix} \quad , \quad \nu_j^2 \equiv \frac{k}{m_j}$$

The characteristic polynomial is

$$P(\omega^2) = \det(\omega^2 T - V) \equiv (kR^2)^3 \cdot \tilde{P}(\omega^2)$$

$$\tilde{P}(\omega^2) = \frac{\omega^6}{\nu_1^2\nu_2^2\nu_3^2} - 2\left(\frac{1}{\nu_1^2\nu_2^2} + \frac{1}{\nu_2^2\nu_3^2} + \frac{1}{\nu_3^2\nu_1^2}\right)\omega^4 + 3\left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} + \frac{1}{\nu_3^2}\right)\omega^2$$

This is cubic in ω^2, but since there is no $(\omega^2)^0$ term, ω^2 divides $\tilde{P}(\omega^2)$, i.e. $\tilde{P}(\omega^2) = \omega^2\,\tilde{Q}(\omega^2)$, where $\tilde{Q}(\omega^2)$ is a quadratic function of its argument. Thus the normal mode frequencies are

$$\omega_1^2 = 0$$

$$\omega_{2,3}^2 = \nu_1^2 + \nu_2^2 + \nu_3^2 \pm \frac{1}{4}\sqrt{(\nu_1^2-\nu_2^2)^2 + (\nu_2^2-\nu_3^2) + (\nu_3^2-\nu_1^2)^2}$$

To find the modal matrix, set $(\omega_j^2 T - V)\,\psi^{(j)} = 0$:

$$\begin{pmatrix} \frac{\omega_j^2}{\nu_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\nu_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\nu_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0$$

which yields $\psi_\sigma^{(j)} = C_j \Big/ \left(3 - \frac{\omega_j^2}{\nu_\sigma^2}\right)$, where

$$C_j = \left[\sum_{\sigma=1}^{3} m_\sigma \left(3 - \frac{\omega_j^2}{\nu_\sigma^2}\right)^{-2}\right]^{-1/2}$$ for normalization.

Note for the zero mode ($j=1$) we have

$$A_{\sigma 1} = \psi_\sigma^{(1)} = \frac{C_1}{3} = (m_1+m_2+m_3)^{-1/2} \quad \forall\, \sigma \in \{1,2,3\}$$

Thus,

$$\xi_1 = A_{1\sigma} T_{\sigma\sigma'} \eta_{\sigma'}$$
$$= (m_1+m_2+m_3)^{-1/2} R^2 (m_1 \eta_1 + m_2 \eta_2 + m_3 \eta_3)$$

is the normalized zero mode. This is consistent with Noether's theorem, which says

$$\Lambda = \sum_{\sigma=1}^{3} \frac{\partial L}{\partial \dot{\phi}_\sigma} \frac{\partial \tilde{\phi}_\sigma}{\partial \zeta} = R^2 (m_1 \dot{\phi}_1 + m_2 \dot{\phi}_2 + m_3 \dot{\phi}_3)$$

with $\dot{\Lambda} = 0$. Note that $\dot{\Lambda} = 0$ always, and not only in the limit of small deviations from static equilibrium.

- Chain of identical masses and springs

tension

$$L = \frac{1}{2} m \sum_\sigma \dot{x}_\sigma^2 - \frac{1}{2} k \sum (x_{\sigma+1} - x_\sigma - a)^2 + \tau \sum_n (x_{\sigma+1} - x_\sigma)$$

Clearly $p_\sigma = \frac{\partial L}{\partial \dot{x}_\sigma} = m \dot{x}_\sigma$. If the chain is finite, with n running from 1 to N, then

$$F_1 = \frac{\partial L}{\partial x_1} = k(x_2 - x_1 - a) - \tau$$

$$F_N = \frac{\partial L}{\partial x_N} = -k(x_N - x_{N-1} - a) + \tau$$

$$F_\sigma = \frac{\partial L}{\partial x_\sigma} = k(x_{\sigma+1} + x_{\sigma-1} - 2x_j) \qquad \sigma \in \{2, \dots, N-1\}$$

The last equation says that $F_\sigma = 0 \;\forall\, \sigma \in \{1, \ldots, N\}$ if

$$x_{\sigma+1} - x_\sigma = b \quad , \quad \sigma \in \{1, \ldots, N-1\}$$

where b is a constant. Plugging this into the first equations then yields $b = a + k^{-1}\tau$.

If the chain is a periodic ring with $x_{N+1} \equiv x_1 + C$, then $b = C/N$ is the only solution. We'll solve the problem in this case of periodic boundary conditions (PBCs). In the limit $N \to \infty$, the bulk behavior wont differ between the two cases. Writing

$$x_\sigma = \sigma b + u_\sigma + \xi \qquad \sigma \in \{1, \ldots, N\}$$

we have

$u_{N+1} \equiv u_1$

$$L = \tfrac{1}{2} m \sum_{\sigma=1}^{N} \dot{u}_\sigma^2 - \tfrac{1}{2} k \sum_{\sigma=1}^{N} (u_{\sigma+1} - u_\sigma)^2 - k(b-a)C - \tfrac{1}{2} N k (b-a)^2$$

The last two terms arise when $b \neq a$ due to the fact that the springs are all (equally) stretched in the static equilibrium configuration. These terms are both constants which we henceforth drop. The EL equations are then

$$m\ddot{u}_\sigma = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_\sigma}\right) = \frac{\partial L}{\partial u_\sigma} = k(u_{\sigma+1} + u_{\sigma-1} - 2u_\sigma)$$

with $u_{N+1} \equiv u_1$. These N coupled ODEs may easily be solved

$$k(u_{\sigma+1} - u_\sigma) - k(u_\sigma - u_{\sigma-1})$$

σ σ

by transforming to Fourier space coordinates, viz.

$$u_\sigma = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i j\sigma/N} \hat{u}_j \iff \hat{u}_j = \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} u_\sigma$$

Note that $\hat{u}_j$ is complex, with

$$\hat{u}_{N-j} = \frac{1}{\sqrt{N}} \sum^{N} e^{2\pi i j\sigma/N} u_\sigma = \hat{u}_j^*$$

Let's count degrees of freedom. The set $\{u_1, \ldots, u_N\}$ constitutes N real degrees of freedom. For N even, $\hat{u}_N$ and $\hat{u}_{N/2}$ are real, while $\hat{u}_j$ for $j \in \{1, \ldots, \frac{1}{2}N-1\}$ are complex and satisfy $\mathrm{Re}\,\hat{u}_{N-j} = \mathrm{Re}\,\hat{u}_j$ and $\mathrm{Im}\,\hat{u}_{N-j} = -\mathrm{Im}\,\hat{u}_j$. The number of real degrees of freedom is then

$$\mathrm{DOF} = 2 + 2 \times \left(\tfrac{1}{2}N - 1\right) = N \quad \checkmark$$

If N is odd, then $\hat{u}_N$ is again real, but there is no mode $\hat{u}_j$ with $j = \frac{1}{2}N$. We again have $\hat{u}_{N-j} = \hat{u}_j^*$, this time for $j \in \{1, \ldots, \frac{1}{2}(N-1)\}$. The number of real degrees of freedom is

$$\mathrm{DOF} = 1 + 2 \times \tfrac{1}{2}(N-1) = N \quad \checkmark$$

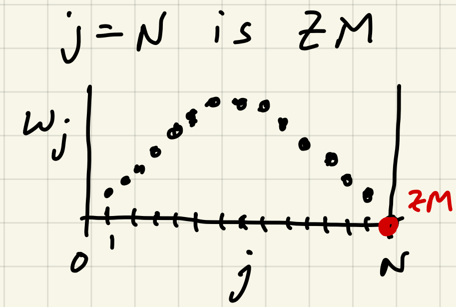
We now have

$$m \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} \ddot{u}_\sigma = k \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} (u_{\sigma+1} + u_{\sigma-1} - 2u_\sigma)$$

$$m\ddot{\hat{u}}_j = -2k\left[1 - \cos\left(\frac{2\pi j}{N}\right)\right] \hat{u}_j$$

Thus we may write $\ddot{\hat{u}}_j = -\omega_j^2 \hat{u}_j$ with

$$\omega_j = 2\sqrt{\frac{k}{m}}\,\left|\sin\left(\frac{\pi j}{N}\right)\right|$$



The solution for each normal mode is

$$\hat{u}_j(t) = C_j\, e^{-i\omega_j t}\, e^{i\delta_j}$$

where $C_{N-j} = C_j$ and $\delta_{N-j} = -\delta_j$ for all $j \notin \{\frac{N}{2}, N\}$, and $\delta_{N/2} = \delta_N = 0$. The $\{C_j, \delta_j\}$ are all real constants

The modal matrix is then $A_{\sigma j} = \frac{1}{\sqrt{Nm}}\, e^{2\pi i j\sigma/N}$, where we have now included the $m^{-1/2}$ factor. Note

$$T_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'}$$
$$V_{\sigma\sigma'} = 2k\,\delta_{\sigma\sigma'} - k\,\delta_{\sigma',\sigma+1} - k\,\delta_{\sigma',\sigma-1}$$

the Kronecker deltas are understood to be modulo N, i.e.

$$\delta_{\sigma\sigma'} = \begin{cases} 1 & \text{if } \sigma' = \sigma \bmod N \\ 0 & \text{otherwise} \end{cases}$$

Thus, the matrix forms of T and V are

$$T = \begin{pmatrix} m & 0 & & & \\ 0 & m & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & m & 0 \\ & & & 0 & m \end{pmatrix}, \quad V = \begin{pmatrix} 2k & -k & 0 & \cdots & 0 & -k \\ -k & 2k & -k & 0 & \cdots & 0 \\ 0 & -k & 2k & -k & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & -k & 2k & -k \\ -k & 0 & \cdots & 0 & -k & 2k \end{pmatrix}$$

Using the equation $\frac{1}{N}\sum_{\sigma=1}^{N} e^{2\pi i(j-j')\sigma/N} = \delta_{jj'}$ we can prove that $A^t T A = \mathbb{1}$ and $A^t V A = \mathrm{diag}(\omega_1^2, \dots, \omega_N^2)$.

Continuum limit: We take

$$u_\sigma(t) \to u(x=\sigma b, t)$$

and

$$u_{\sigma+1} - u_\sigma = u(x+b) - u(x) = b\,\frac{\partial u}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial x^2} + \dots$$

Thus,

$$T = \frac{1}{2}m\sum_\sigma \dot{u}_\sigma^2 \longrightarrow \frac{1}{2}m\int\frac{dx}{b}\left(\frac{\partial u}{\partial t}\right)^2$$

$$V = \frac{1}{2}k\sum_\sigma (u_{\sigma+1}-u_\sigma)^2 \longrightarrow \frac{1}{2}k\int\frac{dx}{b}\left(b\frac{\partial u}{\partial x}\right)^2 + \cdots$$

and we may write

$$S = \int dt\, L(\{u_\sigma\},\{\dot{u}_\sigma\},t) = \int dt\int dx\,\mathcal{L}(u,\partial_x u,\partial_t u,t)$$

where

$$\mathcal{L}(u,\partial_x u,\partial_t u,t) = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2$$

with $\rho = m/b$ = mass density and $\tau = kb$ = "tension" is the **Lagrangian density**. Suppose the Lagrangian is of the form

$$L = \sum_\sigma L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, t\Big)$$

We have

$$L = \sum_\sigma L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, t\Big)$$

The EL equns are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_\sigma}\right) = \frac{\partial L}{\partial u_\sigma} = \frac{\partial L_\sigma}{\partial u_\sigma} + \frac{1}{b}\frac{\partial L_{\sigma-1}}{\partial u'_\sigma} - \frac{1}{b}\frac{\partial L_\sigma}{\partial u'_\sigma}$$

Now

$$\frac{(\partial L_\sigma/\partial u'_\sigma) - (\partial L_{\sigma-1}/\partial u'_\sigma)}{b} = \frac{\partial}{\partial x}\frac{\partial L_\sigma}{\partial u'_\sigma} + \cdots$$

and writing

$$L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1}-u_\sigma}{b}, t\Big) \equiv \frac{1}{b}\mathcal{L}\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, \overbrace{\sigma b}^{x}, t\Big)$$

$$= \frac{1}{b}\mathcal{L}(u, \partial_t u, \partial_x u, x, t)$$

we have

$$S = \int dt \int dx\, \mathcal{L}(u, \partial_t u, \partial_x u, x, t)$$

and the equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \partial_t u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \partial_x u}\right) = \frac{\partial \mathcal{L}}{\partial u}$$

More about this in chapter 9 of the lecture notes.

Lecture 8 (Oct. 28)

- Small oscillations summary:

(0) linearize about equilibrium $\left.\dfrac{\partial U}{\partial q_\sigma}\right|_{\bar{q}} = 0$; $q_\sigma = \bar{q}_\sigma + \eta_\sigma$, $\sigma \in \{1, \ldots, n\}$

may be multiple solutions

(1) obtain T and V matrices:

$$T_{\sigma\sigma'} = \left.\frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}}\right|_{\bar{q}} \;,\; V_{\sigma\sigma'} = \left.\frac{\partial^2 U}{\partial q_\sigma \partial q_{\sigma'}}\right|_{\bar{q}}$$

both real, symmetric

Lagrangian is then

$$L = \tfrac{1}{2}\dot{\eta}_\sigma T_{\sigma\sigma'}\dot{\eta}_{\sigma'} - \tfrac{1}{2}\eta_\sigma V_{\sigma\sigma'}\eta_{\sigma'} + \cancel{\mathcal{O}(\eta^3, \eta^2\dot{\eta}, \ldots)}$$

(2) Solve $P(\omega) \equiv \det(\omega^2 T - V) = 0$ for normal mode frequencies ω_i^2. $P(\omega) = a_n\,\omega^{2n} + a_{n-1}\,\omega^{2(n-1)} + \ldots + a_0$.

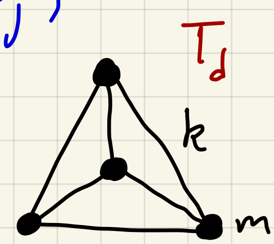
(3) For each ω_i^2, solve $(\omega_i^2 T - V)\vec{\psi}^{(i)} = 0$. The overall <u>length</u> of $\vec{\psi}^{(i)}$ is as yet undetermined.

(4) Necessarily, if $\omega_i^2 \neq \omega_j^2$, then

$$\langle \vec{\psi}^{(i)} | \vec{\psi}^{(j)} \rangle \equiv \psi_\sigma^{(i)} T_{\sigma\sigma'} \psi_{\sigma'}^{(j)} = 0 \quad (\omega_i^2 \neq \omega_j^2)$$

Degenerate eigenvalues: use Gram-Schmidt.

Now normalize: $\langle \vec{\psi}^{(i)} | \vec{\psi}^{(j)} \rangle = \delta_{ij}$



(5) Modal matrix is $A_{\sigma j} = \psi_\sigma^{(j)}$ $\qquad A^{-1} = A^t T$

Normal modes: $\eta_\sigma = A_{\sigma j}\,\xi_j$; $\xi_j = A^{-1}_{j\sigma}\,\eta_\sigma$

Also: $A^t T A = \mathbb{1}$, $A^t V A = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$

$4 \times 3 = 12$
$6 = 3 + 3$ ZM
$1 + 2 + 3$ remaining

(6) L in terms of normal modes: $\eta = A\xi$

$$L = \frac{1}{2}\dot{\eta}^t T \dot{\eta} - \frac{1}{2}\eta^t V \eta$$
$$= \frac{1}{2}\dot{\xi}^t (A^t T A)\dot{\xi} - \frac{1}{2}\xi^t (A^t V A)\xi$$
$$= \sum_{i=1}^{n} \frac{1}{2}\left(\dot{\xi}_i^2 - \omega_i^2 \xi_i^2\right) \Rightarrow \ddot{\xi}_j = -\omega_j^2 \xi_j$$

So the normal modes are decoupled!

(7) Solution:

$$\xi_j(t) = \xi_j(0)\cos\omega_j t + \omega_j^{-1}\dot{\xi}_j(0)\sin\omega_j t$$
$$\eta_\sigma(0) = A_{\sigma j}\xi_j(0), \quad \dot{\eta}_\sigma(0) = A_{\sigma j}\dot{\xi}_j(0)$$
$$\Rightarrow \quad \xi_j(0) = A^{-1}_{j\sigma}\eta_\sigma(0), \quad \dot{\xi}_j(0) = A^{-1}_{j\sigma}\dot{\eta}_\sigma(0)$$

$$\eta_\sigma(t) = A_{\sigma j}\xi_j(t)$$

$A^{-1} = A^t T$

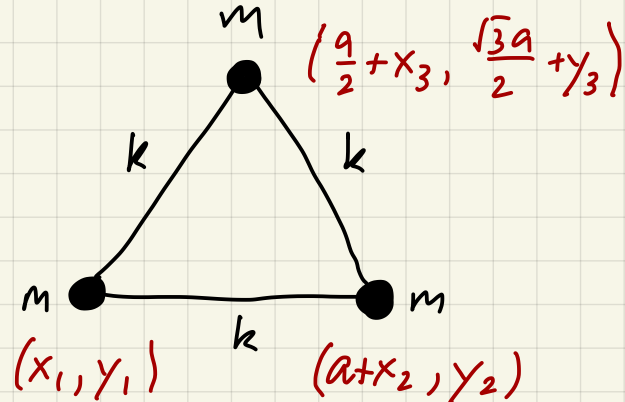
$$= \sum_{j,\sigma'} A_{\sigma j}\cos\omega_j t\, A^{-1}_{j\sigma'}\eta_{\sigma'}(0) + A_{\sigma j}\omega_j^{-1}\sin\omega_j t\, A^{-1}_{j\sigma'}\dot{\eta}_{\sigma'}(0)$$

$$W_{\sigma\sigma'}\ddot{\ddot{\eta}}_{\sigma'} + T_{\sigma\sigma'}\ddot{\eta}_{\sigma'} = V_{\sigma\sigma'}\eta_\sigma$$

eigenvectors

$$\eta_\sigma = \psi_\sigma e^{-i\omega t} \Rightarrow \underbrace{(\omega^4 W - \omega^2 T + V)}_{\det \equiv 0 \Rightarrow \omega_i^2}\vec{\psi} = 0$$

Planar triatomic molecule



DOF : 6 $\overbrace{\{x_1, y_1, x_2, y_2, x_3, y_3\}}^{\vec{\eta}}$

Equilibrium : $\{0,0,0,0,0,0\}$

KE is easy :

$$T = \frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2+\dot{x}_2^2+\dot{y}_2^2+\dot{x}_3^2+\dot{y}_3^2\right) = \frac{1}{2}m\sum_{\sigma=1}^{6}\dot{\eta}_\sigma^2$$

$$T_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'}$$

PE is more challenging : $U = \frac{1}{2}k\left[(d_{12}-a)^2+(d_{23}-a)^2 + (d_{13}-a)^2\right]$

$$d_{12}^2 = (a+x_2-x_1)^2+(y_2-y_1)^2$$
$$d_{23}^2 = \left(-\frac{a}{2}+x_3-x_2\right)^2+\left(\frac{\sqrt{3}}{2}a+y_3-y_2\right)^2$$
$$d_{12}^2 = \left(\frac{a}{2}+x_3-x_1\right)^2+\left(\frac{\sqrt{3}}{2}a+y_3-y_1\right)^2$$

Note : when $x_{1,2,3}=y_{1,2,3}=0$, $d_{ij}^2=a^2 \ \forall\, i\neq j$

Expand to linear order in gd's :

$$d_{12} = a + x_2 - x_1 + \dots$$
$$d_{23} = a - \frac{1}{2}(x_3-x_2) + \frac{\sqrt{3}}{2}(y_3-y_2) + \dots$$
$$d_{13} = a + \frac{1}{2}(x_3-x_1) + \frac{\sqrt{3}}{2}(y_3-y_1) + \dots$$

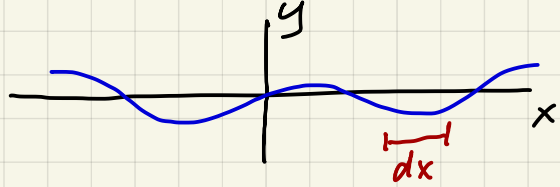
$$U = \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{8}k\left(x_2-x_3+\sqrt{3}\,y_3-\sqrt{3}\,y_2\right)^2$$
$$+\frac{1}{8}k\left(x_3-x_1+\sqrt{3}\,y_3-\sqrt{3}\,y_1\right)^2 + \mathcal{O}(\eta^3)$$

$$U = \frac{1}{2}k(\eta_3 - \eta_1)^2 + \frac{1}{8}k(\eta_3 - \eta_5 + \sqrt{3}\eta_6 - \sqrt{3}\eta_4)^2$$
$$+ \frac{1}{8}k(\eta_5 - \eta_1 + \sqrt{3}\eta_6 - \sqrt{3}\eta_2)^2 + \mathcal{O}(\eta^3)$$

$$V_{\sigma\sigma'} = \left.\frac{\partial^2 U}{\partial\eta_\sigma \partial\eta_{\sigma'}}\right|_{\bar{\eta}} = k\begin{pmatrix} 5/4 & & & \\ & \ddots & & \\ & & & \\ & & & \end{pmatrix}_{6\times6}$$

See §5.9.3 for complete solution.

Lecture 9 (Nov. 2)



System: string of mass density $\mu(x)$ and tension $\tau(x)$. Instantaneous shape is $y(x,t)$.

Differential KE :

$$dT = \frac{1}{2}\mu(x)\left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx$$

Differential PE (relative to $y(x,t) = $ const.) :

$$dU = \tau(x)\, d\ell = \tau(x)\left\{\underbrace{\sqrt{dx^2+dy^2}}_{d\ell} - dx\right\}$$

Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - \tau(x)\left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right]$$

Assuming $\left|\frac{\partial y}{\partial x}\right| \ll 1$, $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{1}{2}\tau y_x^2 + \dots$

Recall that for

$$S[y(x,t)] = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx\, \mathcal{L}(y, y_t, y_x ; x, t)$$

that

$$\delta S = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right)\right]\delta y$$

$$+ \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial y_t}\delta y\right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y_x}\delta y\right]_{x_a}^{x_b}$$

First let's consider what is necessary in order that the boundary terms both vanish. The first boundary term vanishes when $\delta y(x, t_a) = \delta y(x, t_b) = 0$. The second term vanishes when $\frac{\partial \mathcal{L}}{\partial y_x} \delta y$ vanishes at $x = x_{a,b}$ for all times t. For the case $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{1}{2}\tau y_x^2$, we have $\delta\mathcal{L}/\delta y_x = -\tau y_x$, thus, assuming $\tau(x_{a,b}) \neq 0$, the condition $y_x \, \delta y = 0$ at the end points means either (i) $y_x = 0$ or (ii) $\delta y = 0$ at each endpoint $x_{a,b}$. We then have the EL eqn,

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)}_{\to\ -\tau(x)\frac{\partial y}{\partial x} \text{ for our } \mathcal{L}} = 0$$

which for our case yields

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] = \mu(x)\frac{\partial^2 y}{\partial t^2}$$

This equation, plus the spatial boundary conditions, governs the dynamics of the string. The simplest case is when $\mu(x) = \mu$ and $\tau(x) = \tau$ are both constants, whence we obtain the **Helmholtz equation**,

$$\frac{1}{c^2} y_{tt} = y_{xx} \Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) y(x,t) = 0$$

with $c = (\tau/\mu)^{1/2}$, which has units of velocity. This equation may be solved completely, and for arbitrary boundary conditions.

D'Alembert's solution

Define the variables $u \equiv x - ct$ and $v \equiv x + ct$. Then

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}$$

Therefore

$$\underbrace{\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\text{wave operator}} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 - \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2$$

$$= 4\frac{\partial^2}{\partial u \partial v} = 4\frac{\partial}{\partial u}\frac{\partial}{\partial v}$$

Thus

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \Rightarrow y(u,v) = f(u) + g(v)$$

with $f(u)$ and $g(v)$ arbitrary functions as of yet. So:

$$y(x,t) = \underbrace{f(x-ct)}_{\text{right-mover}} + \underbrace{g(x+ct)}_{\text{left-mover}}$$

Now let's apply some initial conditions:

$$y(x,0) = f(x) + g(x)$$
$$c^{-1} y_t(x,0) = -f'(x) + g'(x)$$

Taking the spatial derivative of the first equation yields

$$y_x(x,0) = f'(x) + g'(x)$$

and thus we have

$$f'(\xi) = \frac{1}{2}\,y_x(\xi,0) - \frac{1}{2c}\,y_t(\xi,0)$$

$$g'(\xi) = \frac{1}{2}\,y_x(\xi,0) + \frac{1}{2c}\,y_t(\xi,0)$$

Now all we need to do is integrate $\int_0^{\xi} d\xi'$:

$$f(\xi) = \frac{1}{2}\,y(\xi,0) - \frac{1}{2c}\int_0^{\xi} d\xi'\,y_t(\xi',0) + C$$

$$g(\xi) = \frac{1}{2}\,y(\xi,0) + \frac{1}{2c}\int_0^{\xi} d\xi'\,y_t(\xi',0) - C$$

where $C = f(0) - \frac{1}{2}y(0,0) = \frac{1}{2}y(0,0) - g(0)$. Thus,

$$y(x,t) = \frac{1}{2}\left[y(x-ct,0) + y(x+ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\,y_t(\xi,0)$$

Thus we have a solution for all initial conditions.

Hamiltonian density

We define the momentum density as $\wp = \partial\mathcal{L}/\partial y_t$.
The Hamiltonian density is then $\mathcal{H} = \wp\, y_t - \mathcal{L}$.
Typically $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \mathcal{U}(y, y_x)$, hence $\wp = \mu y_t$ and

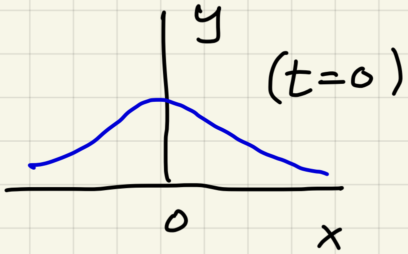
$$\mathcal{H} = \frac{\wp^2}{2\mu} + \mathcal{U}(y, y_x)$$

Expressed in terms of y_t rather than $\wp$, we have

Scratch

$$y(x,t) = \frac{1}{2}\left[y(x-ct,0) + y(x+ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi \, y_t(\xi,0)$$

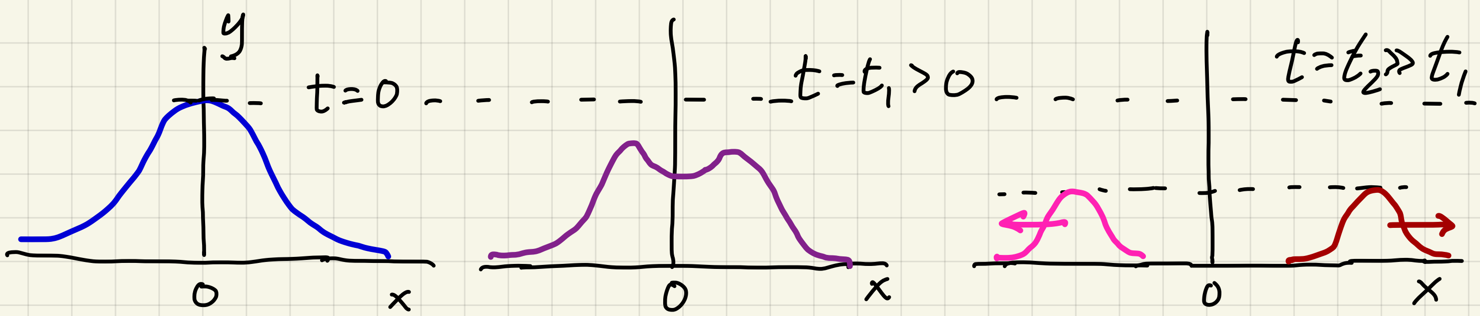
Suppose $y(x,0) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$, $y_t(x,0) = 0$.



Then:

$$y(x,t) = \frac{\gamma/2\pi}{(x-ct)^2+\gamma^2} + \frac{\gamma/2\pi}{(x+ct)^2+\gamma^2}$$

Evolution:



the energy density,

$$\mathcal{E}(x,t) = \tfrac{1}{2}\mu y_t^2 + \mathcal{U}(y, y_x; x)$$

The equations of motion are

$$-\frac{\partial \mathcal{U}}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) = 0$$

Now note that

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= \mu y_t y_{tt} + \frac{\partial \mathcal{U}}{\partial y} y_t + \frac{\partial \mathcal{U}}{\partial y_x} y_{xt} \\
&= \mu y_t y_{tt} - \mu y_t y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) y_t + \frac{\partial \mathcal{U}}{\partial y_x} y_{xt} \\
&= \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{U}}{\partial y_x} y_t\right] \equiv -\frac{\partial j_\varepsilon}{\partial x} \quad ; \quad j_\varepsilon \equiv -\frac{\partial \mathcal{U}}{\partial y_x} y_t
\end{aligned}$$

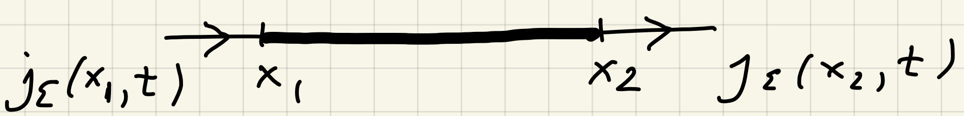
where j_ε is the energy current along the string. For the case $\mathcal{U} = \frac{1}{2}\tau y_x^2$, we have $j_\varepsilon = -\tau y_x y_t$.

Note that

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j_\varepsilon}{\partial x} = 0 \quad ; \quad [\mathcal{E}] = E L^{-1}, \quad [j_\varepsilon] = E T^{-1}$$

which is the **continuity equation** for energy. Thus,

$$\frac{d}{dt}\int_{x_1}^{x_2} dx\, \mathcal{E}(x,t) = -\int_{x_1}^{x_2} dx\, \frac{\partial j_\varepsilon(x,t)}{\partial x} = \underbrace{j_\varepsilon(x_1,t)}_{\text{rate in}} - \underbrace{j_\varepsilon(x_2,t)}_{\text{rate out}}$$



For $\mathcal{U} = \frac{1}{2}\tau y_x^2$ with $\mu(x) = \mu$ and $\tau(x) = \tau$ constant, writing $y(x,t) = f(x-ct) + g(x+ct)$ we find

$$\mathcal{E}(x,t) = \tau\left[f'(x-ct)\right]^2 + \tau\left[g'(x+ct)\right]^2$$
$$j_{\mathcal{E}}(x,t) = c\tau\left[f'(x-ct)\right]^2 - c\tau\left[g'(x+ct)\right]^2$$

which are each sums over right-moving and left-moving contributions.

Example: Klein-Gordon system $\mathcal{U}(y, y_x) = \frac{1}{2}\tau y_x^2 + \frac{1}{2}\beta y^2$

Then $\mathcal{E} = \frac{1}{2}\mu y_t^2 + \frac{1}{2}\tau y_x^2 + \frac{1}{2}\beta y^2$. Eqns of motion:

$$\mathcal{L} = \frac{1}{2}\mu y_t^2 - \mathcal{U}(y, y_x) \Rightarrow$$
$$-\frac{\partial \mathcal{U}}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) = 0$$
$$-\beta y - \mu y_{tt} + \tau y_{xx} = 0$$

Thus we have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) y = m^2 y \quad ; \quad m \equiv \sqrt{\frac{\beta}{\mu}}$$

This is not the Helmholtz eqn (it is the KG eqn). D'Alembert's solution does not pertain. Still,

$$j_{\mathcal{E}} = -\frac{\partial \mathcal{U}}{\partial y_x} y_t = -\tau y_x y_t$$

Momentum flux density and stress-energy tensor:

$$\mathcal{E} = \tfrac{1}{2}\mu y_t^2 + \tfrac{1}{2}\tau y_x^2 \Rightarrow \frac{\partial \mathcal{E}}{\partial x} = \frac{\partial}{\partial t}(\mu y_t y_x)$$

Thus, with

momentum current

momentum flux density

$$J_\pi \equiv \mathcal{E} \quad , \quad \Pi \equiv -\mu y_t y_x = \frac{J_\varepsilon}{c^2}$$

we may write

$$\underbrace{\left(\frac{1}{c}\frac{\partial}{\partial t} \, , \, \frac{\partial}{\partial x}\right)}_{\partial/\partial x^\mu} \overbrace{\begin{pmatrix} c\mathcal{E} & -c\Pi \\ J_\varepsilon & -J_\pi \end{pmatrix}}^{T^\mu{}_\nu} = 0$$

or $\partial_\mu T^\mu{}_\nu = 0$, where $T^\mu{}_\nu$ is the **stress-energy tensor.** Note that while Π and $g = \mu y_t$ have the same dimensions, Π is the momentum density <u>along</u> the string while g is the momentum density <u>transverse</u> to the string. General result:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu y)} \partial_\nu y - \delta^\mu_\nu \mathcal{L}$$

This satisfies $\partial_\mu T^\mu{}_\nu = 0$ for all ν.

Electromagnetism: $\mathcal{E} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \Rightarrow$

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right)$$

$$= \frac{1}{4\pi}\vec{E}\cdot(c\,\vec{\nabla}\times\vec{\nabla} - 4\pi\vec{J}) + \frac{1}{4\pi}\vec{B}\cdot(-c\,\vec{\nabla}\times\vec{E})$$

$$= -\vec{E}\cdot\vec{J} - \vec{\nabla}\cdot\vec{S}$$

where $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$ = Poynting vector. The stress-energy tensor is

$$T^\mu{}_\nu = \begin{pmatrix} \mathcal{E} & -c^{-1}S_x & -c^{-1}S_y & -c^{-1}S_z \\ c^{-1}S_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ c^{-1}S_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ c^{-1}S_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

with

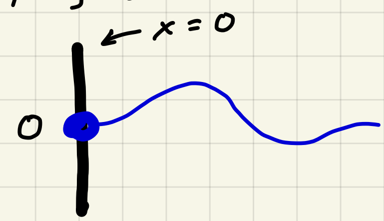
$$\sigma_{ij} = \frac{1}{4\pi}\left\{ -E_iE_j - B_iB_j + \frac{1}{2}(\vec{E}^2+\vec{B}^2)\delta_{ij}\right\}$$

which is the **Maxwell stress tensor**. Now

$$\partial_\mu T^\mu{}_\nu = 0 \quad ; \quad \partial_\mu = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)$$

- Reflection at an interface

Consider a semi-infinite string with $x \in [0,\infty]$ and with $y(0,t) = 0 \;\forall t$. We write



$$y(x,t) = f(x-ct) + g(x+ct)$$

and impose the boundary condition at $x=0$:

$$f(-ct) + g(ct) = 0 \Rightarrow f(\xi) = -g(-\xi) \;\forall \xi$$

$\xi \equiv -ct$

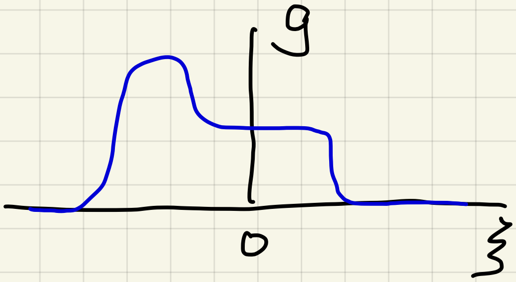
Therefore, we have

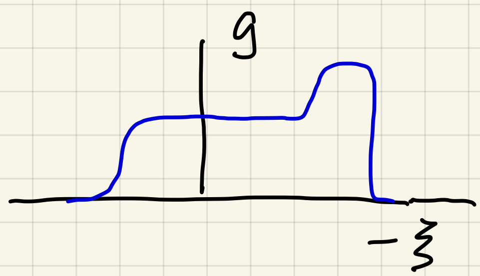
$$y(x,t) = g(ct+x) \underbrace{- g(ct-x)}_{f(x-ct)}$$

This is the general solution. Now suppose $g(\xi)$ resembles a pulse localized around $\xi \approx 0$.

In the distant past, $t \to -\infty \Rightarrow ct - x \to -\infty$

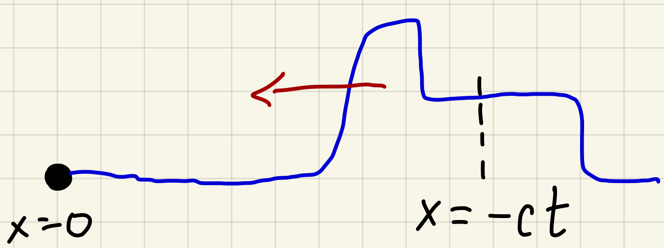
Hence no contribution from right mover.





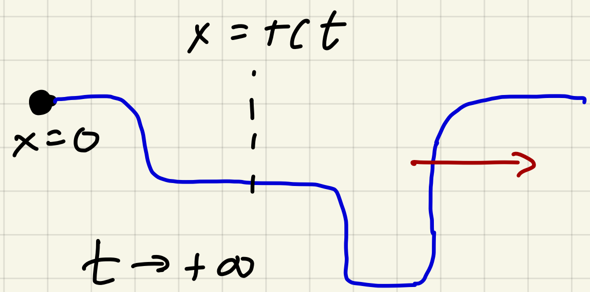
How about the left-mover? Set $ct + x \approx 0 \Rightarrow x \approx -ct \in [0, \infty]$. I.e. incoming left-mover at $x \approx -ct$. For $t \to +\infty$, $ct + x \to +\infty \Rightarrow$ left-mover is gone. $ct - x \approx 0 \Rightarrow x \approx ct \in [0, \infty]$

I.e. outgoing right mover at $x \approx ct$. Sketch:



$t \to -\infty$

incident wave



$t \to +\infty$

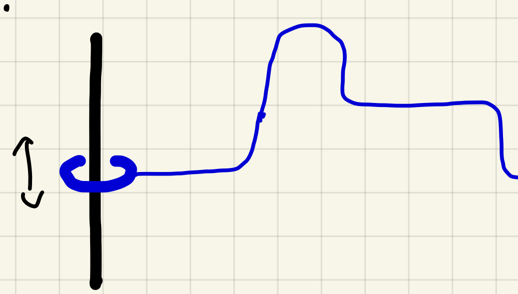
reflected wave

Suppose instead $y_x(0,t) = 0 \;\forall t$.

From $\delta S = \cdots - \dfrac{\partial \mathcal{L}}{\partial y_x} \delta y \Big|_0$

must vanish

free



$$\partial \mathcal{L}/\partial y_x = -\tau y_x \Rightarrow y_x(0,t) = 0 \quad \forall t$$

Shape of string:

$$y(x,t) = f(x-ct) + g(x+ct)$$
$$y_x(x,0) = f'(-ct) + g'(ct)$$

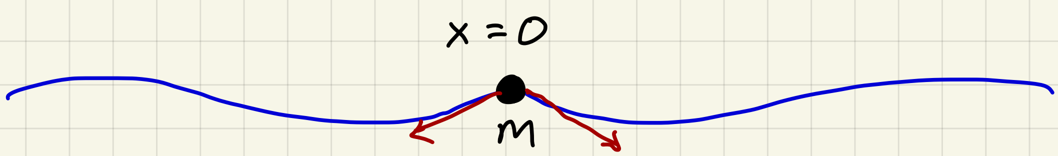
Thus $f'(\xi) = -g'(-\xi)$. Integrate to get

$$f(\xi) = g(-\xi)$$

So the shape is

$$y(x,t) = g(ct+x) + g(ct-x)$$
$$y_x(x,t) = g'(ct+x) - g'(ct-x)$$
$$= 0 \text{ when } x = 0$$

- Mass point on a string:



$x<0$: $y(x,t) = f(ct-x) + g(ct+x)$

$x>0$: $y(x,t) = h(ct-x)$

Interpretation: f = incident wave

g = reflected wave

h = transmitted wave

Newton's law for mass at $x=0$:

$$m\ddot{y}(0,t) = \tau y'(0^+,t) - \tau y'(0^-,t)$$

Discontinuous $y'(0,t) = y_x(0,t) \Rightarrow$ acceleration of m.

Furthermore:

$$y'(0^-,t) = -f'(ct) + g'(ct)$$
$$y'(0^+,t) = h'(ct)$$

Continuity $\Rightarrow y(0^-,t) = y(0^+,t) \Rightarrow$

$$h(ct) = f(ct) + g(ct)$$

Let $\xi = ct \Rightarrow$

$$h(\xi) = f(\xi) + g(\xi)$$
$$f''(\xi) + g''(\xi) = -\frac{2\tau}{mc^2} g'(\xi)$$

From these, get $g(\xi)$ and $h(\xi)$ in terms of $f(\xi)$.

Fourier transforms:

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} \quad , \quad \hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ik\xi}$$

Derivatives wrt ξ replaced by $ik \times \hat{f}(k)$ etc.

Then we have

$$(-k^2 + iQk)\hat{g}(k) = k^2 \hat{f}(k)$$

$$\hat{h}(k) = \hat{f}(k) + \hat{g}(k)$$

with $Q \equiv 2\tau/mc^2 = 2\mu/m$; $[Q] = L^{-1}$.

Solution:

$$\hat{g}(k) = \hat{r}(k)\,\hat{f}(k) \quad , \quad \hat{h}(k) = \hat{t}(k)\,\hat{f}(k)$$

with

$$\hat{r}(k) = -\frac{k}{k-iQ} \quad , \quad \hat{t}(k) = -\frac{iQ}{k-iQ}$$

Note $t = 1+r$ since $h = f+g$.

Shape of transmitted wave:

$$h(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\hat{t}(k)\,\hat{f}(k)$$

$$= \int_{-\infty}^{\infty} d\xi'\; t(\xi-\xi')\,f(\xi')$$

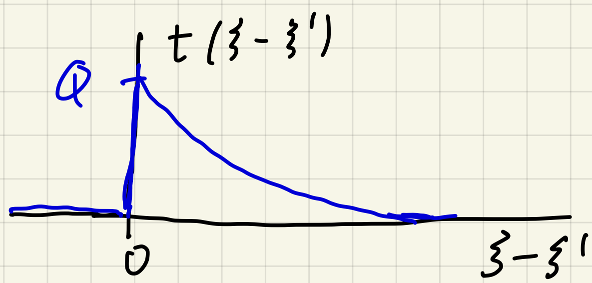
$$t(\xi-\xi') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\hat{t}(k)\,e^{ik(\xi-\xi')}$$

and for

$$\hat{t}(k) = -\frac{iQ}{k-iQ}$$

find

$$t(\xi-\xi') = Q\,e^{-Q(\xi-\xi')}\,\Theta(\xi-\xi')$$



Lecture 10 (Nov. 4)

Recall we were discussing the dynamics of a string (mass density μ, tension τ) with an attached point mass m at $x=0$. We wrote

$$y(x,t) = \underbrace{f(ct-x)}_{\text{incident}} + \underbrace{g(ct+x)}_{\text{reflected}} \qquad (x<0)$$
$$= \underbrace{h(ct-x)}_{\text{transmitted}} \qquad (x>0)$$

At $x=0$, we have $F=ma$ for the mass point, i.e.

$$m\ddot{y}(0,t) = \tau y'(0^+,t) - \tau y'(0^-,t)$$

as well as continuity $y(0^-,t) = y(0^+,t)$. Expressed in terms of the functions $f, g,$ and h, we have

$$f''(\xi) + g''(\xi) = -\frac{2\tau}{mc^2} g'(\xi)$$
$$f(\xi) + g(\xi) = h(\xi)$$

which we solved by going to Fourier space:

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k) e^{ik\xi} \quad , \quad \hat{f}(k) = \int_{-\infty} d\xi\, f(\xi) e^{-ik\xi}$$

etc. Note $\hat{f}(-k) = \hat{f}(k)^*$ since $f(\xi) \in \mathbb{R}$. We found

$$\hat{g}(k) = \hat{r}(k)\hat{f}(k) \quad , \quad \hat{h}(k) = \hat{t}(k)\hat{f}(k)$$

where, with $Q = \dfrac{2\tau}{mc^2} = \dfrac{2\mu}{m}$, $\qquad [Q] = L^{-1}$

$$\hat{r}(k) = -\frac{k}{k - iQ} \quad , \quad \hat{t}(k) = -\frac{iQ}{k - iQ}$$

are, respectively, the **reflection** and **transmission amplitudes**. Note that $\hat{t}(k) = 1 + \hat{r}(k)$, which follows directly from the continuity relation $h = f + g$. Another result is that

$$|\hat{r}(k)|^2 + |\hat{t}(k)|^2 = 1$$

$\hat{r}(k) = -1$

$\hat{t}(k) = 0$ $\qquad m = \infty$

We call $R(k) \equiv |\hat{r}(k)|^2$ and $T(k) \equiv |\hat{t}(k)|^2$ the reflection and transmission **coefficients**. These are the modulus squared, respectively, of the reflection and transmission **amplitudes**. By the way, note that $\hat{r}(-k) = \hat{r}(k)^*$ and $\hat{t}(-k) = \hat{t}(k)^*$.

Energy

The energy in the string is

$$\begin{aligned} E_{\text{string}}(t) &= \int\limits_{-\infty}^{\infty} dx \left\{ \tfrac{1}{2}\mu \dot{y}^2 + \tfrac{1}{2}\tau y'^2 \right\} \\ &= \tau \int\limits_{ct}^{\infty} d\xi \, [f'(\xi)]^2 + \tau \int\limits_{-\infty}^{ct} d\xi \left([g'(\xi)]^2 + [h'(\xi)]^2 \right) \end{aligned}$$

The total energy of the system is $E = E_{\text{string}} + E_{\text{mass}}$, with

$$E_{\text{mass}}(t) = \tfrac{1}{2} mc^2 \left[h'(ct) \right]^2$$

Scratch

$$E_{\text{string}}(t) = \int_{-\infty}^{0} dx \left\{ \frac{1}{2}\mu \left[c f'(ct-x) + c g'(ct+x) \right]^2 + \frac{1}{2}\tau \left[-f'(ct-x) + g'(ct+x) \right]^2 \right\} + \int_0^\infty dx \, \frac{1}{2}(\mu c^2 + \tau) \left[h'(ct) \right]^2$$

But $\mu c^2 = \tau$! Thus

$$E_{\text{string}}(t) = \tau \int_{-\infty}^{0} dx \left\{ \left[f'(ct-x) \right]^2 + \left[g'(ct+x) \right]^2 \right\} + \tau \int_0^\infty dx \left[h'(ct-x)^2 \right]$$

$$= \tau \int_{ct}^{\infty} d\xi \left[f'(\xi) \right]^2 + \tau \int_{-\infty}^{ct} d\xi \left(\left[g'(\xi) \right]^2 + \left[h'(\xi) \right]^2 \right)$$

$x < 0$: $\quad \xi = ct - x \in [ct, \infty]$

$\xi = ct + x \in [-\infty, ct]$

$x > 0$: $\quad \xi = ct - x \in [-\infty, ct]$

$$\int_{-\infty}^{\infty} d\xi \left[f'(\xi) \right]^2 = \int_{-\infty}^{\infty} d\xi \left[\frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} \right] \left[\frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \hat{f}^*(k') e^{-ik'\xi} \right]$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} (ik)(-ik') \hat{f}(k) \hat{f}^*(k') \underbrace{\int_{-\infty}^{\infty} d\xi \, e^{i(k-k')\xi}}_{2\pi\delta(k-k')}$$

Let's evaluate the total energy in the limits $t \to \pm\infty$. For $|t| \to \infty$, $E_{mass} \to 0$ because we assume the mass starts from rest, and by late times it has shaken off all the energy it acquired into vibrations of the string. So we have

$$E_{string}(-\infty) = \tau \int_{-\infty}^{\infty} d\xi \, [f'(\xi)]^2 = \tau \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 |\hat{f}(k)|^2$$

$$E_{string}(+\infty) = \tau \int_{-\infty}^{\infty} d\xi \left([g'(\xi)]^2 + [h'(\xi)]^2 \right) = \tau \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 \left(|\hat{g}(k)|^2 + |\hat{h}(k)|^2 \right)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 \underbrace{\left(|\hat{r}(k)|^2 + |\hat{t}(k)|^2 \right)}_{=1} |\hat{f}(k)|^2 = E_{string}(-\infty)$$

In fact, we can show with a bit more work that $E(t) = E_{string}(-\infty)$ for all times $t \in \mathbb{R}$, including the contribution from $E_{mass}(t)$. I.e. total energy is conserved.

- Back to real space!

We have

$$h(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) \hat{f}(k) e^{ik\xi} = \int_{-\infty}^{\infty} d\xi' \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) e^{ik(\xi - \xi')} \right] f(\xi')$$

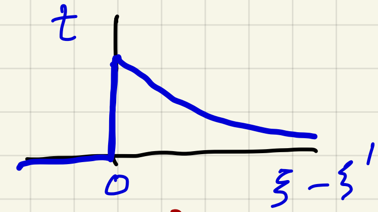
$$= \int_{-\infty}^{\infty} d\xi' \, t(\xi - \xi') f(\xi')$$

where $t(\xi - \xi') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) e^{ik(\xi - \xi')}$

is the transmission kernel in real space. For our case,

$$\hat{t}(k) = \frac{-iQ}{k-iQ} \Rightarrow t(\xi-\xi') = Q e^{-Q(\xi-\xi')}\,\Theta(\xi-\xi')$$

Note that for a δ-function pulse $f(\xi) = C\delta(\xi)$ we have that

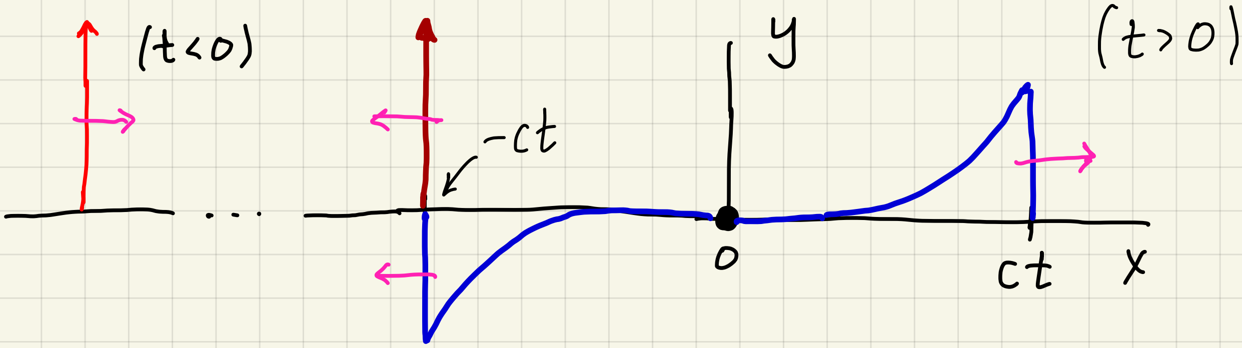


$$f(\xi) = C\,\delta(\xi) \Rightarrow h(\xi) = C\,t(\xi)$$
$$g(\xi) = C\{\delta(\xi) - t(\xi)\}$$

So for our example,

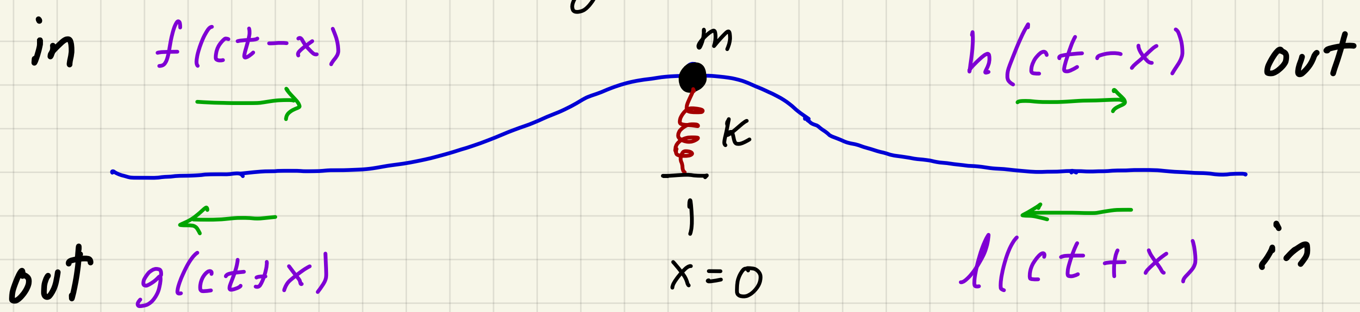
$$h(ct-x) = CQ\,e^{-Q(ct-x)}\,\Theta(ct-x)$$

so the late time shape of $y(x,t)$ looks like this



- S-matrix

Consider a more general state of affairs:



Continuity at $x=0$ says $f(\xi)+g(\xi)=h(\xi)+\ell(\xi)$.
Newton's law $F=ma$ for the mass point is now

$$m\ddot{y}(0,t) = \tau\left[y'(0^+,t) - y'(0^-,t)\right] - K\,y(0,t)$$

which says

$$mc^2\left[f''(\xi)+g''(\xi)\right] = \tau\left[\ell'(\xi)-h'(\xi)-g'(\xi)+f'(\xi)\right] - K\left[f(\xi)+g(\xi)\right]$$

Now take the FT:

$$\hat{f}(k)+\hat{g}(k) = \hat{h}(k)+\hat{\ell}(k)$$

$$-mc^2k^2\left[\hat{f}(k)+\hat{g}(k)\right] = i\tau k\left[\hat{\ell}(k)-\hat{h}(k)-\hat{g}(k)+\hat{f}(k)\right] - K\left[\hat{f}(k)+\hat{g}(k)\right]$$

Divide now by $\frac{1}{2}mc^2$, with

$$Q \equiv \frac{2\tau}{mc^2} \quad , \quad P^2 \equiv \frac{K}{mc^2}$$

units: $[Q]=[P]=L^{-1}$

to obtain (suppressing k in $\hat{f}(k)$ etc.)

$$-k^2\left[\hat{f}+\hat{g}+\hat{h}+\hat{\ell}\right] = iQk\left[\hat{\ell}-\hat{h}-\hat{g}+\hat{f}\right] - P^2\left[\hat{f}+\hat{g}+\hat{h}+\hat{\ell}\right]$$

The **S-matrix** relates outgoing states ($\hat{h}$ and $\hat{g}$) to the incoming ones ($\hat{f}$ and $\hat{\ell}$). We have

$$\text{(i)} \quad \hat{f}-\hat{\ell} = \hat{h}-\hat{g}$$

and

$$\text{(ii)}\quad \overbrace{(k^2+iQk-P^2)}^{\Lambda(k)}(\hat{f}+\hat{\ell}) = -\overbrace{(k^2-iQk-P^2)}^{\Lambda^*(k)}(\hat{h}+\hat{g})$$

In matrix form,

$$\begin{pmatrix} 1 & -1 \\ \Lambda & \Lambda \end{pmatrix}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -\Lambda^* & -\Lambda^* \end{pmatrix}\begin{pmatrix} \hat{h} \\ \hat{g} \end{pmatrix}$$

where $\Lambda(k) \equiv k^2+iQk-P^2$. Thus

$$\begin{pmatrix} \hat{h} \\ \hat{g} \end{pmatrix} = -\frac{1}{2\Lambda^*}\begin{pmatrix} -\Lambda^* & 1 \\ \Lambda^* & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \Lambda & \Lambda \end{pmatrix}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix}$$

$$= \underbrace{-\frac{1}{2\Lambda^*}\begin{pmatrix} \Lambda-\Lambda^* & \Lambda+\Lambda^* \\ \Lambda+\Lambda^* & \Lambda-\Lambda^* \end{pmatrix}}_{S(k) \,=\, \text{"scattering matrix"}}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix}$$

Hence

$$S(k) = \begin{pmatrix} \hat{t}(k) & \hat{r}'(k) \\ \hat{r}(k) & \hat{t}'(k) \end{pmatrix}$$

with

$$\hat{r}(k) = \hat{r}'(k) = -\frac{k^2-P^2}{k^2-iQk-P^2} \xrightarrow[P\to 0]{} \frac{-k}{k-iQ}$$

$$\hat{t}(k) = \hat{t}'(k) = -\frac{iQk}{k^2-iQk-P^2} \xrightarrow[P\to 0]{} \frac{-iQ}{k-iQ}$$

Here $\hat{r}=\hat{r}'$ and $\hat{t}=\hat{t}'$ due to time-reversal symmetry.

Note: (i) $\hat{t}(k) = 1 + \hat{r}(k)$

(ii) $|\hat{r}(k)|^2 + |\hat{t}(k)|^2 = 1$

The first of these again comes from continuity of $y(x,t)$ at $x=0$, which says

$$f(\xi) + g(\xi) = h(\xi) + \ell(\xi) \Rightarrow \hat{f}(k) + \hat{g}(k) = \hat{h}(k) + \hat{\ell}(k)$$

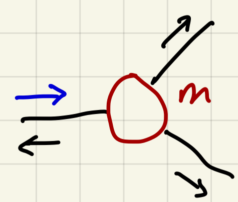
But since $\hat{h} = \hat{t}\hat{f} + \hat{r}'\hat{\ell}$ and $\hat{g} = \hat{r}\hat{f} + \hat{t}'\hat{\ell}$ we have

$$(1 + \hat{r} - \hat{t})\hat{f} = (1 - \hat{r}' - \hat{t}')\hat{\ell}$$

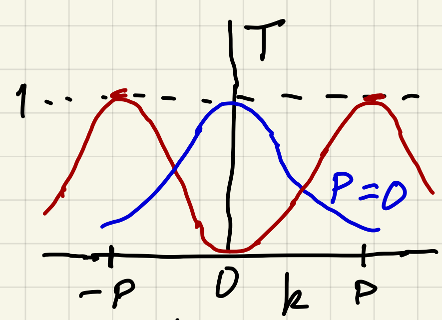
Since the inputs $\hat{f}$ and $\hat{\ell}$ are arbitrary, we must have

$$\hat{t}(k) = 1 + \hat{r}(k) \quad , \quad \hat{t}'(k) = 1 + \hat{r}'(k)$$

for all values of k. The reflection and transmission coefficients are



$$R+T=1 \begin{cases} R(k) = |\hat{r}(k)|^2 = \dfrac{(k^2-P^2)^2}{(k^2-P^2)^2 + Q^2k^2} \\[2ex] T(k) = |\hat{t}(k)|^2 = \dfrac{Q^2k^2}{(k^2-P^2)^2 + Q^2k^2} \end{cases}$$



Note that setting $P \to 0$ recovers our previous results.

Also note that maximizing $T(k)$ with respect to k yields $k^2 = P^2$, and that $T(k=\pm P) = 1$.

- Finite strings: Bernoulli's method

Let $x_L = 0$ and $x_R = L$, with $y(0,t) = y(L,t) = 0$ (fixed ends). Again we write

$$y(x,t) = f(x-ct) + g(x+ct)$$

Invoking the BC at $x=0$ yields $f(\xi) = -g(-\xi)$, hence we have

$$y(x,t) = g(ct+x) - g(ct-x)$$

We next demand $y(L,t) = 0$, which yields

$$g(ct+L) = g(ct-L) \Rightarrow g(\xi+2L) = g(\xi)$$

which says that $g(\xi)$ is periodic with period $2L$. Any such periodic function may be expressed as a Fourier series, viz.

$$g(\xi) = \sum_{n=1}^{\infty} \left\{ \tilde{A}_n \cos\left(\frac{n\pi\xi}{L}\right) + \tilde{B}_n \sin\left(\frac{n\pi\xi}{L}\right) \right\}$$

The full, time-dependent solution is then given by

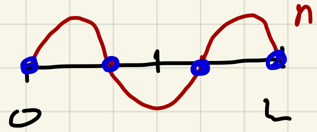
$$A_n \equiv \sqrt{2\mu L}\,\tilde{B}_n$$
$$B_n \equiv -\sqrt{2\mu L}\,\tilde{A}_n$$

$$y(x,t) = g(ct+x) - g(ct-x)$$
$$= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ \underbrace{A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)}_{\equiv\, C_n \cos\left(\frac{n\pi c t}{L} + \phi_n\right)} \right\}$$

We define

$$k_n \equiv \frac{n\pi}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad \psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$$

for $n \in \{1, 2, \ldots, \infty\}$. Thus, $\psi_n(x) = (2/\mu L)^{1/2} \sin(k_n x)$ has $(n+1)$ nodes, located at $x_{j,n} = jL/n$, for $j \in \{0, \ldots, n\}$. We further define the inner product,



$$\langle \phi \,|\, \chi \rangle \equiv \mu \int_0^L dx\, \phi(x)\, \chi(x)$$

where ϕ and χ are real functions of $x \in [0, L]$ that satisfy $\phi(0) = \phi(L) = \chi(0) = \chi(L) = 0$. Our basis functions $\psi_n(x)$ are orthonormal with respect to this IP:

$$\langle \psi_m | \psi_n \rangle = \frac{2}{L} \int_0^L dx\, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \delta_{mn}$$

Furthermore, this basis is <u>complete</u>, i.e.

$$\mu \sum_{n=1}^{\infty} \psi_n(x)\, \psi_n(x') = \delta(x - x')$$

We may express the constants $\{A_n, B_n\}$ in terms of our initial conditions, viz.

$$y(x,0) = \sum_n A_n \psi_n(x) \quad , \quad \dot{y}(x,0) = \sum_{n=1}^{\infty} \omega_n B_n \psi_n(x)$$

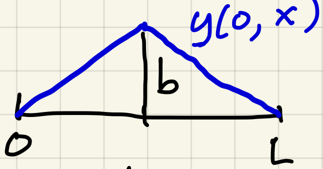
Multiplying by $\mu\psi_m(x)$ and integrating over $[0,L]$,

$$A_m = \mu \int_0^L dx\, y(x,0)\, \psi_m(x) \quad , \quad B_m = \mu\omega_n^{-1} \int_0^L dx\, \dot{y}(x,0)\, \psi_m(x)$$

Example : $y(x,0) = \begin{cases} 2bx/L & \text{if } x \in [0, \frac{1}{2}L] \\ 2b(L-x)/L & \text{if } x \in [\frac{1}{2}L, L] \end{cases}$

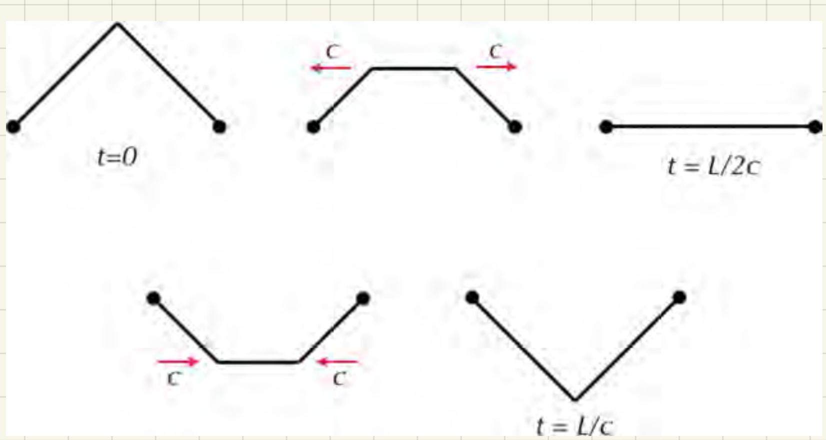
and $\dot{y}(x,0) = 0$ (release string from rest). Find

$$A_n = (2\mu L)^{1/2} \frac{4b}{\pi^2 n^2} \sin\left(\tfrac{1}{2}n\pi\right)$$



i.e. $A_{2k} = 0$ and $A_{2k+1} = (2\mu L)^{1/2} \cdot \frac{4b}{\pi^2} \cdot \frac{(-1)^k}{(2k+1)^2}$.

Also $B_n = 0 \;\forall n$. Note that $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is odd under reflection about the midpoint $x = \frac{L}{2}$, whereas our initial condition $y(x,0) = y(L-x,0)$ was even. Here's a set of images of the evolution:



This is the d'Alembert solution, extending $g(x)$ to the entire real line, with $g(x) = g(x+2L) = -g(-x)$.

Lecture 11 (Nov. 9)

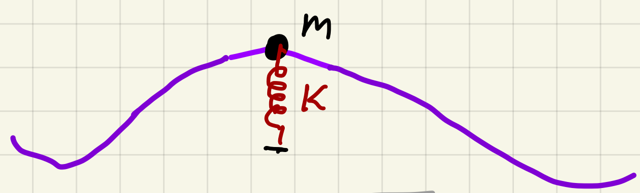
Start with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu(x)\dot{y}^2 - \frac{1}{2}\tau(x)y'^2 - \frac{1}{2}v(x)y^2$$

The last term corresponds to a harmonic potential attracting the string at each x value to $(x, y=0)$. In fact, if

$$\mu(x) = \mu_0 + m\delta(x) \quad , \quad v(x) = K\delta(x)$$

then we recover the problem of a string with an attached point mass that is connected to the point $(0,0)$ by a spring. The EL equations are found to be



$$\boxed{-\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] + v(x)\,y = -\mu(x)\frac{\partial^2 y}{\partial t^2}}$$

This equation is time-translation invariant because the coefficients are autonomous (i.e. $\tau(x)$, $v(x)$, and $\mu(x)$ do not depend on time t). This means that the partial differential operator (PDO)

$$\hat{Q} = \mu(x)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\tau(x)\frac{\partial}{\partial x} + v(x)$$

for which $\hat{Q}\, y(x,t) = 0$, commutes with the PDO $\partial/\partial t$: $[\hat{Q}, \partial/\partial t] = 0$. This means that the solutions to $\hat{Q}\, y(x,t) = 0$ may be written as

$$y(x,t) = \psi(x)\, e^{-i\omega t}$$

Furthermore, since $y^*(x,t)$ is a solution, then we may write

$$y(x,t) = \psi(x) \cos(\omega t + \phi)$$

We are left with the equation

$$\hat{K}\, \psi(x) = \mu(x)\, \omega^2 \psi(x)$$

where

$$\hat{K} = -\frac{d}{dx}\, \tau(x) \frac{d}{dx} + v(x)$$

is an ordinary differential operator (ODO). The equation

$$\hat{K}\, \psi(x) = -\frac{d}{dx}\left[\tau(x) \frac{d\psi(x)}{dx}\right] + v(x)\, \psi(x) = \mu(x) \omega^2 \psi(x)$$

is known as the **Sturm-Liouville equation.**

The simplest example is when $\tau(x) = \tau$ and $\mu(x) = \mu$ are constants, and $v(x) = 0$. Then $\hat{K} = -\tau \frac{d^2}{dx^2}$, and the solutions to the SL eqn are of the form

$$\psi(x) = A e^{ikx}$$

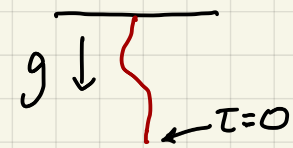
where $k^2 = \mu \omega^2/\tau = \omega^2/c^2$ with $c = (\tau/\mu)^{1/2}$ = wave speed.

I.e. $\psi(x) = A e^{\pm i\omega x/c}$, so $y(x) = f(ct-x) + g(ct+x)$.

- **Boundary conditions** – We consider four classes:

① Fixed endpoints : $\psi(x) = 0$ for $x = x_{L,R}$

② Natural : $\tau(x)\psi'(x) = 0$ for $x = x_{L,R}$



③ Periodic : $\psi(x+L) = \psi(x)$ where $L = x_R - x_L$
[Also require $\tau(x) = \tau(x+L)$.]

④ Mixed homogeneous : $\alpha\psi(x) + \beta\psi'(x) = 0$ for $x = x_{L,R}$
[Same α, β at both endpoints.]

- **Eigenfunction properties** :

The SL equation is an eigenvalue equation:

$$-\frac{d}{dx}\Big(\tau(x)\,\psi_n'(x)\Big) + v(x)\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x) \qquad (A)$$

for a given choice of BCs. Suppose we have a second solⁿ,

$$-\frac{d}{dx}\Big(\tau(x)\,\psi_m'(x)\Big) + v(x)\,\psi_m(x) = \omega_m^2\,\mu(x)\,\psi_m(x) \qquad (B)$$

Multiply (B) by $\psi_n^*(x)$ and (A^*) by $\psi_m(x)$ and subtract :

$$\psi_n^* \frac{d}{dx}\left[\tau \psi_m'\right] - \psi_m \frac{d}{dx}\left[\tau \psi_n^{*\prime}\right] = (\omega_n^{*2} - \omega_m^2)\,\mu\,\psi_m\,\psi_n^*$$

$$= \frac{d}{dx}\left[\tau\,\psi_n^*\psi_m' - \tau\,\psi_m\,\psi_n^{\prime *}\right]$$

Now integrate from x_L to x_R:

$$(\omega_n^{*2} - \omega_m^2)\int_{x_L}^{x_R} dx\,\mu(x)\,\psi_n^*(x)\psi_m(x) = \tau(x)\left[\psi_n^*(x)\psi_m'(x) - \psi_m(x)\,\psi_n^{*\prime}(x)\right]_{x_L}^{x_R}$$

$$= 0$$

because the term in square brackets vanishes for any of the four boundary conditions. Thus,

$$(\omega_n^{*2} - \omega_m^2)\langle\psi_n|\psi_m\rangle = 0$$

where the inner product is

$$\langle\psi|\phi\rangle = \int_{x_L}^{x_R} dx\,\mu(x)\,\psi^*(x)\,\phi(x)$$

Since $\langle\psi_n|\psi_n\rangle > 0$, we have that $\omega_n^2 \in \mathbb{R}$. (Note this does not preclude $\omega_n^2 < 0$ in which case $\omega_n \in i\mathbb{R}$.) When $\omega_m^2 \neq \omega_n^2$, we have $\langle\psi_n|\psi_m\rangle = 0$. For degenerate eigenvalues, we may invoke the Gram-Schmidt method, which orthogonalizes the eigenfunctions within a degenerate subspace. Since the SLE is linear, we may then demand **orthonormality**:

$$\langle\psi_n|\psi_m\rangle = \delta_{mn}$$

Furthermore when the functions $\mu(x)$, $\tau(x)$, $v(x)$ are all real, and when, in the case of mixed homogeneous BCs, $\alpha/\beta \in \mathbb{R}$, we may choose $\psi_n(x) \in \mathbb{R}\ \forall\ n$.

Another aspect of the eigenspectrum, which is more difficult to prove (so we won't) is **completeness**:

$$\mu(x) \sum_{n=0}^{\infty} \psi_n^*(x)\, \psi_n(x') = \delta(x - x')$$

Note that we have labeled the eigenvalues and eigenfunctions with a discrete integer index $n \in \{0, 1, \ldots, \infty\}$, and we may demand $\omega_0^2 \leq \omega_1^2 \leq \omega_2^2 \leq \ldots$. Any square integrable, or $\mathcal{L}^2$, function $f(x)$, for which $\langle f | f \rangle < \infty$, can be expanded in the eigenfunctions, viz.

$$f(x) = \sum_{n=0}^{\infty} f_n\, \psi_n(x) \quad , \quad f_n = \langle \psi_n | f \rangle = \int_{x_L}^{x_R} dx\, \mu(x)\, \psi_n^*(x)\, f(x)$$

NB: What is true is that $\| f - \sum_{n=0}^{\infty} f_n \psi_n \| = 0$, where $\| h \| = \langle h | h \rangle$ is the norm of h. Note that this does not guarantee that $\sum_{n=0}^{\infty} f_n\, \psi_n(x)$ converges to $f(x)$ pointwise for all $x \in [x_L, x_R]$. Rather, the convergence holds "almost everywhere", which is to say for all $x \in [x_L, x_R]$ except on a set of measure zero.

- **Variational method**

Define the functional $\omega^2[\psi(x)] \equiv \dfrac{N[\psi(x)]}{D[\psi(x)]}$ with

$$N[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx \left\{ \tau(x)\,\psi'(x)^2 + v(x)\,\psi(x)^2 \right\}$$

$$D[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\,\psi(x)^2$$

Then the variation of $\omega^2[\psi]$ is

$$\delta\omega^2 = \frac{\delta N}{D} - \frac{N\,\delta D}{D^2}$$

Thus, if we demand $\delta\omega^2 \equiv 0$, we have

$$\delta N = \frac{N}{D}\,\delta D = \omega^2\,\delta D$$

and since

$$\frac{\delta N}{\delta\psi(x)} = -\frac{d}{dx}\left[\tau(x)\,\psi'(x)\right] + v(x)\,\psi(x)$$

$$\frac{\delta D}{\delta\psi(x)} = \mu(x)\,\psi(x)$$

we see that $\delta\omega^2 \equiv 0$ yields the SLE,

$$\frac{\delta N}{\delta\psi(x)} = -\frac{d}{dx}\left[\tau(x)\,\psi'(x)\right] + v(x)\,\psi(x) = \omega^2\mu(x)\,\psi(x) = \omega^2\frac{\delta D}{\delta\psi(x)}$$

Note also that the variation of δN contains

Scratch

$$N[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx \left\{ \tau(x)\,\psi'(x)^2 + v(x)\,\psi(x)^2 \right\} \equiv \int_{x_L}^{x_R} dx\; L_N(\psi,\psi',x)$$

$$D[\psi(x)] = \frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\,\psi(x)^2 \equiv \int_{x_L}^{x_R} dx\; L_D(\psi,\psi',x)$$

$$L_N(\psi,\psi',x) = \frac{1}{2}\tau(x)\,\psi'^2 + \frac{1}{2}v(x)\,\psi^2$$

$$L_D(\psi,\psi',x) = \frac{1}{2}\mu(x)\,\psi^2$$

$$\frac{\delta N}{\delta\psi(x)} = \frac{\partial L_N}{\partial\psi} - \frac{d}{dx}\frac{\partial L_N}{\partial\psi'} = v(x)\,\psi - \frac{d}{dx}\left[\tau(x)\,\psi'\right]$$

$$\frac{\delta D}{\delta\psi(x)} = \frac{\partial L_D}{\partial\psi} - \cancel{\frac{d}{dx}\frac{\partial L_D}{\partial\psi'}} = \mu(x)\,\psi$$

Fourier analysis : $\psi_n(x) \rightarrow \psi_k(x) = e^{ikx}$

$$f(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\hat{f}(k)\,e^{ikx}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} = \langle \psi_k | f \rangle$$

$$\langle k | k' \rangle = \int_{-\infty}^{\infty} dx\; e^{i(k'-k)x} = 2\pi\,\delta(k-k')$$ replaces $\delta_{kk'}$

completeness : $\delta(x-x') = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ik(x-x')}$

a boundary term $\tau(x)\,\psi'(x)\,\delta\psi(x)\Big]_{x_L}^{x_R}$, which vanishes for any of our first three classes of boundary conditions, i.e. fixed endpoints ($\delta\psi(x_{L,R})=0$), natural ($\tau(x_{L,R})\,\psi'(x_{L,R})=0$), or periodic ($f(x)=f(x+L)$ for $f(x)=\psi(x)$ and $f(x)=\tau(x)$).

In order to accommodate the fourth class of BC, i.e. mixed homogeneous, with $\alpha\psi(x)+\beta\psi'(x)=0$ for $x=x_{L,R}$, if we redefine $\omega^2=\tilde{N}/D$, where

$$\tilde{N}[\psi(x)] = N[\psi(x)] + \frac{\alpha}{2\beta}\left\{\tau(x_R)\,\psi(x_R)^2 - \tau(x_L)\,\psi(x_L)^2\right\}$$

In fact, for all for classes of BC we can take

$$\omega^2[\psi(x)] = \frac{N[\psi(x)]}{D[\psi(x)]} = \frac{\frac{1}{2}\int_{x_L}^{x_R} dx\, \psi(x)\overbrace{\left[-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right]}^{\hat{K}}\psi(x)}{\frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\,\psi^2(x)}$$

Thus, expanding $\psi(x) = \sum_{n=0}^{\infty} C_n\,\psi_n(x)$, we have

$$\omega^2[\psi(x)] = \omega^2(C_0,\dots,C_\infty) = \frac{\frac{1}{2}\sum_{n=0}^{\infty}\omega_n^2 C_n^2}{\frac{1}{2}\sum_{m=0}^{\infty} C_m^2}$$

Then $$\frac{\partial\omega^2}{\partial C_j} = \frac{(\omega_j^2-\omega^2)C_j}{\frac{1}{2}\sum_m C_m^2} = 0 \quad \text{for all } j\in\{0,1,\dots,\infty\}$$

any $\psi(x) \to \omega^2[\psi] \geq \omega_0^2$

Solutions:

$$C_j^{(k)} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$$

with $\omega^2=\omega_k^2$ (k^{th} sol^n)

Example : string with mass point in center

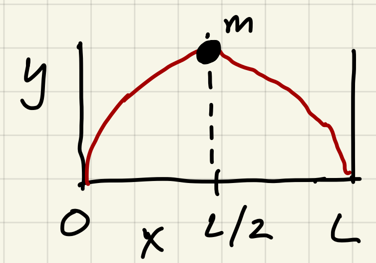
$$\mu(x) = \mu + m\,\delta(x - \tfrac{1}{2}L) \quad ; \quad \tau(x) = \tau \quad ; \quad v(x) = 0$$

Here $x_L = 0$ and $x_R = L$. Then

$$\omega^2[\psi] = \frac{\frac{1}{2}\tau \int_0^L dx\, \psi'^2(x)}{\frac{1}{2}\mu \int_0^L dx\, \psi^2(x) + \frac{1}{2} m\, \psi^2(\frac{1}{2}L)}$$

Now consider a **trial function**

$$\psi(x) = \begin{cases} A x^\alpha & \text{for } x \in [0, \frac{L}{2}] \\ A(L-x)^\alpha & \text{for } x \in [\frac{L}{2}, L] \end{cases}$$



Here we have a single variational parameter, α.

- $\int_0^L dx\, \psi'^2(x) = 2A^2 \int_0^{L/2} dx\, \alpha^2 x^{2\alpha-2} = A^2 \cdot \frac{2\alpha^2}{2\alpha-1} \left(\frac{L}{2}\right)^{2\alpha-1}$
- $\int_0^L dx\, \psi^2(x) = 2A^2 \int_0^{L/2} dx\, x^{2\alpha} = A^2 \cdot \frac{2}{2\alpha+1} \left(\frac{L}{2}\right)^{2\alpha+1}$
- $\psi^2(\frac{1}{2}L) = A^2 \left(\frac{L}{2}\right)^{2\alpha}$

$$c = (\tau/\mu)^{1/2}$$

$$\omega^2[\psi] = \frac{\tau\left(\frac{\alpha^2}{2\alpha-1}\right)\left(\frac{L}{2}\right)^{2\alpha-1}}{\mu\left(\frac{1}{2\alpha+1}\right)\left(\frac{L}{2}\right)^{2\alpha+1} + \frac{1}{2} m \left(\frac{L}{2}\right)^{2\alpha}} = \left(\frac{c}{L}\right)^2 \frac{4\alpha^2(2\alpha+1)}{(2\alpha-1)\left[1 + (2\alpha+1)\frac{m}{M}\right]}$$

$M = \mu L$

Best variational estimate $\Rightarrow$ set $\dfrac{d\omega^2(\alpha)}{d\alpha} = 0$:

$$\frac{d\omega^2}{d\alpha} = 0 \Rightarrow 4\alpha^2 - 2\alpha - 1 + (\alpha-1)(2\alpha+1)^2 \frac{m}{M} = 0$$

This is a cubic equation. For $m/M \to 0$, we have $4\alpha^2 - 2\alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{4}(1+\sqrt{5}) \simeq 0.809$. Find then $\omega^2 \approx 11.09 \frac{c^2}{L^2} \Rightarrow \omega \simeq 3.330 \frac{c}{L}$. The exact result we know is $\psi_0(x) = (2/L)^{1/2} \sin(\pi x/L)$ with $\omega_0 = \pi c/L$, and our variational frequency is about 6.00% higher. For $m/M \to \infty$, the string's inertia is negligible. Then $\psi(x)$ describes an isoceles triangle, and

$$m\ddot{y} = -2\tau \cdot \left(\frac{y}{\frac{1}{2}L}\right) \Rightarrow \omega_0 = 2\sqrt{\frac{\tau}{mL}} = \frac{2}{L}\sqrt{\frac{\tau}{\mu} \cdot \frac{\mu L}{m}} = \frac{2c}{L}\sqrt{\frac{M}{m}}$$

The variational sol$^{\underline{n}}$ yields $\alpha = 1$ and $\omega^2 = \omega_0^2$ exactly. Note $\alpha = 1$ corresponds to a triangular shape

Our example involved just one variational parameter. We could have more, e.g.

$$\psi(x) = A x^\alpha + B x^\beta \qquad (0 \le x \le \tfrac{L}{2})$$
$$\psi(L-x) = \psi(x)$$

Variation parameters: 3 $(\alpha, \beta, B/A)$

Or: $A \equiv C\cos\gamma$, $B \equiv C \sin\gamma \Rightarrow (\alpha, \beta, \gamma)$

Another basis : $\psi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$

$$\cdot \int_0^L dx\, \psi_m(x)\, \psi_n(x) = \delta_{mn}$$

$$\cdot \int_0^L dx\, \psi_m'(x)\, \psi_n'(x) = -\int_0^L dx\, \psi_m(x)\, \underbrace{\psi_n''(x)}_{\psi_n'' = -\left(\frac{n\pi}{L}\right)^2 \psi_n} = \left(\frac{n\pi}{L}\right)^2 \delta_{mn}$$

So take $\psi(x) = \sum_{n=1}^{\infty} C_n\, \psi_n(x)$

variational parameters $\{C_1, \ldots, C_\infty\}$

$$\omega^2[\psi] = \frac{\frac{1}{2}\tau \int_0^L dx\, \psi'^2(x)}{\frac{1}{2}\mu \int_0^L dx\, \psi^2(x) + \frac{1}{2} m\, \psi^2(\frac{1}{2}L)}$$

$$= \frac{\frac{1}{2}\tau \sum_n \left(\frac{n\pi}{L}\right)^2 C_n^2}{\frac{1}{2}\mu \sum_j C_j^2 + \frac{1}{L} m \underbrace{\left(\sum_j C_j \underbrace{\sin\left(\frac{j\pi}{2}\right)}_{(-1)^k \delta_{j,2k-1}}\right)^2}_{\left[\sum_{k=1}^{\infty} (-1)^k C_k\right]^2}}$$

$C_1, \ldots, C_\ell$ finite subset

$$\omega^2(C_1, \ldots, C_\infty) = \frac{\sum_{n=1}^{\infty} n^2 C_n^2}{\sum_{j=1}^{\infty} C_j^2 + \frac{2m}{M}\left[\sum_{k=1}^{\infty} (-1)^k C_k\right]^2} \cdot \left(\frac{\pi c}{L}\right)^2$$

Lecture 12 (Nov. 11)

- Inhomogeneous Sturm–Liouville equation (§9.7):

$$\mu(x)\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[T(x)\frac{\partial y}{\partial x}\right] + v(x)\,y = \mu(x)\,\mathrm{Re}\left[f(x)e^{-i\omega t}\right]$$

Here the string is forced at frequency ω.
We write the solution as

$$y(x,t) = \mathrm{Re}\left[y(x)\,e^{-i\omega t}\right]$$

where

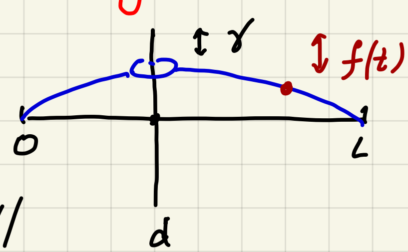
could redefine as $\tilde{f}(x)$ but it is convenient if we include $\mu(x)$

$$\left[\hat{K} - \omega^2\mu(x)\right] y(x) = \overbrace{\mu(x) f(x)}$$

$\longrightarrow y(x,t) = y_{hom}(x,t) + y_{inh}(x,t)$

with

$$\hat{K} = -\frac{d}{dx}T(x)\frac{d}{dx} + v(x)$$



the Sturm–Liouville operator. Recall

$$\hat{K}\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)$$

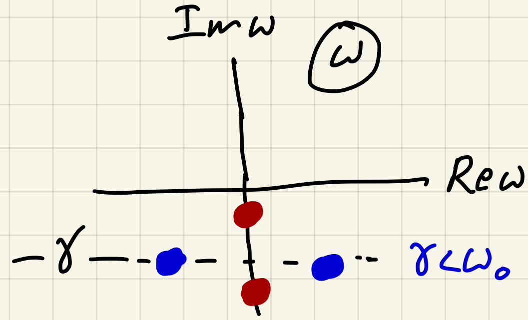
$$\langle \psi_m | \psi_n \rangle = \int_{x_L}^{x_R} dx\, \mu(x)\,\psi_m^*(x)\,\psi_n(x) = \delta_{mn}$$

$$\mu(x)\sum_n \psi_n(x)\,\psi_n^*(x') = \delta(x-x')$$

Taking the inverse of $\hat{K} - \omega^2\mu(x)$, we have that the inhomogeneous solution is

Scratch

Unforced, damped SHO:



$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$$

Sol$^{\underline{n}}$: $x = A e^{-i\omega t} \Rightarrow -\omega^2 - 2i\gamma\omega + \omega_0^2 = 0$

$$\omega^2 + 2i\gamma\omega - \omega_0^2 = 0 \Rightarrow \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

$e^{-i\omega_\pm t} \to 0$ as $t \to \infty$ due to $\gamma > 0$

$\gamma^2 < \omega_0^2 \Rightarrow$ underdamped, $\gamma^2 > \omega_0^2 \Rightarrow$ overdamped

Harmonic forcing: $f(t) = \int \frac{d\Omega}{2\pi} \hat{f}(\Omega) e^{-i\Omega t}$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \hat{f}(\Omega) e^{-i\Omega t}$$

$\hat{x}(\Omega)\, e^{-i\Omega t}$

Sol$^{\underline{n}}$: $x(t) = x_{hom}(t) + x_{inh}(t)$

$$A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t} \to 0$$

$$(\omega_0^2 - 2i\gamma\Omega - \Omega^2)\, \hat{x}(\Omega) = \hat{f}(\Omega)$$

Single frequency: $x_{inh}(t) = A(\Omega) \cos[\Omega t + \delta(\Omega)]$

amplitude: $A(\Omega) = \left[(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2\right]^{-1/2}$

phase shift: $\delta(\Omega) = \tan^{-1}\left(\frac{2\gamma\Omega}{\Omega^2 - \omega_0^2}\right)$

$$y_{inh}(x) = \int_{x_L}^{x_R} dx'\, \mu(x')\, G_\omega(x,x')\, f(x')$$

where $G_\omega(x,x')$ is the **Green's function**, satisfying

$$\left[\hat{K} - \omega^2 \mu(x)\right] G_\omega(x,x') = \delta(x-x')$$

I.e. $G_\omega(x,x') = \left[\hat{K} - \omega^2 \mu\right]^{-1}_{x,x'}$. We may write

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\, \psi_n^*(x')}{\omega_n^2 - \omega^2} \quad , \quad [G_\omega] = \frac{T^2}{M}$$

You can read about how to obtain $G_\omega(x,x')$ without having to do the infinite sum over all the eigenfunctions in §9.7.1. For now, I just quote the result for the case where $\mu(x) = \mu$, $\tau(x) = \tau$, $v(x) = 0$, and $[x_L, x_R] = [0, L]$. Then

$$G_\omega(x,x') = \frac{\sin(\omega x_< / c)\, \sin(\omega(L - x_>)/c)}{(\omega\tau/c)\, \sin(\omega L/c)}$$

where $x_< = \min(x,x')$ and $x_> = \max(x,x')$, $c = \sqrt{\frac{\tau}{\mu}}$

Example: Let $f(x) = f_0\, \delta(x - x_0)$. Then

$$y_{inh}(x) = \mu f_0\, G_\omega(x, x_0)$$

Note that there are no constants of integration.

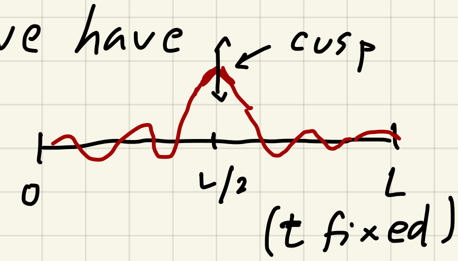
The full sol$^{\underline{n}}$ is then

homogeneous sol$^{\underline{n}}$ (e.g. Bernoulli)

inhomogeneous sol$^{\underline{n}}$

$$y(x,t) = y_{hom}(x,t) + y_{inh}(x,t)$$

The initial conditions enter in $y_{hom}(x,t)$ as we have learned from the Bernoulli solution. If there is some small damping, then at long times we have



$$\begin{aligned} y(x, t \gg \gamma^{-1}) &= y_{inh}(x,t) \\ &= \mu f_0 \, G_\omega(x, x_0) \cos(\omega t) \end{aligned}$$

where γ is the damping rate (i.e. rate of energy loss for unforced system). If $x_0 = \frac{1}{2}L$, then

$$G_\omega(x, \tfrac{1}{2}L) = \frac{c}{2\omega\tau\cos(\omega L/2c)} \times \begin{cases} \sin(\omega x/c) & \text{if } x < L/2 \\ \sin(\omega(L-x)/c) & \text{if } x > L/2 \end{cases}$$

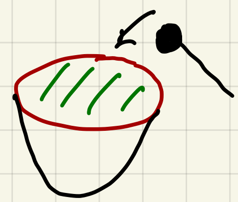
Note that $y_{inh}(x,t)$ is continuous at $x = \frac{1}{2}L$ but its spatial derivative $y'_{inh}(x,t)$ is discontinuous at $x = \frac{1}{2}L$.

- **Continua in higher dimensions** : $h(\vec{x}, t)$ displacement

Generalization of wave operator :

e.g. drum head :

$$\hat{K} = -\frac{\partial}{\partial x^\alpha} \, \tau_{\alpha\beta}(\vec{x}) \frac{\partial}{\partial x^\beta} + v(\vec{x})$$



This arises from

$$\mathcal{L} = \frac{1}{2}\mu(\vec{x})\left(\frac{\partial h}{\partial t}\right)^2 - \frac{1}{2}\tau_{\alpha\beta}(\vec{x})\frac{\partial h}{\partial x^\alpha}\frac{\partial h}{\partial x^\beta} - \frac{1}{2}v(\vec{x})h^2$$

The wave equation is

$$\hat{K}\,h(\vec{x},t) = -\mu(\vec{x})\frac{\partial^2}{\partial t^2}h(\vec{x},t)$$

Since $[\hat{K},\partial_t] = 0$, solutions may be written as

$$h(\vec{x},t) = \mathrm{Re}\left[h(\vec{x})\,e^{-i\omega t}\right]$$

where

$$\left[\hat{K} - \omega^2\mu(\vec{x})\right]h(\vec{x}) = 0$$

This is again an eigenvalue equation, with solutions

$$\psi_n(\vec{x}) \implies \hat{K}\psi_n(\vec{x}) = \omega_n^2\,\mu(\vec{x})\,\psi_n(\vec{x})$$

The eigenfunctions and eigenvalues satisfy

$$\langle\psi_m|\psi_n\rangle = \int_\Omega d^dx\;\mu(\vec{x})\,\psi_m^*(\vec{x})\,\psi_n(\vec{x}) = \delta_{mn}$$

$$\mu(\vec{x})\sum_n \psi_n(\vec{x})\,\psi_n^*(\vec{x}\,') = \delta(\vec{x}-\vec{x}\,')$$

where the medium is confined to a region $\Omega \subset \mathbb{R}^d$. We must also apply boundary conditions of the form

(i) $h(\vec{x})\big|_{\partial\Omega} = 0$, where $\partial\Omega$ = boundary of Ω

(ii) $\tau(\vec{x})\, \hat{n}\cdot\vec{\nabla}h\big|_{\partial\Omega} = 0$, where $\hat{n}$ is normal to $\partial\Omega$

(iii) PBCs , e.g. in a box of dim$^{\underline{ns}}$ $L_1 \times L_2 \times \cdots \times L_d$

(iv) $\left[\alpha\psi(\vec{x}) + \beta\hat{n}\cdot\vec{\nabla}\psi(\vec{x})\right]_{\partial\Omega} = 0$

The Green's function is

$$G_\omega(\vec{x},\vec{x}\,') = \sum_n \frac{\psi_n(\vec{x})\,\psi_n^*(\vec{x}\,')}{\omega_n^2 - \omega^2}$$

with

$$\left[\hat{K} - \omega^2\mu(\vec{x})\right] G_\omega(\vec{x},\vec{x}\,') = \delta(\vec{x}-\vec{x}\,')$$

The variational approach generalizes as well, with

$$\omega^2[\psi(\vec{x})] \equiv \frac{N[\psi(\vec{x})]}{D[\psi(\vec{x})]}$$

and

$$N[\psi(\vec{x})] = \int_\Omega d^dx\; \psi^*(\vec{x}) \overbrace{\left\{-\frac{\partial}{\partial x^\alpha}\tau_{\alpha\beta}(\vec{x})\frac{\partial}{\partial x^\beta} + v(\vec{x})\right\}}^{\hat{K}} \psi(\vec{x})$$

$$D[\psi(\vec{x})] = \int_\Omega d^dx\; \mu(\vec{x})\,\psi^2(\vec{x})$$

Demanding $\delta\omega^2 = 0$ yields the wave equation

$$\hat{K}\psi(\vec{x}) = \omega^2\mu(\vec{x})\,\psi(\vec{x})$$

- **Membranes** : $z = h(x,y)$

The equation of a surface is $F(x,y,z) = z - h(x,y) = 0$. Let the differential surface area be dS. The projection onto the (x,y) plane is then

$$dA = dx\,dy = \hat{n}\cdot\hat{z}\,dS = n^z dS$$

The unit normal is

$$\hat{n} = \frac{\vec{\nabla}F}{|\vec{\nabla}F|} = \frac{\hat{z} - \vec{\nabla}h}{\sqrt{1+(\vec{\nabla}h)^2}} \qquad (\text{note } \hat{z}\cdot\vec{\nabla}h = 0)$$

Thus,

$$dS = \frac{dx\,dy}{\hat{n}\cdot\hat{z}} = \sqrt{1+(\vec{\nabla}h)^2}\,dx\,dy$$

before: $ds = \sqrt{1+h'^2}\,dx$

We consider a model where

$$U[h(x,y,t)] = \int dS\,\sigma = U_0 + \frac{1}{2}\int d^2x\,\sigma(\vec{x})\,(\vec{\nabla}h)^2 + \ldots$$

with σ the surface tension. Other energy functions are possible. The kinetic energy is

$$T[h(x,y,t)] = \frac{1}{2}\int d^2x\,\mu(\vec{x})\left(\frac{\partial h}{\partial t}\right)^2$$

Thus

$$S = \int dt \int d^2x\,\mathcal{L}(h, \partial_t h, \vec{\nabla}h, t, \vec{x})$$

$$\mathcal{L} = \frac{1}{2}\mu(\vec{x})(\partial_t h)^2 - \frac{1}{2}\sigma(\vec{x})(\vec{\nabla}h)^2$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial h} - \underbrace{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \partial_t h}}_{=\,\mu(\vec{x})\frac{\partial^2 h}{\partial t^2}} - \underbrace{\vec{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial \vec{\nabla} h}}_{=\,-\vec{\nabla}\cdot\left[\sigma(\vec{x})\vec{\nabla} h\right]} = 0$$

$$0 - \left(\mu(\vec{x})\frac{\partial^2 h}{\partial t^2}\right) - \left(-\vec{\nabla}\cdot\left[\sigma(\vec{x})\vec{\nabla} h\right]\right) = 0$$

Thus

$$\vec{\nabla}\cdot\left[\sigma(\vec{x})\vec{\nabla} h(\vec{x},t)\right] = \frac{\partial^2 h(\vec{x},t)}{\partial t^2}$$

which is a generalization of the Helmholtz equation. When μ and σ are constants, we get Helmholtz:

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h(\vec{x},t) = 0$$

Note $[\mu] = ML^{-2}$ and $[\sigma] = EL^{-2} = MT^{-2}$, thus with $c \equiv (\sigma/\mu)^{1/2}$ we have $[c] = LT^{-1}$ as before.

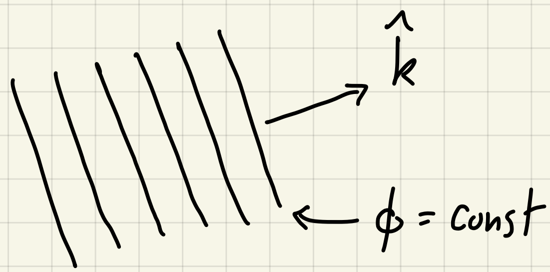
d'Alembert solution:

$$h(\vec{x},t) = f(\hat{k}\cdot\vec{x} - ct)$$

where $\hat{k}$ is a fixed direction in space. These are plane waves (really "line waves"). The locus of points of constant $h(\vec{x},t)$ satisfies

$$\phi(\vec{x},t) = \hat{k}\cdot\vec{x} - ct = \text{constant}$$

and setting $d\phi = 0$ then yields $\hat{k} \cdot \frac{d\vec{x}}{dt} = c$, i.e. the velocity along $\hat{k}$ is c. The component of $\vec{x}$ lying perpendicular to $\hat{k}$ is arbitrary, so constant $\phi(\vec{x},t)$ corresponds to lines orthogonal to $\hat{k}$.



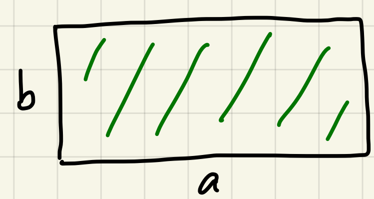
Due to linearity of the wave eqn, we can superpose plane wave solutions to arrive at the general solution,

$$h(\vec{x},t) = \int \frac{d^2k}{(2\pi)^2} \Big[\underbrace{A(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - ckt)}}_{+\hat{k}\text{ mover}} + \underbrace{B(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} + ckt)}}_{-\hat{k}\text{ mover}} \Big]$$

$k = |\vec{k}|$

- Rectangles : $\Omega = [0,a] \times [0,b]$

Separation of variables solves PDE:



$$h(x,y,t) = X(x)\, Y(y)\, T(t)$$

Helmholtz eqn $\frac{1}{h}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) h = 0$ yields

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\text{depends only on } x} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\text{depends only on } y} = \frac{1}{c^2} \cdot \underbrace{\frac{1}{T}\frac{\partial^2 T}{\partial t^2}}_{\text{depends only on } t}$$

So we conclude

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2}$$

Thus, $\omega = c|\vec{k}|$. Most general sol$^{\underline{n}}$:

$$\begin{aligned} X(x) &= A\sin(k_x x + \alpha) \\ Y(y) &= B\sin(k_y y + \beta) \\ T(t) &= C\sin(\omega t + \gamma) \end{aligned} \quad , \quad h(x,y,t) = X(x)Y(y)T(t)$$

but imposing boundary conditions $h(\vec{x},t)|_{\partial\Omega} = 0$ then requires

$$\alpha = \beta = 0 \quad , \quad \sin(k_x a) = \sin(k_y b) = 0 \Rightarrow \begin{cases} k_x = m\pi/a \\ k_y = n\pi/b \end{cases}$$

The most general sol$^{\underline{n}}$ consistent with the BCs is then

$$h(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin(\omega_{mn}t + \gamma_{mn})$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2}$$

and the constants $\{A_{mn}, \gamma_{mn}\}$ are determined by the initial conditions.

- Circles: $\Omega = \{(x,y) \mid x^2+y^2 \leq a^2\}$

It is convenient to work in 2d polar coordinates (r,φ).
The Helmholtz equation takes the form

$$\vec{\nabla}^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \varphi^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2}$$

Separation of variables:

$$h(r,\varphi,t) = R(r)\,\Phi(\varphi)\,T(t)$$

Again we have

$$\frac{1}{R}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\cdot\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}$$

with

$$\Phi(\varphi) = \cos(m\varphi + \beta)$$
$$T(t) = \cos(\omega t + \gamma)$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{m^2}{r^2} - \frac{\omega^2}{c^2}\right)R = 0$$

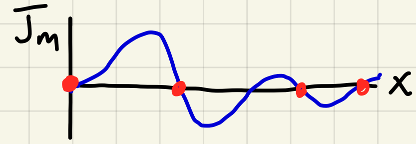
Since $h(r,\varphi+2\pi,t) = h(r,\varphi,t)$, we must have $m \in \mathbb{Z}$.
This is Bessel's equation, with solutions

$$R(r) = A\,J_m\left(\frac{\omega r}{c}\right) + B\,N_m\left(\frac{\omega r}{c}\right)$$

with $J_m(z)$ and $N_m(z)$ the Bessel and Neumann functions of order m, respectively. Since $N_m(z)$ diverges as $z \to 0$ for all m, we must have $B = 0$. (For an annulus, we may have $B \neq 0$.) The boundary condition at $r = a$ yields

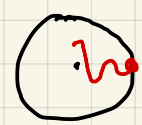
$$J_m\left(\frac{\omega a}{c}\right) = 0 \quad \Rightarrow \quad \omega = \omega_{m\ell} = x_{m\ell} \cdot \frac{c}{a}$$

where $J_m(x_{m\ell}) = 0$, i.e. $x_{m\ell}$ is the ℓ^{th} zero $(\ell = 1, 2, \dots, \infty)$ of $J_m(x)$. Thus,



$$h(r, \varphi, t) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{\infty} A_{m\ell} J_m(x_{m\ell} r/a) \cos(m\varphi + \beta_{m\ell}) \cos(\omega_{m\ell} t + \gamma_{m\ell})$$

The constants $A_{m\ell}$, $\beta_{m\ell}$, and $\gamma_{m\ell}$ are set by the initial conditions. Note $h(r = a, \varphi, t) = 0$ for all φ and for all t.



- Read §9.3.6 (sound in fluids) and §9.4 (dispersion)

- Classical Field Theory

  Independent variables : $\{x^1, \dots, x^n\} \in \Omega \subset \mathbb{R}^n$ or $\{x^0, x^1, \dots, x^d\}$, $n = d+1$

  Real fields : $\{\phi_1, \dots, \phi_K\}$

  Lagrangian density : $\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu \phi_a, x^\mu)$

  Action : $S = \int d^n x\, \mathcal{L}$

  Let's compute the variation of S :

$$\delta S = \int_\Omega d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \delta \phi_a}{\partial x^\mu} \right\}$$

$$= \int_\Omega d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta \phi_a$$

differential surface area

$$+ \oint_{\partial\Omega} d\Sigma \, n^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a$$

The surface term vanishes if we demand

$$\delta \phi_a(\vec{x}) \Big|_{\partial\Omega} = 0 \quad \text{or} \quad n^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Big|_{\partial\Omega} = 0$$

Then we have

$$\frac{\delta S}{\delta \phi_a(\vec{x})} = \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right]_{\vec{x}}$$

evaluate at $\vec{x}$

Thus $\delta S = 0$ entails the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

When $\mathcal{L}$ is independent of the independent variables x^μ, the stress-energy tensor is conserved:

$$\partial_\mu T^\mu{}_\nu = 0 \quad \text{with} \quad T^\mu{}_\nu = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}$$

This is analogous to $\frac{dH}{dt} = 0$ in particle mechanics.

Maxwell theory

The Lagrangian density, with sources, is

$$\mathcal{L}(A^\nu, \partial_\mu A^\nu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ with $x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$
and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; ; \quad A_\nu = g_{\nu\lambda} A^\lambda \quad , \; g = \mathrm{diag}(+,-,-,-)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = g^{\mu\alpha} g^{\nu\beta} A_{\alpha\beta} \; ; \quad g_{\mu\nu} = g^{\mu\nu}$$

The EL equations are

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

Conserved currents in field theory

In particle mechanics, a one-parameter family of transformations $\tilde{q}_\sigma(q, \zeta)$ which leaves $L(q, \dot{q}, t)$ invariant results in a conserved "charge"

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \; ; \quad \tilde{q}_\sigma(q, \zeta=0) = q_\sigma$$

with $d\Lambda/dt = 0$. We generalize to field theory by taking $q_\sigma(t) \to \phi_a(x^\mu)$. Then

$$\frac{d}{d\zeta}\bigg|_{\zeta=0} \mathcal{L}(\tilde{\phi}_a, \partial_\mu \tilde{\phi}_a, x^\mu) = \frac{\partial \mathcal{L}}{\partial \phi_a} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial}{\partial x^\mu} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0}$$

$$= \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)\bigg|_{\zeta=0},$$

where we have invoked the EL eqns,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)$$

Thus we have

$$\partial_\mu J^\mu = 0 \qquad \text{with} \qquad J^\mu = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0}$$

Let us write $x^\mu = \{x^0, x^1, \ldots, x^d\}$ with $n = d+1$. Then with $x^0 \equiv ct$ and $Q_\Omega \equiv c^{-1}\int d^d x\, J^0$, we have

$$\frac{dQ_\Omega}{dt} = \int_\Omega d^d x\, \partial_0 J^0 = -\int_\Omega d^3 x\, \vec{\nabla}\cdot\vec{J} = -\oint_{\partial\Omega} d\Sigma\, \hat{n}\cdot\vec{J} = 0$$

provided $\hat{n}\cdot\vec{J}\big|_{\partial\Omega} = 0$. Thus, the rate of change of Q_Ω is minus the integrated flux exiting the region Ω.

Example:

$$\mathcal{L}(\psi, \psi^*, \partial_\mu\psi, \partial_\mu\psi^*) = \tfrac{1}{2}K(\partial_\mu\psi^*)(\partial^\mu\psi) - U(\psi^*\psi)$$

The Lagrangian density is invariant under

$$\psi \to \tilde{\psi} = e^{i\zeta}\psi \quad , \quad \psi^* \to \tilde{\psi}^* e^{-i\zeta}\psi$$

We regard ψ and ψ^* as independent fields. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i e^{i\zeta}\psi \quad , \quad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -ie^{-i\zeta}\tilde{\psi}^*$$

and thus

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}(-i\psi^*)$$

$$= \frac{K}{2i}\left(\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*\right) = K\,\mathrm{Im}\left(\psi^*\partial^\mu\psi\right)$$

Note that $U(\tilde{\psi}^*\tilde{\psi}) = U(\psi^*\psi)$ is independent of ζ.

- Gross – Pitaevskii model

This is a model of nonrelativistic interacting bosons, with

$$\mathcal{L} = i\hbar\,\psi^*\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi - g\left(\psi^*\psi - n_0\right)^2$$

Details in §9.5.3 of the notes. The EL equations are

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + 2g\left(|\psi|^2 - n_0\right)\psi$$

and its complex conjugate. This is called the nonlinear Schrödinger equation (NLSE). The one-parameter invariance of $\mathcal{L}$ is again

$$\psi(\vec{x},t) \to \tilde{\psi}(\vec{x},t) \equiv e^{-i\zeta}\,\psi(\vec{x},t)$$

$$\psi^*(\vec{x},t) \to \tilde{\psi}^*(\vec{x},t) \equiv e^{+i\zeta}\,\psi^*(\vec{x},t)$$

The conserved current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0}$$

with components

$$J^0 = \hbar|\psi|^2 \equiv \hbar\rho$$

$$\vec{J} = \frac{\hbar^2}{2im}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) \equiv \hbar\vec{j}$$

Thus,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \text{(continuity eqn.)}$$

In this example, $x^\mu = x_\mu$ and there is no difference between raised and lowered indices.

Lecture 13 (Nov. 16)

- **Hamiltonian mechanics**

Recall that $H(q,p,t) = \sum_{\sigma=1}^{n} p_\sigma \dot{q}_\sigma - L(q,\dot{q},t)$ is a Legendre transform:

$$dH = \sum_\sigma \left(\cancel{p_\sigma d\dot{q}_\sigma} + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma}\right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(-\frac{\partial L}{\partial q_\sigma} dq_\sigma + \dot{q}_\sigma dp_\sigma\right) - \frac{\partial L}{\partial t} dt$$

We conclude

$$\frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -\dot{p}_\sigma \quad , \quad \frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma$$

as well as

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Note:

(i) If $\partial L/\partial t = 0$, then $dH/dt = 0$, i.e. H is a constant of the motion.

(ii) To express $H = H(q,p,t)$, we must invert the relation $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = p_\sigma(q,\dot{q})$ to obtain $\dot{q}_\sigma(q,p)$. This requires that the Hessian,

$$\frac{\partial p_\sigma}{\partial \dot{q}_{\sigma'}} = \frac{\partial^2 L}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}}$$

be nonsingular. (cf. inverse function theorem)

(iii) Define the rank $2n$ vector $\vec{\xi}$ by

$$\vec{\xi} \equiv \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \Rightarrow \quad \xi_i \equiv \begin{cases} q_i & \text{if } 1 \leq i \leq n \\ p_{i-n} & \text{if } n < i \leq 2n \end{cases}$$

Then we may write Hamilton's equations of motion as

$$\left.\begin{aligned} \dot{q}_\sigma &= \frac{\partial H}{\partial p_\sigma} \\ \dot{p}_\sigma &= -\frac{\partial H}{\partial q_\sigma} \end{aligned}\right\} \Rightarrow \quad \dot{\xi}_i = J_{ij} \frac{\partial H}{\partial \xi_j} \quad ; \quad J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

Note that J is an antisymmetric rank $2n$ matrix. The coordinates $\{\xi_1, \ldots, \xi_{2n}\} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ define a $2n$-dimensional **phase space**. If $\partial H/\partial t = 0$, then the equations of motion specify a rank $2n$ **dynamical system**, $\dot{\xi}_i = V_i(\vec{\xi})$, where

$$V_i(\xi) = J_{ij} \frac{\partial H(\vec{\xi})}{\partial \xi_j} = \text{velocity vector in phase space}$$

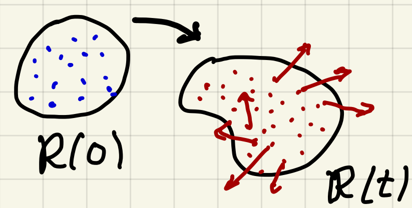
[If $\partial H/\partial t \neq 0$, define $\xi_0 = t$ and we have a rank $(2n+1)$ DS with $\dot{\xi}_0 = 1$ and $\dot{\xi}_i = V_i(\xi_0, \xi_1, \ldots, \xi_{2n})$.]

– Incompressible flow in phase space

Consider the (autonomous) dynamical system

$$\frac{d\vec{\xi}}{dt} = \vec{V}(\vec{\xi})$$

where $\vec{\xi}(t) \in \mathbb{R}^N$. Consider now the evolution of a compact region $R(t)$, each point in which evolves according to our DS. We have



$$\mathcal{R}(t) = \left\{ \vec{\xi}(t) \,\middle|\, \vec{\xi}(0) \in \mathcal{R}(0) \right\}$$

Now define $\Omega(t) = \text{vol}\, \mathcal{R}(t) = \int_{\mathcal{R}(t)} d\mu$, where

$$d\mu = d\xi_1 \cdots d\xi_N$$

Then

$$\Omega(t+dt) = \int_{\mathcal{R}(t+dt)} d\mu' = \int_{\mathcal{R}(t)} d\mu \, \underbrace{\left\| \frac{\partial \xi_i(t+dt)}{\partial \xi_j(t)} \right\|}_{\text{Jacobian}}$$

where

$$\left\| \frac{\partial \xi_i(t+dt)}{\partial \xi_j(t)} \right\| = \frac{\partial(\xi_1', \ldots, \xi_N')}{\partial(\xi_1, \ldots, \xi_N)} = \det \frac{\partial \xi_i(t+dt)}{\partial \xi_j(t)}$$

i.e. the determinant of the Jacobian. Now

$$\xi_i(t+dt) = \xi_i(t) + V_i(\vec{\xi}(t))\, dt + \mathcal{O}(dt^2)$$

and therefore

$$\frac{\partial \xi_i(t+dt)}{\partial \xi_j(t)} = \delta_{ij} + \left. \frac{\partial V_i}{\partial \xi_j} \right|_{\vec{\xi}(t)} dt + \mathcal{O}(dt^2)$$

We now invoke the identity $\ln\det A = \mathrm{Tr}\,\ln A$ for any matrix A, which is easily demonstrated when A is put in diagonal form. Thus, with $A \equiv \mathbb{1} + \epsilon M$

$$\det(\mathbb{1}+\epsilon M) = \exp \mathrm{Tr}\ln(\mathbb{1}+\epsilon M)$$
$$= \exp \mathrm{Tr}\left(\epsilon M - \tfrac{1}{2}\epsilon^2 M^2 + \ldots\right)$$
$$= 1 + \epsilon\,\mathrm{Tr}\,M + \tfrac{1}{2}\epsilon^2\left[(\mathrm{Tr}\,M)^2 - \mathrm{Tr}\,M^2\right] + \ldots$$

and with $\epsilon = dt$ and $M_{ij}(\vec{\xi}) = \left.\dfrac{\partial V_i}{\partial \xi_j}\right|_{\vec{\xi}}$, we have

$$\Omega(t+dt) = \Omega(t) + \int\limits_{\mathcal{R}(t)} d\mu\, \vec{\nabla}\cdot\vec{V}\, dt + \mathcal{O}(dt^2)$$

i.e. the rate of change of $\Omega(t) = \mathrm{vol}\,\mathcal{R}(t)$ is given by

$$\frac{d\Omega}{dt} = \int\limits_{\mathcal{R}(t)} d\mu\, \vec{\nabla}\cdot\vec{V}$$

where $\vec{\nabla}\cdot\vec{V} = \sum\limits_{i=1}^{N} \dfrac{\partial V_i}{\partial \xi_i}$ = divergence of phase space velocity.

Alternate derivation: Let $\rho(\vec{\xi},t)$ be the density of some collection of points in phase space. This must satisfy the continuity equation,

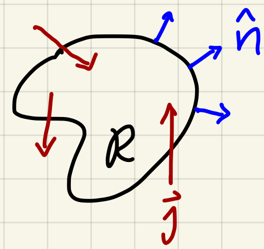
$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$$

Integrate over a region $\mathcal{R}$:

$$\frac{d}{dt}\int\limits_{\mathcal{R}} d\mu\,\rho = -\int\limits_{\mathcal{R}} d\mu\,\vec{\nabla}\cdot(\rho\vec{V}) = -\int\limits_{\partial\mathcal{R}} dS\,\hat{n}\cdot\rho\vec{V}$$

where $\partial\mathcal{R}$ is the boundary of $\mathcal{R}$. It is perhaps useful to think of ρ as a number or charge density and $\vec{j} \equiv \rho\vec{V}$ as the corresponding current density. Then if $Q_{\mathcal{R}} = \int_{\mathcal{R}} d\mu\, \rho$, then

$$\frac{dQ_{\mathcal{R}}}{dt} = -\int_{\partial\mathcal{R}} dS\, \underbrace{\hat{n}\cdot\vec{j}}_{(-\text{flux})}$$



Note that the Leibniz rule says

$$\frac{\partial\rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{V} = 0$$

and if $\vec{\nabla}\cdot\vec{V} = 0$, then

$$\frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right)\rho = 0$$

We call $\frac{D\rho}{Dt}$ the **convective derivative**, as it tells us the rate of change of ρ in a frame comoving with the local velocity $\vec{V}$. Thus,

$\dot{\vec{\xi}} = \vec{V}$

$$\frac{d}{dt}\rho(\vec{\xi}(t),t) = \frac{\partial\rho}{\partial t} + \dot{\vec{\xi}}\cdot\vec{\nabla}\rho = \frac{D\rho}{Dt}$$

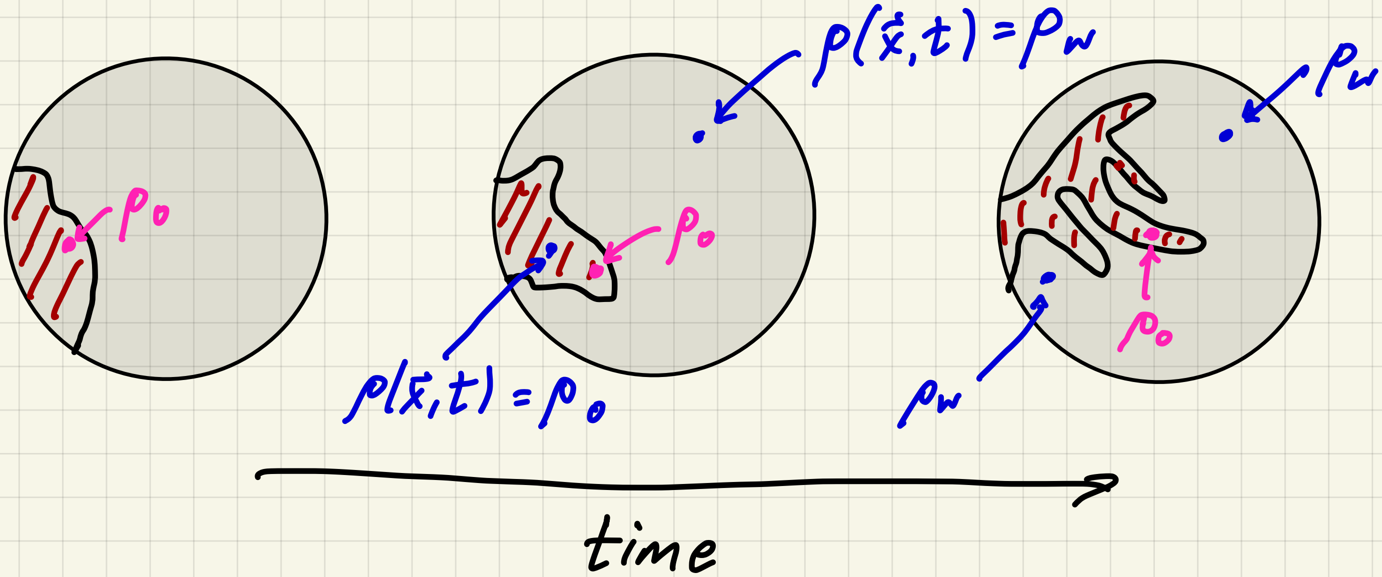
If we define

$$\rho(\vec{\xi},t=0) = \begin{cases} 1 & \text{if } \vec{\xi}\in\mathcal{R}_0 \\ 0 & \text{if } \vec{\xi}\notin\mathcal{R}_0 \end{cases}$$

i.e. the "characteristic function" of $\mathcal{R}_0$, then the

Scratch

Immiscible fluids (e.g. oil and water):



Two possible values of $\rho(\vec{x},t)$: ρ_w and ρ_o

Volume of red region is preserved by dynamics.

vanishing of the convective derivative says that $\rho(\vec{\xi}(t), t)$ is a constant, hence the image $R(t)$ of the set $R(0) \equiv R_0$ always has the same volume. In other words, the phase space flow is **incompressible**. Hamiltonian evolution is always incompressible:

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_i}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(J_{ij} \frac{\partial H}{\partial \xi_j}\right) = J_{ij} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j} = 0$$

- Poisson brackets

Consider the time evolution of any function $F(\vec{\xi}(t), t)$. We have

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{\sigma=1}^{n} \left\{ \frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right\}$$

$$\equiv \frac{\partial F}{\partial t} + \{F, H\}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where

$$\{A, B\} \equiv \sum_{\sigma=1}^{n} \left(\frac{\partial A}{\partial q_\sigma} \frac{\partial B}{\partial p_\sigma} - \frac{\partial A}{\partial p_\sigma} \frac{\partial B}{\partial q_\sigma} \right) = \sum_{i,j=1}^{2n} J_{ij} \frac{\partial A}{\partial \xi_i} \frac{\partial B}{\partial \xi_j}$$

is the **Poisson bracket** of A and B. Properties of the PB:

- Antisymmetry : $\{A, B\} = -\{B, A\}$
- Bilinearity : for constant λ,

$$\{A + \lambda B, C\} = \{A, C\} + \lambda \{B, C\}$$

- Associativity:

$$\{AB, C\} = A\{B,C\} + B\{A,C\}$$

- Jacobi identity:

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

We also have

- If $\{A,H\} = 0$ and $\partial A/\partial t = 0$, then $dA/dt = 0$, i.e. $A(q,p)$ is a constant of the motion.

- If $\{A,H\} = 0$ and $\{B,H\} = 0$, then by the Jacobi identity we have $\{\{A,B\},H\} = 0$, and if $\partial A/\partial t = 0$ and $\partial B/\partial t = 0$ (or, more weakly, if $\partial\{A,B\}/\partial t = 0$), then $\{A,B\}(q,p)$ is a constant of the motion.

- It is easily established that

$$\{q_\sigma, q_{\sigma'}\} = \{p_\sigma, p_{\sigma'}\} = 0 \quad , \quad \{q_\sigma, p_{\sigma'}\} = \delta_{\sigma\sigma'}$$

– Any density function $\rho(q,p,t)$ must satisfy continuity, hence

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \{\rho, H\} = 0 \;\Rightarrow\; \frac{\partial\rho}{\partial t} = -\{\rho,H\} = +\{H,\rho\}$$

Liouville eqn.

Consider a distribution $\rho(q,p,t) = \rho(\Lambda_1, \ldots, \Lambda_k)$ where each Λ_a is conserved, i.e. $\Lambda_a = \Lambda_a(q,p)$ with

$$\frac{d\Lambda_a}{dt} = \sum_\sigma \left(\frac{\partial \Lambda_a}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial \Lambda_a}{\partial p_\sigma} \dot{p}_\sigma \right) = \{\Lambda_a, H\} = 0 .$$

Then $\rho(\Lambda_1, \dots, \Lambda_k)$ is a stationary sol^{n} to Liouville's equation, i.e.

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} = 0$$

Examples:

- microcanonical distribution:

$$\rho(q,p) = \delta(E - H(q,p))/D(E)$$

where the density of states $D(E)$ fixes the normalization

$$\int_{\mathbb{R}^{2n}} d\mu \, \rho(q,p) = 1 \Rightarrow D(E) = \int_{\mathbb{R}^{2n}} d\mu \, \delta(E - H(q,p))$$

- ordinary canonical distribution:

$$\rho(q,p) = \frac{1}{Z(\beta)} e^{-\beta H(q,p)}$$

with

$$Z(\beta) = \int_{\mathbb{R}^{2n}} d\mu \, e^{-\beta H(q,p)}$$

for normalization. You may know $\beta = 1/k_B T$. temperature

– Aside: It is conventional to define the **Liouvillean operator** $\hat{L}$ by $\hat{L}\,\bullet = i\{H, \bullet\}$, where $\bullet$ = anything. Thus,

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} = -i\hat{L}\rho$$

which bears a resemblance to the Schrödinger equation.

- **Poincaré recurrence theorem** $\quad g_\tau \vec{\xi}(t) = \vec{\xi}(t+\tau)$

  Let g_τ be the "τ-advance mapping" which evolves time by τ, i.e. integrate the dynamical system $\dot{\xi}_i = V_i(\vec{\xi})$ forward by a time $\Delta t = \tau$. We assume three conditions:

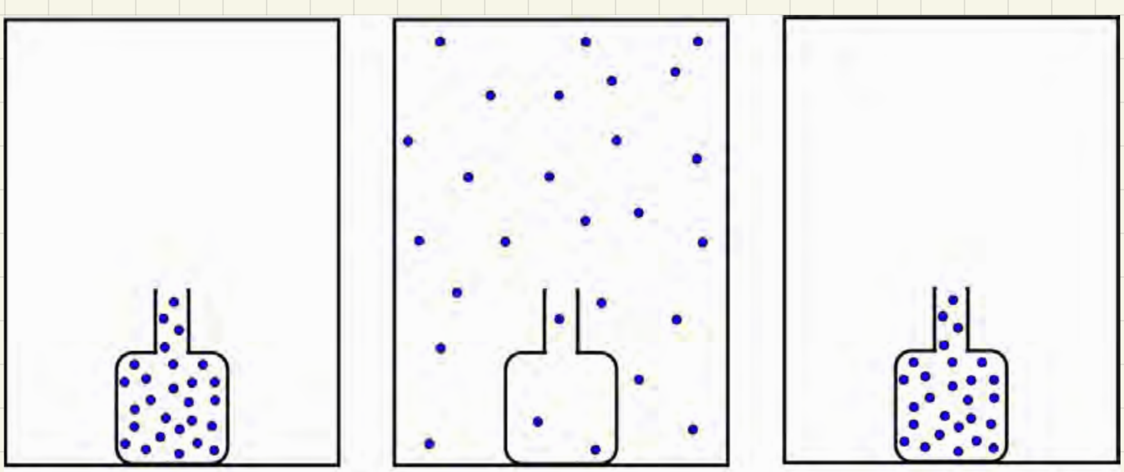
  (i) g_τ is invertible (integrate DS backward by $-\tau$)
  (ii) g_τ is volume-preserving (evolution is Hamiltonian)
  (iii) accessible phase space volume is finite, e.g.

$$\int_{\mathbb{R}^{2n}} d\mu\, \Theta\big(E + \Delta E - H(q,p)\big)\, \Theta\big(H(q,p) - E\big) = \int_E^{E+\Delta E} dE'\, D(E') < \infty$$
$$\approx D(E)\,\Delta E$$

We will henceforth refer to the $(2n-1)$-dimensional hypersurface Γ defined by $H(q,p) = E$ as the "phase space" for Hamiltonian evolution.

Theorem: In any finite neighborhood $R_0 \subset \Gamma$ there exists a point $\vec{\xi}_0$ which returns to R_0 after finitely many applications of g_τ.

Before proving the theorem, let's consider first its remarkable consequences. Suppose we had a bottle of perfume which we open at time $t=0$ in an evacuated room. Initially all the perfume molecules are inside the bottle, with CM positions $\vec{R}_a(0)$ and orientations (for diatomic or polyatomic molecules) $\{\phi_a(0), \theta_a(0), \psi_a(0)\}$. The initial conditions also specify the corresponding velocities $\{\dot{X}_a(0), \dot{Y}_a(0), \dot{Z}_a(0), \dot{\phi}_a(0), \dot{\theta}_a(0), \dot{\psi}_a(0)\}$. With N polyatomic molecules, there are $6N$ coordinates and $6N$ velocities $\Rightarrow$ $12N$-dim^{l} phase space. We choose $\mathcal{R}_0$ to be a ball in this space of arbitrarily small but finite size. The theorem says that there is an initial condition within the ball $\mathcal{R}_0$ which will repeat after a finite time $m\tau$, where $m \in \mathbb{Z}$. Thus, all the molecules return to the bottle, and to within $\mathcal{R}_0$ of their initial configuration! (However, this **recurrence time** may be much, much



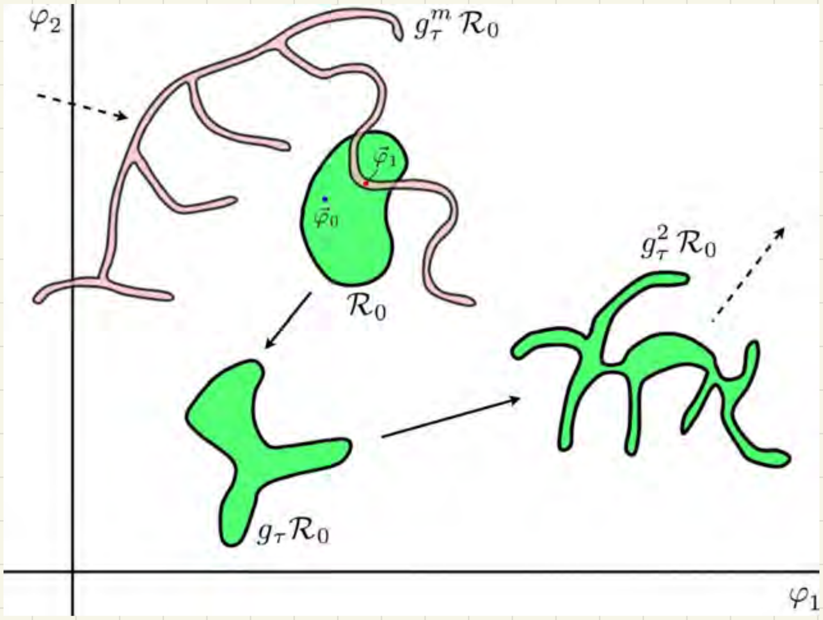
$t=0$ $\qquad$ $t = 1\,\mathrm{s}$ $\qquad$ $t >> \mathrm{H}^{-1}$

greater than the age of the universe!)

Proof: Assume the theorem fails and there is no recurrence. We will prove this results in a contradiction. Consider the union $\Delta = \bigcup_{k=0}^{\infty} g_\tau^k R_0$ of all the images of $g_\tau^k R_0$, where $k \in \{0, 1, \dots, \infty\}$. Suppose all these images are disjoint. Then

$$\mathrm{vol}(\Delta) = \sum_{k=0}^{\infty} \mathrm{vol}(g_\tau^k R_0) = \sum_{k=0}^{\infty} \mathrm{vol}(R_0) = \infty$$

where we have used that g_τ is **volume-preserving**. Since $\mathrm{vol}(\Gamma) < \infty$, we contradict **finite volume**. Therefore the sets $\{g_\tau^k R_0 \mid k \in \mathbb{Z}_{\geq 0}\}$ cannot be disjoint, i.e. there must exist two finite integers k and l with $k \neq l$ such that $g_\tau^k R_0 \cap g_\tau^l R_0 \neq \emptyset$. Due to **invertibility**, the inverse map g_τ^{-1} exists. Assume wolog that $k > l$ and apply



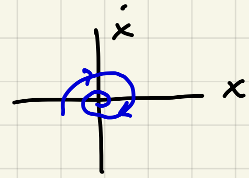
the map $(g_\tau^{-1})^\ell$ to this relation, obtaining

$$R_0 \cap g_\tau^m R_0 \neq \phi$$

where $m = k - \ell > 0$. Now choose any point $\vec{\xi}_1 \in R_0 \cap g_\tau^m R_0$. Then $\vec{\xi}_0 \equiv (g_\tau^{-1})^m \vec{\xi}_1 \in R_0$ lies within R_0 and we have proven the theorem!

Each of the three conditions – volume preservation, invertibility, and finite phase space volume – are essential here, and if any one doesn't hold the proof fails, viz.

- g_τ not volume-preserving: E.g. damped oscillator with $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. Then with $\vec{\xi} = (x, \dot{x})$ we have $\vec{V} = (\dot{x}, -2\beta\dot{x} - \omega_0^2 x)$ and



$$\vec{\nabla}\cdot\vec{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial(-2\beta\dot{x} - \omega_0^2 x)}{\partial \dot{x}} = -2\beta$$

Thus phase space volumes collapse: $\Omega(t) = e^{-2\beta t}\,\Omega(0)$. The set Δ can be of finite volume even if all the $g_\tau^k R_0$ are distinct, because

$$\sum_{k=0}^{\infty} \Omega(k\tau) = \sum_{k=0}^{\infty} e^{-2k\beta\tau}\,\Omega_0 = \frac{\Omega_0}{1 - e^{-2\beta\tau}} < \infty$$

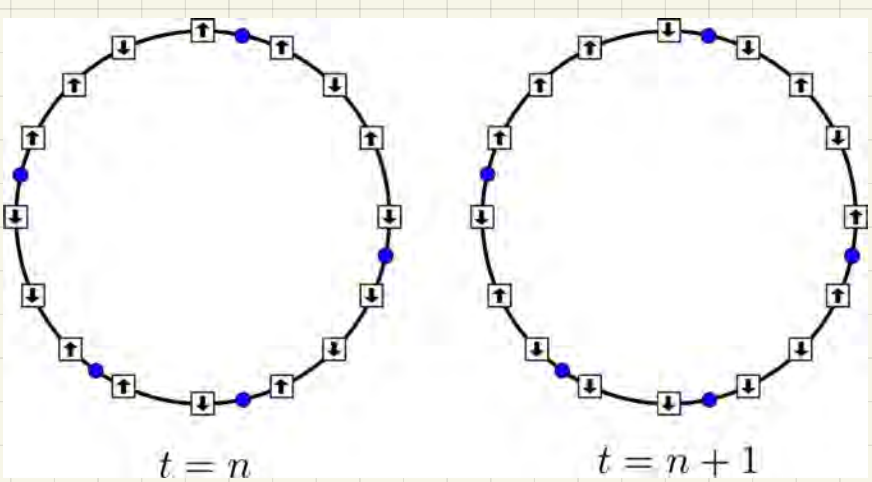
The phase space orbits all spiral into the origin and will not be recurrent. Note g_τ is invertible and phase space is of finite total volume.

- g_τ not invertible: Let $g: \mathbb{R} \to [0,1)$ with $g(x) = \text{frac}(x)$, the fractional part of x. Acting on sets of volume (length) less than one, this map is volume preserving, but obviously g is not invertible, so the proof fails.

- Γ not finite: Let $g: \mathbb{R} \to \mathbb{R}$ with $g(x) = x + a$. Clearly this is invertible and volume-preserving, but not recurrent.

- Kac ring model Lecture 14 (Nov. 18)

Can a system exhibit both equilibration and recurrence? Formally no, but practically yes. We noted how for the case of the open perfume bottle, the recurrence time could be vastly longer than age of the universe. A nice example due to Mark Kac shows how both equilibration and recurrence can be present, on different but accessible time scales. Consider N spins $\uparrow$ or $\downarrow$ on a ring which evolve by rotating clockwise. There are thus N sites and N links. Along F of these links are



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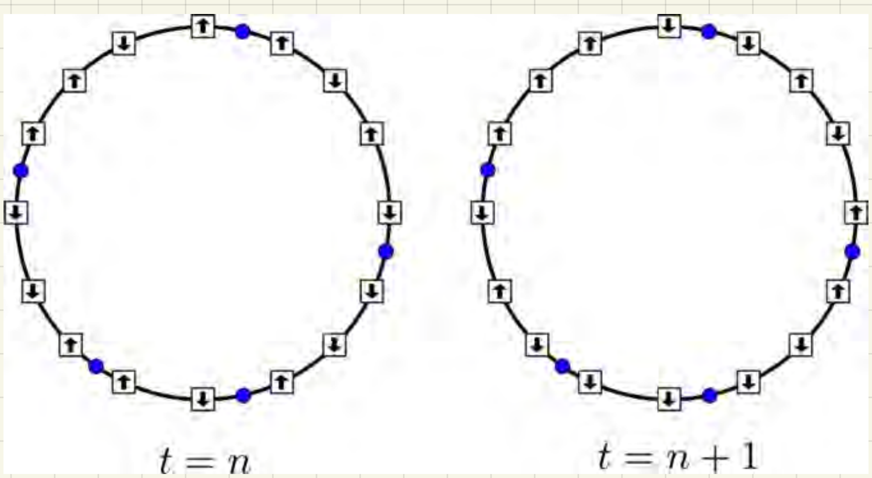


flippers which flip each spin from ↑ to ↓ or from ↓ to ↑ as it passes by. The configuration of flippers is frozen in from the start ("quenched randomness"). See the above figure. The number of possible spin configurations is finite and given by $\mathrm{vol}(\Gamma) = 2^N$.

Consider the evolution of a single spin, and let p_n be the probability the spin is up at time n (units of τ). Let $x = F/N$ be the fraction of flippers. If the flippers were to move about randomly, we would write

$$p_{n+1} = (1-x)\,p_n + x\,(1-p_n) \qquad \text{"Stosszahlanzatz"}$$

probability up at time n and did not pass flipper

probability down at time n and passed by a flipper

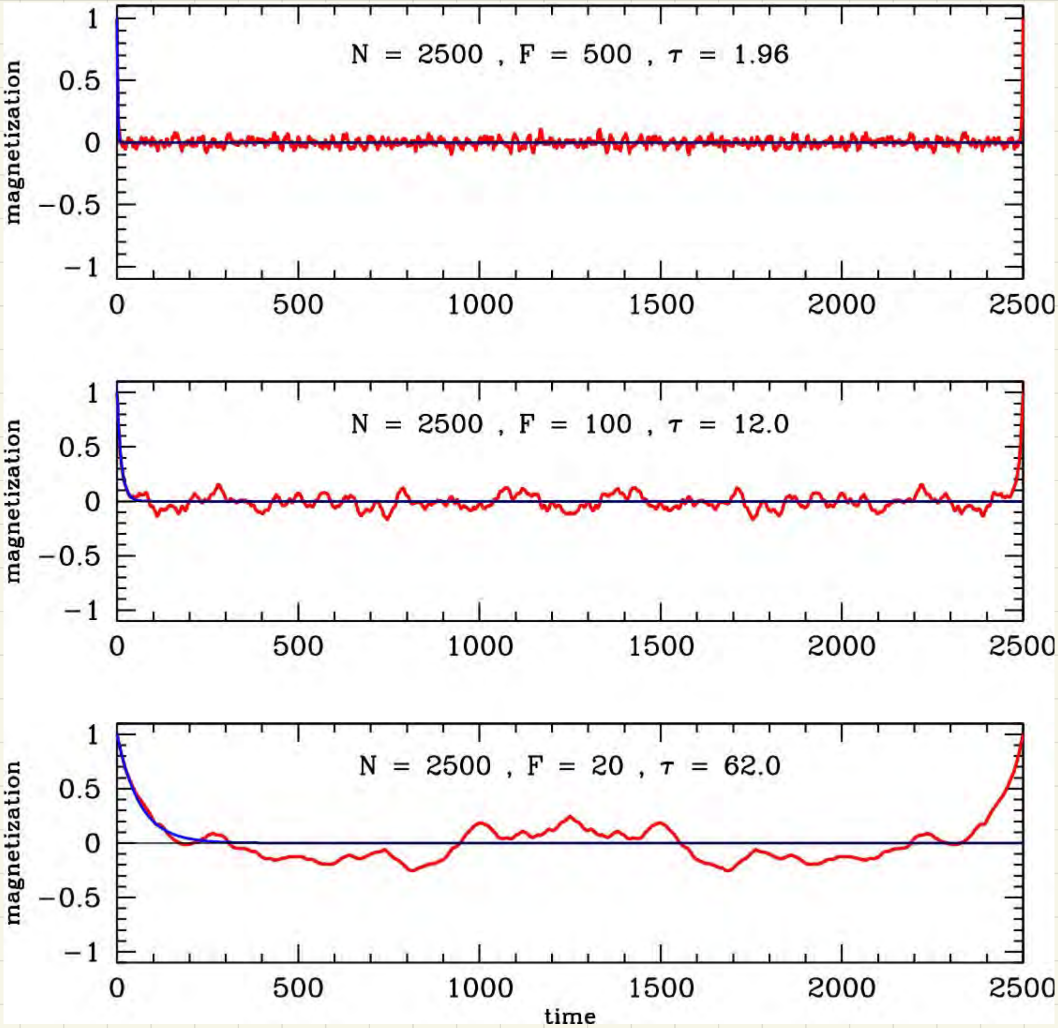
We can solve this easily: $u_n \equiv p_n - \frac{1}{2}$; $u_{n+1} = (1-2x)\,u_n = (1-2x)^n u_1$

$$p_{n+1} - \tfrac{1}{2} = (1-2x)\left(p_n - \tfrac{1}{2}\right) \Rightarrow p_n = \tfrac{1}{2} + (1-2x)^n\left(p_0 - \tfrac{1}{2}\right)$$

Thus there is exponential convergence to the equilibrium state $p_{n\to\infty} = \frac{1}{2}$ on a time scale $\tau^* = -1/\ln|1-2x|$. Note $\tau^*(0) = \tau^*(1) = 0$ while $\tau^*(1/2) = 0$. We identify $\tau^*(x)$ as the microscopic relaxation time over which local equilibrium is established.

$$|1-2x| \equiv e^{-1/\tau^*(x)}$$

$$|1-2x|^n = e^{-n/\tau^*(x)}$$



$x = \frac{500}{2500} = \frac{1}{5}$

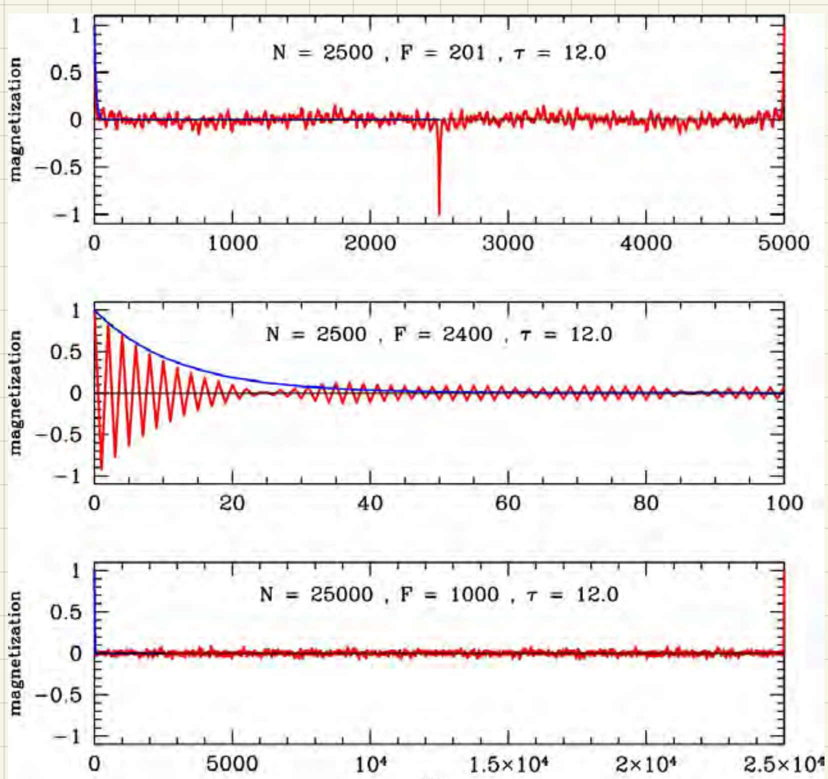
$x = \frac{100}{2500} = \frac{1}{25}$

$x = \frac{20}{2500} = \frac{1}{125}$

In the figure, we simulate the Kac ring model dynamics for rings of size $N = 2{,}500$ with $F = 20, 100,$ and 500. The initial conditions are that every spin is in the $\uparrow$ state. Note how there is an initial exponential relaxation of the magnetization $m = (N_\uparrow - N_\downarrow)/N = 2p-1$ to the equilibrium value $m_{eq} = 0$, about which m fluctuates. But at time $n = N = 2500$, we have $m = 1$ once again, and all the spins have returned to their initial $\uparrow$ state! It is easy to see why: after $n = N$ time steps, each spin will have gone completely around the ring and encountered all F flippers. If F is even, each spin will have flipped an even number of times, thus returning to its initial state. Thus $m_N = m_0$. If F is odd, each spin flips an odd number of times after N steps, and $m_N = -m_0$. But then $m_{2N} = m_0$ and the recurrence time is $2N$. We emphasize that not only does the magnetization repeat, but the entire initial configuration $\{\sigma_1, \ldots, \sigma_N\}$, where $\sigma_j = \pm 1$, has repeated, and this is true for all 2^N initial conditions. Note that the KRM satisfies the conditions for recurrence:

- map is volume-preserving (one configuration $\vec{\sigma}$ maps to a unique image $\vec{\sigma}\,'$)
- map is invertible (just run counterclockwise!)
- phase space volume is finite ($\mathrm{vol}(\Gamma) = 2^N$)

- F odd $\Rightarrow m_N = -m_0$
- $x > \frac{1}{2} \Rightarrow m_n$ oscillates
- $N = 25{,}000$: still recurrent!



- **Canonical transformations**

In Lagrangian mechanics, we are free to redefine our generalized coordinates, viz.

$$Q_\sigma = Q_\sigma(q_1, \ldots, q_n, t)$$

This is called a "point transformation". It is locally invertible provided $\det(\partial Q_\alpha / \partial q_\beta) \neq 0$. Assuming the transformation is everywhere invertible, so we can write $q_\sigma = q_\sigma(Q, t)$, the Lagrangian is

$$\tilde{L}(Q, \dot{Q}, t) = L(q(Q,t), \dot{q}(Q, \dot{Q}, t), t) + \frac{d}{dt} F(q(Q,t), t)$$

Note that $q = q(Q,t) \Rightarrow \dot{q} = \dot{q}(q, \dot{q}, t)$. For example,

$$\phi(x,y) = \tan^{-1}(y/x)$$
$$\dot{\phi}(x,y,\dot{x},\dot{y}) = (x\dot{y} - y\dot{x})/(x^2+y^2)$$

We can always add to L a total derivative of any function of coordinates and time. If $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0 \ \forall \sigma$, then $\delta Q_\sigma(t_a) = \delta Q_\sigma(t_b) = 0 \ \forall \sigma$, and Hamilton's principle,

$$\delta \int_{t_a}^{t_b} dt\, \tilde{L}(Q, \dot{Q}, t) = 0$$

yields the EL eqns

$$\frac{\partial\tilde{L}}{\partial Q_\sigma} - \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{Q}_\sigma}\right) = 0$$

This may also be derived starting with the EL eqns for the original generalized coordinates (see Eqns 15.36-37) in the notes.

In Hamiltonian mechanics, we deal with a much broader class of transformations. These are called **canonical transformations** (CTs). The word "canonical" means "conforming to a general rule or accepted procedure" (Webster). What is canonical about CTs is that they preserve a particular structure, namely that of the Poisson bracket. The general form of a CT is

$$q_\sigma = q_\sigma(Q_1,\ldots,Q_n,P_1,\ldots,P_n,t)$$
$$p_\sigma = p_\sigma(Q_1,\ldots,Q_n,P_1,\ldots,P_n,t)$$

We may write this as

\xi   \Xi

$$\xi_i = \xi_i(\Xi_1,\ldots,\Xi_{2n},t) \quad ; \quad \vec{\xi} = \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix} \quad , \quad \vec{\Xi} = \begin{pmatrix}\vec{Q}\\ \vec{P}\end{pmatrix}$$

where $i \in \{1,\ldots,2n\}$. We shall see that the transformed Hamiltonian is

$$\tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial}{\partial t}F(q,Q,t)$$

where $F(q, Q, t)$ is a function of the old and new coordinates, and of time.

We know that $\dot{\xi}_j = J_{jk} \frac{\partial H}{\partial \xi_k}$. Now consider a canonical transformation to new phase space coordinates $\Xi_a = \Xi_a(\vec{\xi}, t)$. We have

$$J = \begin{pmatrix} 0_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0_{n\times n} \end{pmatrix}$$

$$\frac{d\Xi_a}{dt} = \frac{\partial \Xi_a}{\partial \xi_j} J_{jk} \frac{\partial H}{\partial \xi_k} + \frac{\partial \Xi_a}{\partial t}$$

But if the transformation is canonical, we must have

$$\frac{d\Xi_a}{dt} = J_{ab} \frac{\partial \tilde{H}}{\partial \Xi_b} = J_{ab} \frac{\partial \xi_k}{\partial \Xi_b} \frac{\partial}{\partial \xi_k} \left(H(\vec{\xi}, t) + \frac{\partial}{\partial t} F(\vec{q}, \vec{Q}, t) \right)$$

$$= J_{ab} \frac{\partial \xi_k}{\partial \Xi_b} \frac{\partial H}{\partial \xi_k} + J_{ab} \frac{\partial^2}{\partial t \, \partial \Xi_b} F(\vec{q}, \vec{Q}, t)$$

Now define the matrix

$$M_{aj} M^{-1}_{jb} = \frac{\partial \Xi_a}{\partial \xi_j} \frac{\partial \xi_j}{\partial \Xi_b} = \frac{\partial \Xi_a}{\partial \Xi_b} = \delta_{ab}$$

$$M_{aj} \equiv \frac{\partial \Xi_a}{\partial \xi_j} \Rightarrow M^{-1}_{kb} = \frac{\partial \xi_k}{\partial \Xi_b} = (M^t)^{-1}_{bk}$$

Equating the two expressions for $d\Xi_a/dt$, we have

$$M_{aj} J_{jk} \frac{\partial H}{\partial \xi_k} + \frac{\partial \Xi_a}{\partial t} = J_{ab} (M^t)^{-1}_{bk} \frac{\partial H}{\partial \xi_k} + J_{ab} \frac{\partial^2 F}{\partial t \, \partial \Xi_b}$$

Since $\vec{\xi}$ is arbitrary, the coefficients of $\frac{\partial H}{\partial \xi_k}$ on each side must match, which says

$$MJ = J(M^t)^{-1} \Rightarrow \mathbf{MJM^t = J}$$

What about the terms in blue? We must also have

$$\frac{\partial \Xi_a}{\partial t} = J_{ab} \frac{\partial}{\partial \Xi_b} \frac{\partial}{\partial t} F(\vec{q}(\Xi), \vec{Q}, t)$$

This is true, but the proof requires results from the next section on generating functions. For now, let's focus on the result $\mathbf{MJM^t = J}$. (Note this entails $\mathbf{M^tJM = J}$ (exercise!)). An $N \times N$ real-valued matrix R which satisfies $R^tR = \mathbb{1}$ is called orthogonal, and $N \times N$ orthogonal matrices form a Lie group, $O(N)$. Thus $R^tR = \mathbb{1} \Leftrightarrow R \in O(N)$. A $2n \times 2n$ real-valued matrix M satisfying $\mathbf{M^tJM = J}$ with $J = \begin{pmatrix} 0_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0_{n\times n} \end{pmatrix}$ is called **symplectic**, and we write $M \in Sp(2n)$, the Lie group of real symplectic matrices of rank $2n$. With $M_{aj} = \partial \Xi_a / \partial \xi_j$, the Poisson bracket is preserved:

$$\{A,B\}_\xi = J_{ij} \frac{\partial A}{\partial \xi_i} \frac{\partial B}{\partial \xi_j} = J_{ij} \frac{\partial A}{\partial \Xi_a} \overbrace{\frac{\partial \Xi_a}{\partial \xi_i}}^{M_{ai}} \frac{\partial B}{\partial \Xi_b} \overbrace{\frac{\partial \Xi_b}{\partial \xi_j}}^{M_{bj} = M^t_{jb}}$$

$$= M_{ai} J_{ij} M^t_{jb} \frac{\partial A}{\partial \Xi_a} \frac{\partial B}{\partial \Xi_b} = J_{ab} \frac{\partial A}{\partial \Xi_a} \frac{\partial B}{\partial \Xi_b} = \{A,B\}_\Xi$$

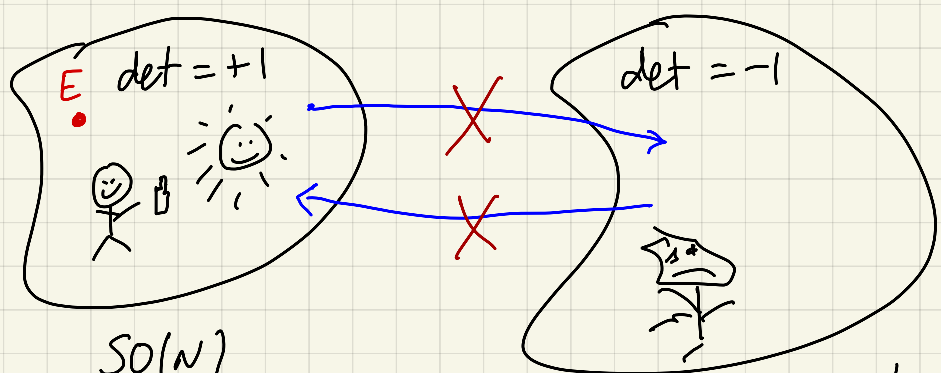
We next consider how to manufacture a canonical transformation. But before doing so, let us first show that Hamiltonian evolution itself generates a CT.

Scratch

$O(N)$: $R^t R = \mathbb{1} \Rightarrow \det R = \pm 1$

$SO(N)$: $R^t R = \mathbb{1}$ and $\det R = +1$

$O(N) \subset GL(N, \mathbb{R})$



SO(N)
(proper rotations)

unhappy island
of improper rotations

$M^t J M = J \rightarrow \det M = \pm 1$

$\det M = -1$ excluded (no unhappy island)

$$\mathrm{Pf}\, A = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2n-1)\sigma(2n)}$$

$2n \times 2n$

$$\det A = (\mathrm{Pf}\, A)^2$$

$$\mathrm{Pf}(A^t J A) = \det A \;\mathrm{Pf}\, J$$

$$M \in Sp(2N) \Rightarrow \mathrm{Pf}(M^t J M) = \cancel{\mathrm{Pf}\, J} = \det M \cancel{\det J}$$

- Proof Hamiltonian evolution generates a CT

We consider an infinitesimal evolution:

$$\xi_i(t) \to \xi_i(t+dt) = \xi_i(t) + J_{ik}\frac{\partial H}{\partial \xi_k}\bigg|_{\vec{\xi}(t)} dt + O(dt^2)$$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ \xi_i & & \xi'_i \end{array}$$

We have that $M_{ij} = \dfrac{\partial \xi'_i}{\partial \xi_j} = \delta_{ij} + J_{ir}\dfrac{\partial^2 H}{\partial \xi_j \partial \xi_r} dt + O(dt^2)$

Thus $M^t_{kl} = \delta_{kl} + J_{ls}\dfrac{\partial^2 H}{\partial \xi_k \partial \xi_s} dt$ and

$$M_{ij} J_{jk} M^t_{kl} = \left(\delta_{ij} + J_{ir}\frac{\partial^2 H}{\partial \xi_j \partial \xi_r} dt\right) J_{jk} \left(\delta_{kl} + J_{ls}\frac{\partial^2 H}{\partial \xi_k \partial \xi_s} dt\right)$$

$$= J_{il} + \left(J_{ir} J_{jl}\frac{\partial^2 H}{\partial \xi_j \partial \xi_r} + \underbrace{J_{ik} J_{ls}\frac{\partial^2 H}{\partial \xi_k \partial \xi_s}}_{\text{take } k\to r,\ s\to j} dt\right) + O(dt^2)$$

$$= J_{il} + O(dt^2)$$

Lecture 15 (November 23)

- **Generating functions for canonical transformations**

For a transformation to be canonical, we require

$$\delta \int_{t_a}^{t_b} dt \left[p_\sigma \dot{q}_\sigma - H(\vec{q},\vec{p},t)\right] = 0 = \delta \int_{t_a}^{t_b} dt \left[P_\sigma \dot{Q}_\sigma - \tilde{H}(\vec{Q},\vec{P},t)\right]$$

This is satisfied for all motions provided

$$p_\sigma \dot{q}_\sigma - H(\vec{q},\vec{p},t) = \lambda \left[P_\sigma \dot{Q}_\sigma - \tilde{H}(\vec{Q},\vec{P},t) + \frac{d}{dt} F(\vec{q},\vec{Q},t)\right]$$

where λ is a constant. We can always rescale coordinates and momenta to achieve $\lambda=1$, which we henceforth assume. Therefore,

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + P_\sigma \dot{Q}_\sigma - p_\sigma \dot{q}_\sigma + \overbrace{\frac{\partial F}{\partial Q_\sigma}\dot{Q}_\sigma + \frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial t}}^{dF/dt}$$

To eliminate the terms proportional to $\dot{Q}_\sigma$ and $\dot{q}_\sigma$, demand

$$\frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial q_\sigma} = +p_\sigma$$

We then have

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F(\vec{q},\vec{Q},t)}{\partial t}$$

This is called a "type I canonical transformation". By making Legendre transformations, we can extend this to a family of four types of CTs:

$$F(\vec{q},\vec{Q},t) = \begin{cases} F_1(\vec{q},\vec{Q},t) & \text{with} \quad p_\sigma = \dfrac{\partial F_1}{\partial q_\sigma} \, , \quad P_\sigma = -\dfrac{\partial F_1}{\partial Q_\sigma} \\ F_2(\vec{q},\vec{P},t) - P_\sigma Q_\sigma & \text{with} \quad p_\sigma = \dfrac{\partial F_2}{\partial q_\sigma} \, , \quad Q_\sigma = \dfrac{\partial F_2}{\partial P_\sigma} \\ F_3(\vec{p},\vec{Q},t) + p_\sigma q_\sigma & \text{with} \quad q_\sigma = -\dfrac{\partial F_3}{\partial p_\sigma} \, , \quad P_\sigma = -\dfrac{\partial F_3}{\partial q_\sigma} \\ F_4(\vec{p},\vec{P},t) + p_\sigma q_\sigma - P_\sigma Q_\sigma & \text{with} \quad q_\sigma = -\dfrac{\partial F_4}{\partial p_\sigma} \, , \quad Q_\sigma = \dfrac{\partial F_4}{\partial P_\sigma} \end{cases}$$

In each case, we have

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F_\gamma}{\partial t} \quad , \quad \gamma \in \{1,2,3,4\}$$

Examples of CTs from generating functions

- Consider the type-II transformation generated by

$$F_2(\vec{q},\vec{P}) = A_\sigma(\vec{q})\,P_\sigma$$

where $A_\sigma(\vec{q})$ is an arbitrary function of $\{q_1,\cdots,q_n\}$. Then

$$Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} = A_\sigma(\vec{q}) \quad , \quad p_\sigma = \frac{\partial F_2}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} P_\alpha = \frac{\partial Q_\alpha}{\partial q_\sigma} P_\alpha$$

which is equivalent to: $Q_\sigma = A_\sigma(\vec{q})$, $P_\sigma = \dfrac{\partial q_\alpha}{\partial Q_\sigma} p_\alpha$

This is in fact the general point transformation discussed previously. For linear point transformations,

$$Q_\alpha = M_{\alpha\sigma}\, q_\sigma \quad , \quad P_\beta = p_{\sigma'} M^{-1}_{\sigma'\beta}$$

$$\{Q_\alpha, P_\beta\} = M_{\alpha\sigma} M^{-1}_{\sigma'\beta} \underbrace{\{q_\sigma, p_{\sigma'}\}}_{\delta_{\sigma\sigma'}} = \delta_{\alpha\beta}$$

Note that $F_2(\vec{q},\vec{P}) = q_1 P_3 + q_3 P_1$ exchanges the labels 1 and 3: $Q_1 = \partial F_2/\partial P_1 = q_3$, $p_1 = \partial F_2/\partial q_1 = P_3$

$Q_3 = \partial F_2/\partial P_3 = q_1$, $p_3 = \partial F_2/\partial q_3 = P_1$

- Next, consider the type-I transformation generated by $F_1(\vec{q},\vec{Q}) = A_\sigma(\vec{q})\,Q_\sigma$. We then have

$$p_\sigma = \frac{\partial F_1}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} Q_\alpha \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} = -A_\sigma(\vec{q})$$

Thus, $F_1(\vec{q},\vec{Q}) = q_\sigma Q_\sigma$, for which $A_\sigma(\vec{q}) = q_\sigma$, generates

$$p_\sigma = Q_\sigma \quad , \quad P_\sigma = -q_\sigma$$

$$\vec{\xi} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \rightarrow \begin{pmatrix} -\vec{P} \\ +\vec{Q} \end{pmatrix} = \vec{\Xi}$$

- A mixed generator:

$$F(\vec{q},\vec{Q}) = q_1 Q_1 + (q_3 - Q_2) P_2 + (q_2 - Q_3) P_3$$

which is type-I wrt index $\sigma = 1$ and type II wrt $\sigma = 2, 3$. This generates

$$Q_1 = p_1 \,,\; Q_2 = q_3 \,,\; Q_3 = q_2 \,,\; P_1 = -q_1 \,,\; P_2 = p_3 \,,\; P_3 = p_2$$

(swaps p, q for label 1, swaps labels 2,3)

- $d=1$ simple harmonic oscillator: $H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2$

If we could find a CT for which

$$p = \sqrt{2mf(P)}\,\cos Q \quad , \quad q = \sqrt{\frac{2f(P)}{k}}\,\sin Q$$

then we'd have $\tilde{H}(Q,P) = f(P)$, which is cyclic in Q. The equations of motion are then $\dot{P} = -\partial\tilde{H}/\partial Q = 0$ and $\dot{Q} = \partial\tilde{H}/\partial P = f'(P)$. Taking the ratio gives

$$p = \sqrt{mk}\, q \,\mathrm{ctn}\, Q = \frac{\partial F}{\partial q}$$

This suggests a type-I transformation

$$F_1(q,Q) = \frac{1}{2}\sqrt{mk}\, q^2 \operatorname{ctn} Q$$

for which

$$p = \frac{\partial F_1}{\partial q} = \sqrt{mk}\, q \operatorname{ctn} Q$$

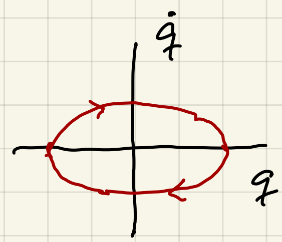
$$P = -\frac{\partial F_1}{\partial Q} = \frac{\sqrt{mk}\, q^2}{2\sin^2 Q}$$

Thus,

$$q = \frac{(2P)^{1/2}}{(mk)^{1/4}} \sin Q \Rightarrow f(P) = \sqrt{\frac{k}{m}}\, P \equiv \omega P$$

where $\omega = (k/m)^{1/2}$ is the oscillation frequency. We also have $\tilde{H}(Q,P) = \omega P = E$, the conserved energy, i.e. $P = \frac{E}{\omega}$. The equations of motion are $\dot{P} = 0$ and $\dot{Q} = f'(P) = \omega$, so the motion is $Q(t) = \omega t + \phi_0$, $P(t) = P = E/\omega \Rightarrow$

$$q(t) = \sqrt{\frac{2f(P)}{k}} \sin Q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \phi_0)$$



- **Hamilton-Jacobi theory**

General form of CT:

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F(\vec{q},\vec{Q},t)}{\partial t}$$

with

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \; , \; \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \; , \; \frac{\partial F}{\partial p_\sigma} = \frac{\partial F}{\partial P_\sigma} = 0$$

Let's be audacious and demand $\tilde{H}(\vec{Q},\vec{P},t)=0$!

This entails

$$\frac{\partial F}{\partial t}=-H \quad , \quad \frac{\partial F}{\partial q_\sigma}=p_\sigma$$

$\frac{\partial S}{\partial q_\sigma}=p_\sigma \quad , \quad \frac{\partial S}{\partial t}=-H$

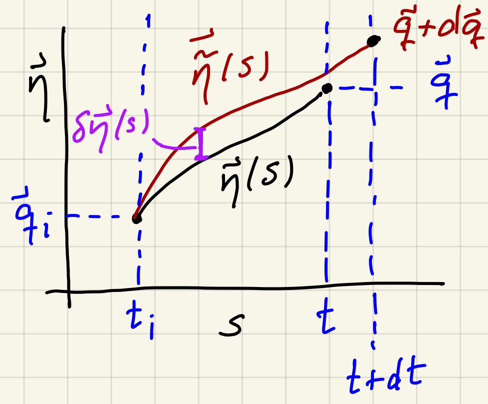
The remaining functional dependence of F may either be on $\vec{Q}$ (type I) or on $\vec{P}$ (type II). It turns out that the function we seek is none other than the action, S, expressed as a function of its endpoint values.

- **Action as a function of coordinates and time**

Consider a path $\vec{\eta}(s)$ interpolating between $(\vec{q}_i,t_i)$ and $(\vec{q},t)$ which satisfies

$$\frac{\partial L}{\partial \eta_\sigma}-\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right)=0$$

Now consider a new path $\tilde{\vec{\eta}}(s)$ starting at $(\vec{q}_i,t_i)$ but ending at $(\vec{q}+d\vec{q},t+dt)$, which also satisfies the equations of motion. We wish to compute the differential



$$dS=S[\tilde{\vec{\eta}}(s)]-S[\vec{\eta}(s)]$$

$$=\int_{t_i}^{t+dt}ds\,L(\tilde{\vec{\eta}},\dot{\tilde{\vec{\eta}}},s)-\int_{t_i}^{t}ds\,L(\vec{\eta},\dot{\vec{\eta}},s)$$

$$=L(\tilde{\vec{\eta}}(t),\dot{\tilde{\vec{\eta}}}(t),t)\,dt+\int_{t_i}^{t}ds\left\{\frac{\partial L}{\partial \eta_\sigma}\left[\tilde{\eta}_\sigma-\eta_\sigma\right]+\frac{\partial L}{\partial \dot{\eta}_\sigma}\left[\dot{\tilde{\eta}}_\sigma-\dot{\eta}_\sigma\right]\right\}$$

$$= L(\vec{\tilde{\eta}}(t), \dot{\vec{\tilde{\eta}}}(t), t)\,dt + \left.\frac{\partial L}{\partial \dot{\eta}_\sigma}\right|_t [\tilde{\eta}_\sigma(t) - \eta_\sigma(t)]$$

$$+ \int_{t_i}^{t} ds \, \underbrace{\left\{ \frac{\partial L}{\partial \eta_\sigma} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right)\right\}}_{=0} [\tilde{\eta}_\sigma(s) - \eta_\sigma(s)]$$

$$= L(\vec{\eta}(t), \dot{\vec{\eta}}(t), t)\,dt + \pi_\sigma(t)\,\delta\eta_\sigma(t) + \mathcal{O}(\delta\vec{q}\,dt)$$

where $\pi_\sigma \equiv \partial L/\partial\dot{\eta}_\sigma$ and $\delta\eta_\sigma(s) \equiv \tilde{\eta}_\sigma(s) - \eta_\sigma(s)$.

Note that

$$\begin{aligned} dq_\sigma &= \tilde{\eta}_\sigma(t+dt) - \eta_\sigma(t) \\ &= \tilde{\eta}_\sigma(t+dt) - \tilde{\eta}_\sigma(t) + \overbrace{\tilde{\eta}_\sigma(t) - \eta_\sigma(t)}^{\delta\eta_\sigma(t)} \\ &= \dot{\tilde{\eta}}_\sigma(t)\,dt + \delta\eta_\sigma(t) \\ &= \dot{\eta}_\sigma(t)\,dt + \delta\eta_\sigma(t) + \overbrace{[\dot{\tilde{\eta}}_\sigma(t) - \dot{\eta}_\sigma(t)]}^{\delta\dot{\eta}_\sigma(t)}dt \end{aligned}$$

and therefore

$$\delta\eta_\sigma(t) = dq_\sigma - \dot{\eta}_\sigma(t)\,dt - \cancel{\delta\dot{\eta}_\sigma(t)\,dt}$$

Thus, we have

$$dS = \pi_\sigma(t)\,dq_\sigma + \left[L(\vec{\eta}(t), \dot{\vec{\eta}}(t), t) - \pi_\sigma(t)\,\dot{\eta}_\sigma(t)\right]dt$$

$$= p_\sigma\,dq_\sigma - H\,dt$$

We then conclude

$$\frac{\partial S}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial S}{\partial t} = -H \quad , \quad \frac{dS}{dt} = L$$

What about the lower limit at t_i? Clearly there are $(n+1)$ constants associated with this limit, viz.

$$\{q_1(t_i), \ldots, q_n(t_i); t_i\}$$

We'll call these constants $\{\Lambda_1, \ldots, \Lambda_{n+1}\}$ and write

$$S = S(q_1, \ldots, q_n; \Lambda_1, \ldots, \Lambda_n; t) + \Lambda_{n+1}$$

We may regard each Λ_σ as either Q_σ or P_σ, i.e. that S is in general a mixed type I - type II generator. That is to say, for $\sigma \in \{1, \ldots, n\}$,

$$\Gamma_\sigma \equiv \frac{\partial S}{\partial \Lambda_\sigma} = \begin{cases} -P_\sigma & \text{if } \Lambda_\sigma = Q_\sigma \\ +Q_\sigma & \text{if } \Lambda_\sigma = P_\sigma \end{cases}$$

The last constant Λ_{n+1} will be associated with time translation.

- **Hamilton-Jacobi equation**

  Since $S(\vec{q}, \vec{\Lambda}, t)$ generates a CT for which $\tilde{H}(\vec{Q}, \vec{P}, t) = 0$, we must have $\partial F/\partial t = -H \Rightarrow$

$$H\Big(q_1, \ldots, q_n, \overbrace{\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}}^{p_\sigma}, t\Big) + \frac{\partial S}{\partial t} = 0$$

which is known as the Hamilton-Jacobi equation (HJE). The HJE is a PDE in $(n+1)$ variables $\{q_1, \ldots, q_n, t\}$.

Since $\tilde{H}(\vec{Q},\vec{P},t)=0$, the equations of motion are utterly trivial:

$$Q_\sigma(t)=\text{const.} \quad , \quad P_\sigma(t)=\text{const.} \quad \forall\sigma \ !$$

How can this yield any nontrivial dynamics? Well what we really want is the motion $\{q_\sigma(t)\}$, and to obtain this we must **invert** the relation

$$\Gamma_\sigma = \frac{\partial S(\vec{q},\vec{\Lambda},t)}{\partial \Lambda_\sigma}$$

in order to arrive at $q_\sigma(\vec{Q},\vec{P},t)$. This is possible only if

$$\det\left(\frac{\partial^2 S}{\partial q_\alpha \partial \Lambda_\beta}\right) \neq 0$$

known as the **Hessian condition**.

Example

Consider $H=\frac{p^2}{2m}$, i.e. a free particle in $d=1$ dimension. The HJE is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{\partial S}{\partial t}=0$$

One solution is

$$S(q,\Lambda,t)=\frac{m(q-\Lambda)^2}{2t}$$

$$\frac{\partial S}{\partial q}=\frac{m(q-\Lambda)}{t}$$

$$\frac{\partial S}{\partial t}=-\frac{m(q-\Lambda)^2}{2t^2}$$

for which we obtain

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \frac{m}{t}(\Lambda - q)$$

Inverting, we obtain the motion

$$q(t) = \Lambda - \frac{\Gamma t}{m} = q(0) + pt/m$$

We identify $\Lambda = q(0)$ as the initial value of q, and $\Gamma = -p$ as minus the (conserved) momentum.

The HJE may have many solutions, all yielding the same motion. For example,

$$S(q, \Lambda, t) = \sqrt{2m\Lambda}\, q - \Lambda t \quad \begin{array}{l} \nearrow \dfrac{\partial S}{\partial q} = \sqrt{2m\Lambda} \\ \searrow \dfrac{\partial S}{\partial t} = -\Lambda \end{array}$$

This yields

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \sqrt{\frac{m}{2\Lambda}}\, q - t \Rightarrow q(t) = \sqrt{\frac{2\Lambda}{m}}\,(t + \Gamma)$$

Here $\Lambda = E$ is the energy and $q(0) = \sqrt{\frac{2\Lambda}{m}}\,\Gamma$.

- **Time-independent Hamiltonians** Lecture 15 (Wed. Nov. 25)

When $\partial H/\partial t = 0$, we may reduce the order of the HJE by writing

$$S(\vec{q}, \vec{\Lambda}, t) = W(\vec{q}, \vec{\Lambda}) + T(t, \vec{\Lambda})$$

The HJE then becomes

$$H\left(\vec{q}, \frac{\partial W}{\partial \vec{q}}\right) = -\frac{\partial T}{\partial t}$$

Since the LHS is independent of t and the RHS is independent of q, each side must be equal to the same constant, which we may take to be Λ_1. Therefore

$$S(\vec{q}, \Lambda, t) = W(\vec{q}, \Lambda) - \Lambda_1 t$$

We call $W(\vec{q}, \vec{\Lambda})$ **Hamilton's characteristic function.** The HJE now takes the form

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \Lambda_1$$

Note that adding an additional constant Λ_{n+1} to S simply shifts the time variable : $t \to t - \Lambda_{n+1}/\Lambda_1$.

One-dimensional motion

Consider the Hamiltonian $H(q,p) = \frac{p^2}{2m} + U(q)$. The HJE is

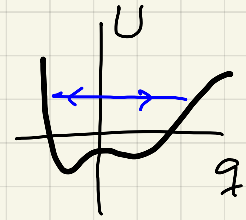
$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + U(q) = \Lambda \quad \leftarrow \text{clearly } \Lambda = E$$

with $\Lambda = \Lambda_1$. This may be recast as

$$\frac{\partial W}{\partial q} = \pm\sqrt{2m\left[\Lambda - U(q)\right]}$$

with a double-valued solution

$$W(q,\Lambda) = \pm\sqrt{2m}\int^{q} dq'\,\sqrt{\Lambda - U(q')}$$



The action (generating function) is $S(q,\Lambda,t) = W(q,\Lambda) - \Lambda t$.
The momentum is

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\left[\Lambda - U(q)\right]}$$

and

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \frac{\partial W}{\partial \Lambda} - t = \pm\sqrt{\frac{m}{2}}\int^{q(t)} dq'\,\frac{1}{\sqrt{\Lambda - U(q')}} - t$$

Thus the motion $q(t)$ is obtained by inverting

$$t + \Gamma = \pm\sqrt{\frac{m}{2}}\int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q)}} = I(q(t))$$

The lower limit on the integral is arbitrary and merely shifts t by a constant. Motion : $q(t) = I^{-1}(t+\Gamma)$

- **Separation of Variables**

If the characteristic function can be written as the sum

$$W(\vec{q},\vec{\Lambda}) = \sum_{\sigma=1}^{n} W_\sigma(q_\sigma,\vec{\Lambda})$$

the HJE is said to be completely separable. (A system may also be only partially separable.) In this case, each $W_\sigma(q_\sigma, \vec{\Lambda})$ is the solution of an equation of the form

$$H_\sigma\left(q_\sigma, \frac{\partial W_\sigma}{\partial q_\sigma}\right) = \Lambda_\sigma \,, \qquad p_\sigma = \frac{\partial W}{\partial q_\sigma} = \frac{\partial W_\sigma}{\partial q_\sigma}$$

NB: $H_\sigma(q_\sigma, p_\sigma)$ may depend on all the $\{\Lambda_1, \ldots, \Lambda_n\}$.

As an example, consider

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + \overbrace{A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\theta}}^{U(r,\theta,\phi)}$$

This is a real mess to tackle using the Lagrangian formalism. We seek a characteristic function of the form

$$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

The HJE then takes the form

$$\frac{1}{2m}\Big(\underbrace{\frac{\partial W_r}{\partial r}}_{p_r}\Big)^2 + \frac{1}{2mr^2}\Big(\underbrace{\frac{\partial W_\theta}{\partial \theta}}_{p_\theta}\Big)^2 + \frac{1}{2mr^2\sin^2\theta}\Big(\overbrace{\frac{\partial W_\phi}{\partial \phi}}^{p_\phi}\Big)^2$$

$$+ A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\theta} = \Lambda_1 = E$$

Multiply through by $r^2\sin^2\theta$ to obtain

$$\underbrace{\frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi)}_{\text{depends only on } \phi} = -\sin^2\theta\left\{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta)\right\}$$

$$\underbrace{- r^2\sin^2\theta\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\}}_{\text{depends only on } r,\,\theta}$$

Thus we must have

$$(\phi) \qquad \frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) = \Lambda_2 = \text{constant}$$

Now replace the LHS of the penultimate equation by Λ_2 and divide by $\sin^2\theta$ to get

$$\underbrace{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\theta}}_{\text{depends only on } \theta} = \underbrace{-r^2\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\}}_{\text{depends only on } r}$$

Same story. We set

$$(\theta) \qquad \frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\theta} = \Lambda_3 = \text{constant}$$

We are now left with

$$(r) \qquad \frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) + \frac{\Lambda_3}{r^2} = \Lambda_1$$

Thus,

$$\begin{aligned} S(\vec{q}, \vec{\Lambda}, t) = \sqrt{2m}\int^r dr' \sqrt{\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2}} \\ + \sqrt{2m}\int^\theta d\theta' \sqrt{\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'}} \\ + \sqrt{2m}\int^\phi d\phi' \sqrt{\Lambda_2 - C(\phi')} - \Lambda_1 t \end{aligned}$$

Now differentiate with respect to $\Lambda_{1,2,3}$ to obtain

(1) $$\Gamma_1 = \frac{\partial S}{\partial \Lambda_1} = \sqrt{\frac{m}{2}} \int^{r(t)} dr' \left[\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2}\right]^{-1/2} - t$$

(2) $$\Gamma_2 = \frac{\partial S}{\partial \Lambda_2} = -\sqrt{\frac{m}{2}} \int^{\theta(t)} \frac{d\theta'}{\sin^2\theta'} \left[\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'}\right]^{-1/2} + \sqrt{\frac{m}{2}} \int^{\phi(t)} d\phi' \left[\Lambda_2 - C(\phi')\right]^{-1/2}$$

(3) $$\Gamma_3 = \frac{\partial S}{\partial \Lambda_3} = -\sqrt{\frac{m}{2}} \int^{r(t)} \frac{dr'}{(r')^2} \left[\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2}\right]^{-1/2} + \sqrt{\frac{m}{2}} \int^{\theta(t)} d\theta' \left[\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'}\right]^{-1/2}$$

Order of solution:

1. Invert (1) to obtain $r(t)$.

2. Insert this result for $r(t)$ into (3), then invert to obtain $\theta(t)$.

3. Insert $\theta(t)$ into (2) and invert to obtain $\phi(t)$.

NB: Varying the lower limits on the integrals in (1, 2, 3) just redefines the constants $\Gamma_{1,2,3}$.

- ## Action-Angle Variables

In a system which is "completely integrable", the HJE may be solved by separation of variables. Each momentum p_σ is then a function of its conjugate coordinate q_σ plus constants : $p_\sigma = \dfrac{\partial W_\sigma}{\partial q_\sigma} = p_\sigma(q_\sigma, \vec{\Lambda})$. This satisfies $H_\sigma(q_\sigma, p_\sigma) = \Lambda_\sigma$. The level sets of each $H_\sigma(q_\sigma, p_\sigma)$ are curves $C_\sigma(\vec{\Lambda})$, which describe projections of the full motion onto the (q_σ, p_σ) plane. We will assume in general that the motion is <u>bounded</u>, which means only two types of projected motion are possible :

librations : periodic oscillations about an equilibrium

rotations : in which an angular coordinate advances by 2π in each cycle

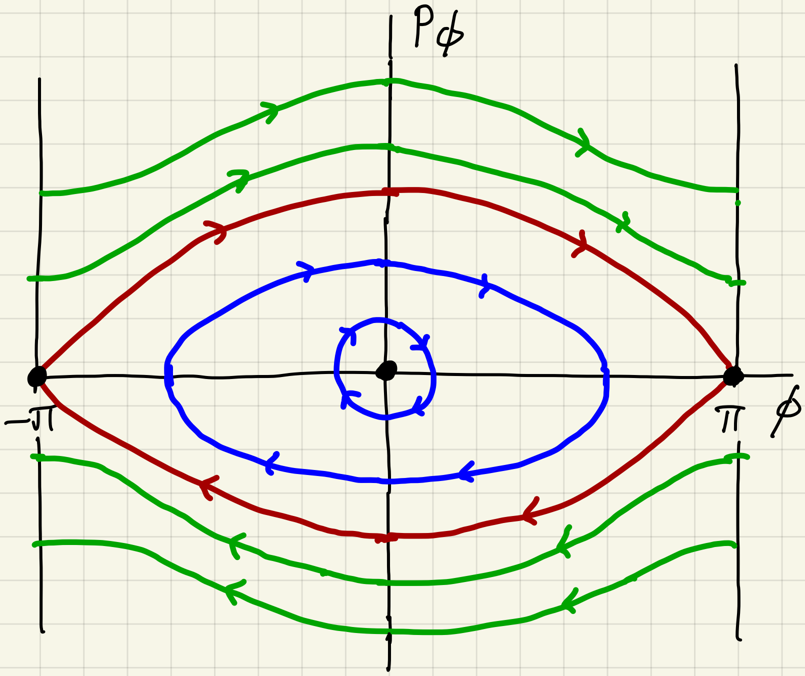
Example : simple pendulum $H(\phi, p_\phi) = \dfrac{p_\phi^2}{2I} + \frac{1}{2} I\omega^2 (1-\cos\phi)$

rotations : $E > I\omega^2$

librations : $0 < E < I\omega^2$

separatrix : $E = I\omega^2$

Generically, each $C_\sigma(\vec{\Lambda})$ is either a libration or a rotation.



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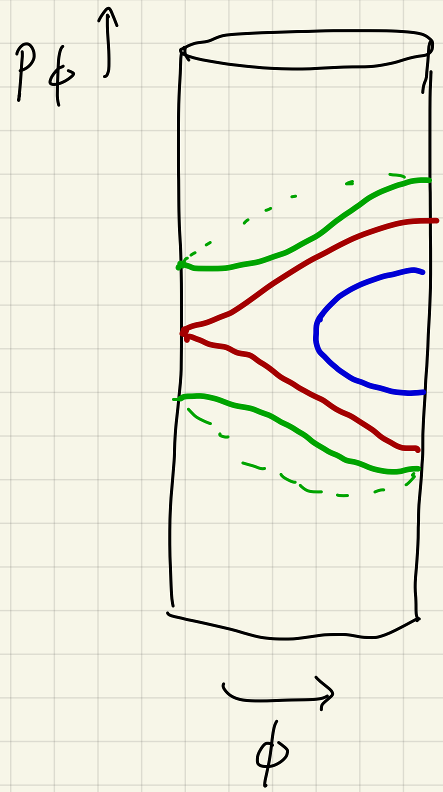
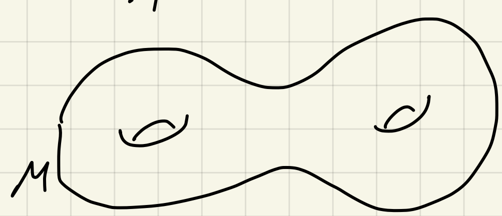


$$\int_M dK = -4\pi$$

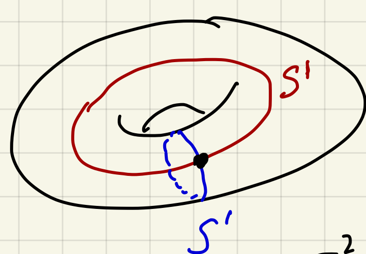


$g = 2$

$\mathbb{R}/\mathbb{Z} \cong S^1$

$S^1 \times \mathbb{R} = \text{cylinder}$

$S^1 \times S^1 = T^2$



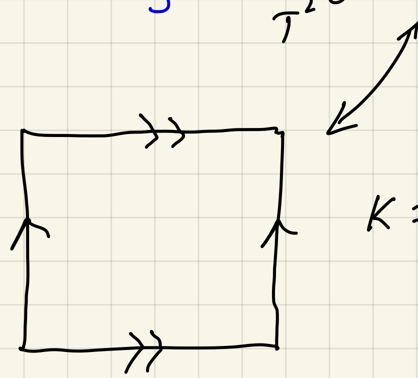
$g = 1$

$$\int_{T^2} dK = 0$$

$$\int_M d\Sigma\, K = 2\pi(2-2g)$$

$g = \#$ holes

$\chi \equiv 2-2g$



$K = 0$

$$\int_{S^2} d\Sigma \frac{1}{R^2} = \int_{S^2} d\hat{n} = 4\pi$$

$aba^{-1}b^{-1} \neq 1$

Topologically, both librations and rotations are homotopic to (= "can be continuously distorted to") a circle, S^1. Note though that they cannot be continuously distorted into each other, since librations can continuously be deformed to the point of static equilibrium, while rotations cannot. For a system with n freedoms, the motion is thus confined to n-tori:

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$$

$C_1(\vec{\lambda})$ $\quad$ $C_n(\vec{\lambda})$

These are called **invariant tori**, because for a given set of initial conditions, the motion is confined to one such n-torus. **Invariant tori never intersect!** Note that phase space is of dimension 2n, while the invariant tori, which fill phase space, are of dimension n. (Think about the phase space for the simple pendulum, which is topologically a cylinder, covered by librations and rotations which themselves are topologically circles.)

Action-angle variables $(\vec{\phi}, \vec{J})$ are a set of coordinates $(\vec{\phi})$ and momenta $(\vec{J})$ which cover phase space with invariant n-tori. The n **actions** $\{J_1, \ldots, J_n\}$ specify a particular n-torus, and the n **angles** $\{\phi_1, \ldots, \phi_n\}$

coordinatize each such torus. Invariance of the tori means that

$$\dot{J}_\sigma = -\frac{\partial H}{\partial \phi_\sigma} = 0 \Rightarrow H = H(\vec{J})$$

Each coordinate ϕ_σ describes the projected motion around C_σ, and is normalized so that

$$\oint_{C_\sigma} d\phi_\sigma = 2\pi \quad (\text{once around } C_\sigma)$$

The dynamics of the angle variables are given by

$$\dot{\phi}_\sigma = \frac{\partial H}{\partial J_\sigma} = \nu_\sigma(\vec{J})$$

Thus $\phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\vec{J})\, t$. The n frequencies $\{\nu_\sigma(\vec{J})\}$ describe the rates at which the circles C_σ are traversed.

(topologically!)

Lecture 17 (Nov. 30)

- **Canonical transformation to action-angle variables**

These AAVs sound great! Very intuitive! But how do we find them? Since the $\{J_\sigma\}$ determine the $\{C_\sigma\}$ and since each q_σ determines a point (two points, in the case of librations) on C_σ, this suggests a type-II

CT with generator $F_2(\vec{q}, \vec{J})$:

$$p_\sigma = \frac{\partial F_2}{\partial q_\sigma} \quad , \quad \phi_\sigma = \frac{\partial F_2}{\partial J_\sigma}$$

Now

$$2\pi = \oint_{C_\sigma} d\phi_\sigma = \oint_{C_\sigma} d\left(\frac{\partial F_2}{\partial J_\sigma}\right) = \oint_{C_\sigma} dq_\sigma \frac{\partial^2 F_2}{\partial J_\sigma \partial q_\sigma} = \frac{\partial}{\partial J_\sigma} \oint dq_\sigma \, p_\sigma$$

we are led to define

$$J_\sigma = \frac{1}{2\pi} \oint_{C_\sigma} dq_\sigma \, p_\sigma$$

Procedure :

(1) Separate and solve the HJE for $W(\vec{q}, \vec{\Lambda}) = \sum_\sigma W_\sigma(q_\sigma, \vec{\Lambda})$.

(2) Find the orbits $C_\sigma(\vec{\Lambda})$, i.e. the level sets satisfying the conditions $H_\sigma(q_\sigma, p_\sigma ; \vec{\Lambda}) = \Lambda_\sigma$.

(3) Invert the relation $J_\sigma(\vec{\Lambda}) = \frac{1}{2\pi} \oint_{C_\sigma} dq_\sigma \, p_\sigma$ to obtain $\vec{J}(\vec{\Lambda}) \xrightarrow{\text{(invert)}} \vec{\Lambda}(\vec{J})$

(4) The type-II generator to AAVs is

$$F_2(\vec{q}, \vec{J}) = \sum_\sigma W_\sigma(q_\sigma, \vec{\Lambda}(\vec{J}))$$

Let's now work through some examples.

Harmonic oscillator

Our Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$, so the HJE equation is

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2} m\omega_0^2 q^2 = \Lambda$$

We have

$$p = \frac{\partial W}{\partial q} = \pm\sqrt{2m\Lambda - m^2\omega_0^2 q^2}$$

Simplify by defining

$$q = \sqrt{\frac{2\Lambda}{m\omega_0^2}}\sin\theta \Rightarrow p = \sqrt{2m\Lambda}\cos\theta$$

and so

$$J = \frac{1}{2\pi}\oint dq\, p = \frac{1}{2\pi}\cdot\frac{2\Lambda}{\omega_0}\int_0^{2\pi} d\theta\cos^2\theta = \frac{\Lambda}{\omega_0}$$

We still must solve the HJE:

$$\frac{dW}{d\theta} = \frac{dW}{dq}\cdot\frac{\partial q}{\partial\theta} = \sqrt{2m\Lambda}\cos\theta\cdot\sqrt{\frac{2\Lambda}{m\omega_0^2}}\cos\theta = 2J\cos^2\theta$$

Integrate to get

$$W(\theta, J) = J\theta + \frac{1}{2}J\sin 2\theta + \text{const.}$$

$$\theta = \cos^{-1}\left[q/\sqrt{2m\Lambda(J)}\right] \longrightarrow W(q, J)$$

Then

$$\phi = \frac{\partial W}{\partial J}\bigg|_q = \theta + \tfrac{1}{2}\sin 2\theta + J(1+\cos 2\theta)\,\frac{\partial\theta}{\partial J}\bigg|_q$$

Now $q = \sqrt{2J/m\omega_0}\,\sin\theta$ so

$$dq = \frac{\sin\theta}{\sqrt{2m\omega_0 J}}\,dJ + \sqrt{\frac{2J}{m\omega_0}}\,\cos\theta\, d\theta \implies \frac{\partial\theta}{\partial J}\bigg|_q = -\frac{1}{2J}\tan\theta$$

Plugging into our expression for ϕ, we obtain $\phi = \theta$. (Not much of a surprise.) Thus, the full CT is

$$q = \left(\frac{2J}{m\omega_0^2}\right)\sin\phi \quad , \quad p = \sqrt{2m\omega_0 J}\,\cos\phi$$

and the Hamiltonian is $H(\phi, J) = \omega_0 J$. The equations of motion are

call it $H \equiv \tilde{H}$

$$\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0 \quad , \quad \dot{J} = -\frac{\partial H}{\partial \phi} = 0$$

with solution

$$\phi(t) = \phi(0) + \omega_0 t$$
$$J(t) = J(0)$$

and of course $\nu(J) = \omega_0$ (independent of J).

- Please read § 15.5.5 (AAV for particle in a box)

- **Integrability and motion on invariant tori**

Recall that a **completely integrable** system may be solved by separation of variables, and that

$$H(\vec{q},\vec{p}) \rightarrow \tilde{H}(\vec{\phi},\vec{J}) = \tilde{H}(\vec{J})$$

$$\dot{J}_\sigma = -\frac{\partial \tilde{H}}{\partial \phi_\sigma} = 0 \Rightarrow J_\sigma(t) = J_\sigma(0)$$

$$\dot{\phi}_\sigma = +\frac{\partial \tilde{H}}{\partial J_\sigma} = \nu_\sigma(\vec{J}) \Rightarrow \phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\vec{J})\, t$$

Thus, the angle variables wind around the invariant torus at constant rates $\nu_\sigma(\vec{J})$. While each $\phi_\sigma(t)$ winds around its own circle, the motion of the system as a whole will not be periodic unless the frequencies $\nu_\sigma(\vec{J})$ are commensurate, which means that there exists a time T (i.e. the period) such that $\nu_\sigma T = 2\pi k_\sigma$ with $k_\sigma \in \mathbb{Z}$ $\forall\, \sigma \in \{1,\ldots,n\}$. Thus

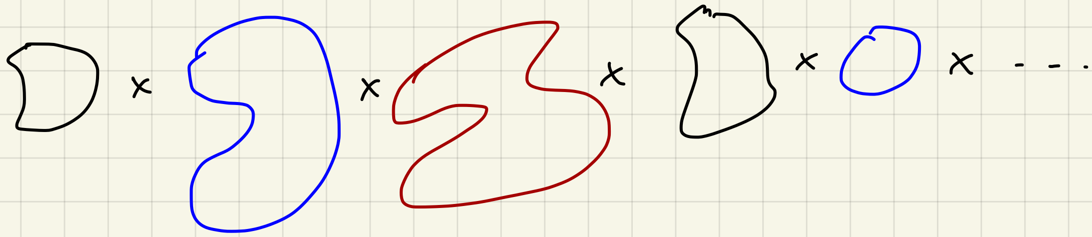
$$\frac{\nu_\alpha}{\nu_\beta} = \frac{k_\alpha}{k_\beta} \in \mathbb{Q} \quad \forall\ \alpha,\beta \in \{1,\ldots,n\}$$

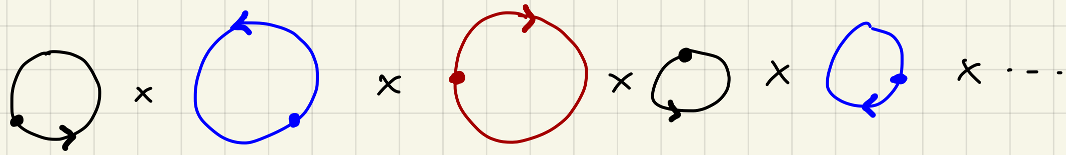
T is the smallest such period if $\{k_1,\ldots,k_n\}$ have no common factors. On a given torus, either all orbits are periodic or none is periodic.

In terms of the original $\{q_1,\ldots,q_n\}$ coordinates,

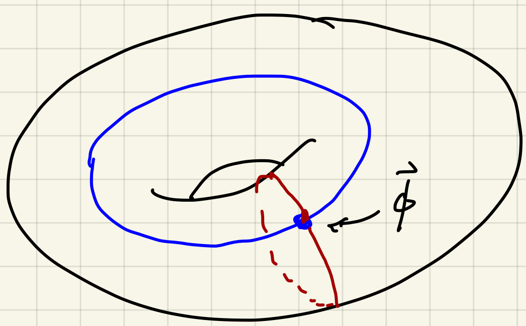
Scratch

n-torus : $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$

$T^n =$ 

$=$ 

$\vec{\phi}(t) = \{ \phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t), \ldots \}$



There are two possibilities:

(i) libration: $q_\sigma(t) = \sum_{\vec{l}\in\mathbb{Z}^n} A^{(\sigma)}_{l_1\cdots l_n}\, e^{il_1\phi_1(t)}\cdots e^{il_n\phi_n(t)}$

(ii) rotation: $q_\sigma(t) = q^0_\sigma\, \phi_\sigma(t) + \sum_{\vec{l}\in\mathbb{Z}^n} B^{(\sigma)}_{l_1\cdots l_n}\, e^{il_1\phi_1(t)}\cdots e^{il_n\phi_n(t)}$

where a complete rotation results in $\Delta q_\sigma = 2\pi\, q^0_\sigma$.

- **Liouville-Arnol'd Theorem**

This is another statement of what it means for a Hamiltonian system to be integrable. Suppose a Hamiltonian $H(\vec{q},\vec{p})$ has n **first integrals** $I_k(\vec{q},\vec{p})$, where $k\in\{1,\ldots,n\}$. This means

$$\frac{dI_k}{dt} = \sum_{\sigma=1}^{n}\left(\frac{\partial I_k}{\partial q_\sigma}\frac{dq_\sigma}{dt} + \frac{\partial I_k}{\partial p_\sigma}\frac{dp_\sigma}{dt}\right) = \underbrace{\{I_k, H\}}_{\text{Poisson bracket}} = 0$$

If the $\{I_k\}$ are independent functions, meaning that $\{\vec{\nabla} I_k\}$ form a set of n linearly independent vectors at almost every point in phase space M, and if all the first integrals commute with respect to the Poisson bracket, i.e. $\{I_k, I_l\} = 0$ for all k, l ($\Leftrightarrow$ I_k and I_l **in involution**), then:

(i) The space $\underbrace{M_I}_{\dim(M_I)=n} \equiv \{(\vec{q},\vec{p})\in \underbrace{M}_{\dim(M)=2n} \mid I_k(\vec{q},\vec{p}) = C_k \;\forall\, k\in\{1,\ldots,n\}\}$ is diffeomorphic to an n-torus $T^n = S^1\times S^1\times\cdots\times S^1$, on which one can introduce action-angle variables on a set of overlapping patches whose union contains M_I, where the angle variables are coordinates on M_I and the action variables are the first integrals.

(ii) The transformed Hamiltonian is $\tilde{H} = \tilde{H}(\vec{I})$, hence

$$\dot{I}_k = -\frac{\partial \tilde{H}}{\partial \phi_k} = 0$$

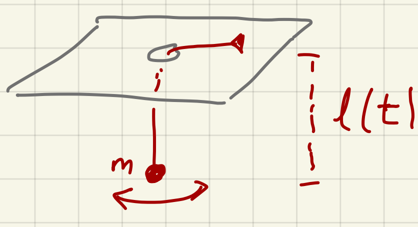
$$\dot{\phi}_k = +\frac{\partial \tilde{H}}{\partial I_k} = \nu_k(\vec{I}) \Rightarrow \phi_k(t) = \phi_k(0) + \nu_k(\vec{I})\, t$$

Note this does not require $\tilde{H} = \sum_k \tilde{H}_k(I_k)$.

- **Adiabatic invariants**

Adiabatic processes in thermodynamics are ones in which no heat is exchanged between a system and its environment. In mechanics, adiabatic perturbations are slow, smooth changes to a Hamiltonian system's parameters. A typical example: slowly changing the length $\ell(t)$ of a pendulum.

General setting: $H = H(\vec{q}, \vec{p}; \lambda(t))$. All explicit time dependence in H is through $\lambda(t)$. If ω_0 is a characteristic frequency of the motion when λ is constant, then



$$\epsilon \equiv \omega_0^{-1} \left| \frac{d \ln \lambda}{dt} \right|$$

provides a dimensionless measure of the rate of change of $\lambda(t)$. We require $\epsilon \ll 1$ for adiabaticity. Under such conditions, the action variables are preserved to exponential accuracy. (We will see just what this means.) For the SHO, the energy, action, and oscillation frequency are related according to $J = E/\nu$. During an adiabatic process, $E(t)$ and $\nu(t)$ may vary appreciably, but $J(t)$ remains very nearly constant. Thus, if θ_0 is the oscillation amplitude, then assuming small oscillations,

$$E = \frac{1}{2} mg\ell\,\theta_0^2 = \nu J = \sqrt{\frac{g}{\ell}}\,J$$

$$\Rightarrow \theta_0(\ell) \approx \frac{2J}{m\sqrt{g}\,\ell^{3/2}}$$

Adiabatic invariance then says $\theta_0(\ell) \propto \ell^{-3/2}$.

Consider now an $n=1$ system, and suppose that for fixed λ the type-II generator to action-angle variables is $S(q, J\,;\lambda)$. Now let $\lambda = \lambda(t)$, in which case

$$\tilde{H}(\phi, J, t) = H(J\,;\lambda) + \frac{\partial S}{\partial \lambda}\frac{d\lambda}{dt}$$

ϕ - dependence through $S(q(\phi, J;\lambda), J\,;\lambda)$

where

$$H(J\,;\lambda) = H\big(q(\phi, J;\lambda), p(\phi, J;\lambda)\,;\lambda\big)$$

Note that $H(J;\lambda)$ is independent of ϕ, because for fixed λ the function $S(q, J\,;\lambda)$ generates the AAV.

Hamilton's equations are now

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial J} = \nu(J;\lambda) + \frac{\partial^2 S}{\partial\lambda\,\partial J}\frac{d\lambda}{dt}$$

$$\dot{J} = -\frac{\partial \tilde{H}}{\partial \phi} = -\frac{\partial^2 S}{\partial\lambda\,\partial\phi}\frac{d\lambda}{dt}$$

where $\nu(J;\lambda) \equiv \partial H(J;\lambda)/\partial J$ and where

$$S(\phi,J;\lambda) = S(q(\phi,J;\lambda),J;\lambda) \equiv \sum_{m=-\infty}^{\infty} S_m(J;\lambda)\,e^{im\phi}$$

Fourier analyzing the equation for $\dot{J}$, we have

$$\dot{J} = -i\dot{\lambda}\sum_{m=-\infty}^{\infty} m\,\frac{\partial S_m}{\partial\lambda}\,e^{im\phi}$$

Now,

$$\Delta J = J(\infty) - J(-\infty) = \int_{-\infty}^{\infty} dt\,\dot{J}$$

$$= -i\sum_{m=-\infty}^{\infty} m\int_{-\infty}^{\infty} dt\,\frac{\partial S_m(J;\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\,e^{im\phi}$$

($m=0$ term is cancelled)

Now $\phi(t) = \nu t + \phi(0)$ to good accuracy, since $\dot{\lambda}$ is small. So we must evaluate expressions such as

$$m\neq 0: \quad \mathcal{A}_m \equiv \int_{-\infty}^{\infty} dt\,\underbrace{\left\{\frac{\partial S_m(J;\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\right\}}_{f(t)}\,e^{im\nu t}\,e^{im\phi(0)}$$

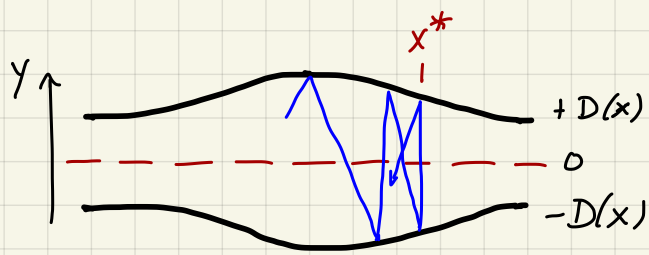
The bracketed term is a smooth function of time t which by assumption varies slowly on the scale ν^{-1}. Call it $f(t)$.

We assume $f(t)$ may be analytically continued off the real t axis, and that its closest singularities in the complex t plane lie at $\mathrm{Im}\, t = \pm\tau$, where $|\nu\tau| \gg 1$. Then $\mathcal{J}_m \sim e^{-|m\nu\tau|} = e^{-|m|/\epsilon}$, which is exponentially small in $|\nu\tau| = \frac{1}{\epsilon}$ (hence only $m = \pm 1$ need be considered). Thus, ΔJ may be kept arbitrarily small if $\lambda(t)$ is varied sufficiently slowly.

$$f(t) = \frac{1}{\pi}\frac{\tau}{t^2+\tau^2} \Rightarrow \int_{-\infty}^{\infty} dt\, f(t)\, e^{im\nu t} = e^{-|m\nu|\tau} \propto e^{-\frac{|m|}{\epsilon}}$$

- **Examples**

Mechanical mirror : A point particle bounces between two curves $y = \pm D(x)$, with $|D'(x)| \ll 1$. The bounce time is $\tau_\perp / 2 v_y$, and we assume $\tau \ll L/v_x$ where L = length.



So there are many bounces, during which the particle samples $D(x)$. The adiabatic invariant is the action,

$$J = \frac{1}{2\pi}\oint dy\, p_y = \frac{2}{\pi} m v_y D(x)$$

The energy is

$$E = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m v_x^2 + \frac{\pi^2 J^2}{8 m D^2(x)}$$

Thus,

$$v_x^2 = \frac{2E}{m} - \left(\frac{\pi J}{2 m D(x)}\right)^2$$

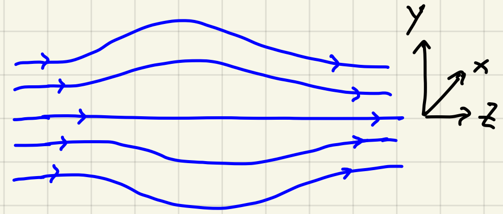
which means the particle turns around when $D(x^*) = \dfrac{\pi J}{\sqrt{8mE}}$.

A pair of such mirrors (when $D(x) = D(-x)$) confines the particle.

Similar physics is present in the magnetic mirror, or "magnetic bottle", discussed in §15.7.3. There the adiabatic invariant is the magnetic moment,

$$M = -\frac{eJ}{mc} = \frac{e^2}{2\pi m c^2}\,\Phi$$

where J = action and Φ = magnetic flux.



magnetic field lines (azimuthally symmetric about the middle line)

- **Resonances**

What happens when $n > 1$? We then have

$$\dot{J}^\alpha = -i\dot{\lambda}\sum_{\vec{m}\in\mathbb{Z}^n} m^\alpha \,\frac{\partial S_{\vec{m}}(J;\lambda)}{\partial\lambda}\, e^{i\vec{m}\cdot\vec{\phi}}$$

and

$$\Delta J^\alpha = -i\sum_{\vec{m}\in\mathbb{Z}^n} m^\alpha \int_{-\infty}^{\infty}\! dt\, \frac{\partial S_{\vec{m}}(\vec{J};\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\, e^{i\vec{m}\cdot\vec{\nu}t}\, e^{i\vec{m}\cdot\vec{\beta}}$$

When $\vec{m}\cdot\vec{\nu}(\vec{J}) = 0$, we have a **resonance**, and the integral grows linearly in the time limits, which is a violation of adiabatic invariance. Resonances may result in the breakdown of invariant tori, and provide a route to chaos. Resonances can thus only occur when two or more frequencies $\nu_\alpha(\vec{J})$ have a ratio which is a rational number. But even if the frequency ratios are all irrational, any such irrational number may be approximated to arbitrary accuracy by some choice of rational number. To understand how to deal with resonances, we need **canonical perturbation theory**.

Lecture 18 (Dec. 2)

• Canonical perturbation theory

Suppose

$$H(\vec{q},\vec{p},t) = H_0(\vec{q},\vec{p},t) + \underset{\text{dimensionless}}{\epsilon} H_1(\vec{q},\vec{p},t)$$

where $|\epsilon| \ll 1$. Let's implement a type-II CT generated by $S(\vec{q},\vec{P},t)$ (not intended to signify Hamilton's principal function):

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial}{\partial t} S(\vec{q},\vec{P},t)$$

Expand everything in sight in powers of ϵ:

$$\begin{aligned}
q_\sigma &= Q_\sigma + \epsilon q_{1,\sigma} + \epsilon^2 q_{2,\sigma} + \cdots \\
p_\sigma &= P_\sigma + \epsilon p_{1,\sigma} + \epsilon^2 p_{2,\sigma} + \cdots \\
\tilde{H} &= \tilde{H}_0 + \epsilon \tilde{H}_1 + \epsilon^2 \tilde{H}_2 + \cdots \\
S &= \underbrace{q_\sigma P_\sigma}_{\text{identity CT}} + \epsilon S_1 + \epsilon^2 S_2 + \cdots
\end{aligned}$$

Then

$$\begin{aligned}
Q_\sigma = \frac{\partial S}{\partial P_\sigma} &= q_\sigma + \epsilon \frac{\partial S_1}{\partial P_\sigma} + \epsilon^2 \frac{\partial S_2}{\partial P_\sigma} + \cdots \\
&= Q_\sigma + \left(q_{1,\sigma} + \frac{\partial S_1}{\partial P_\sigma} \right)\epsilon + \left(q_{2,\sigma} + \frac{\partial S_2}{\partial P_\sigma} \right)\epsilon^2 + \cdots
\end{aligned}$$

We also have

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + \epsilon \frac{\partial S_1}{\partial q_\sigma} + \epsilon^2 \frac{\partial S_2}{\partial q_\sigma} + \dots$$

$$= P_\sigma + \epsilon p_{1,\sigma} + \epsilon^2 p_{2,\sigma} + \dots$$

Thus we conclude, order by order in ϵ,

$$q_{k,\sigma} = -\frac{\partial S_k}{\partial P_\sigma} \quad , \quad p_{k,\sigma} = \frac{\partial S_k}{\partial q_\sigma}$$

Next, expand the Hamiltonian:

$$\tilde{H}(\vec{Q},\vec{P},t) = H_0(\vec{q},\vec{p},t) + \epsilon H_1(\vec{q},\vec{p},t) + \frac{\partial S}{\partial t}$$

$$= H_0(\vec{Q},\vec{P},t) + \frac{\partial H_0}{\partial Q_\sigma}(q_\sigma - Q_\sigma) + \frac{\partial H_0}{\partial P_\sigma}(p_\sigma - P_\sigma) + \dots$$

$$+ \epsilon H_1(\vec{Q},\vec{P},t) + \epsilon \frac{\partial}{\partial t} S_1(\vec{Q},\vec{P},t) + O(\epsilon^2)$$

$$= H_0(\vec{Q},\vec{P},t) + \left(-\frac{\partial H_0}{\partial Q_\sigma}\frac{\partial S_1}{\partial P_\sigma} + \frac{\partial H_0}{\partial P_\sigma}\frac{\partial S_1}{\partial Q_\sigma} + \frac{\partial S_1}{\partial t} + H_1 \right)\epsilon + O(\epsilon^2)$$

Notice we are writing $q_\sigma = Q_\sigma + (q_\sigma - Q_\sigma) = Q_\sigma - \epsilon \frac{\partial S_1}{\partial P_\sigma} + \dots$
so, e.g.

$$S_1(\vec{q},\vec{P},t) = S_1(\vec{Q},\vec{P},t) + (q_\sigma - Q_\sigma)\frac{\partial S_1}{\partial Q_\sigma} + \dots$$

$$= S_1(\vec{Q},\vec{P},t) - \frac{\partial S_1(\vec{Q},\vec{P},t)}{\partial P_\sigma}\frac{\partial S_1(\vec{Q},\vec{P},t)}{\partial Q_\sigma}\epsilon + O(\epsilon^2)$$

Thus, we have

$$\tilde{H}(\vec{Q},\vec{P},t) = H_0(\vec{Q},\vec{P},t) + \left(H_1 + \{S_1, H_0\} + \frac{\partial S_1}{\partial t}\right)\epsilon + \mathcal{O}(\epsilon^2)$$
$$= \tilde{H}_0(\vec{Q},\vec{P},t) + \epsilon \tilde{H}_1(\vec{Q},\vec{P},t) + \mathcal{O}(\epsilon^2)$$

We therefore conclude

$$\tilde{H}_0(\vec{Q},\vec{P},t) = H_0(\vec{Q},\vec{P},t)$$
$$\tilde{H}_1(\vec{Q},\vec{P},t) = \left[H_1 + \{S_1, H_0\} + \frac{\partial S_1}{\partial t}\right]_{\vec{Q},\vec{P},t}$$

We are left with a single equation in two unknowns, i.e. $\tilde{H}_1$ and S_1. The problem is underdetermined. We could at this point demand $\tilde{H}_1 = 0$, but this is just one of many possible choices. Similar story in QM:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = (\hat{H}_0 + \epsilon \hat{H}_1)|\psi\rangle$$

Now define $|\psi\rangle \equiv e^{i\hat{S}/\hbar}|\chi\rangle$ with $\hat{S} = \epsilon \hat{S}_1 + \epsilon^2 \hat{S}_2 + \dots$. Then find

$$i\hbar \frac{\partial}{\partial t}|\chi\rangle = \hat{H}_0|\chi\rangle + \epsilon\left(\hat{H}_1 + \frac{1}{i\hbar}\underbrace{[\hat{S}_1, \hat{H}_0]}_{\text{commutator}} + \frac{\partial \hat{S}_1}{\partial t}\right)|\chi\rangle + \dots$$
$$\equiv \hat{\tilde{H}}|\chi\rangle$$

Typically we choose $\hat{S}_1$ such that the $\mathcal{O}(\epsilon)$ term vanish. But this isn't the only possible choice. (Note here the correspondence $\{A, B\} \leftrightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}]$.)

- **CPT for $n=1$ systems**

Here we demonstrate the implementation of CPT in a general $n=1$ system. We will need to deal with resonances when $n>1$, which we discuss later on.

We assume $H(q,p) = H_0(q,p) + \epsilon H_1(q,p)$ is time-independent.

Let (ϕ_0, J_0) be AAV for H_0, so that

$$\tilde{H}_0(J_0) = H_0(q(\phi_0,J_0), p(\phi_0,J_0))$$

We define

$$\tilde{H}_1(\phi_0,J_0) \equiv H_1(q(\phi_0,J_0), p(\phi_0,J_0))$$

We assume that $\tilde{H} = \tilde{H}_0 + \epsilon \tilde{H}_1$ is integrable, which for $n=1$ is indeed always the case. [Reminder: $H(q,p)=E$ means all motion takes place on the one-dimensional level sets of $H(q,p)$.]

Thus there must be a CT taking $(\phi_0,J_0) \to (\phi,J)$, where

$$\tilde{H}(\phi_0(\phi,J), J_0(\phi,J)) = E(J)$$

We solve by a type-II CT:

$$S(\phi_0,J) = \underbrace{\phi_0 J}_{\text{identity CT}} + \epsilon S_1(\phi_0,J) + \epsilon^2 S_2(\phi_0,J) + \cdots$$

Then

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \epsilon \frac{\partial S_1}{\partial \phi_0} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0} + \cdots$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + \epsilon^2 \frac{\partial S_2}{\partial J} + \cdots$$

We also write

$$E(J) = E_0(J) + \epsilon E_1(J) + \epsilon^2 E_2(J) + \dots$$
$$= \tilde{H}_0(J_0) + \epsilon \tilde{H}_1(\phi_0, J_0) \qquad \text{(no higher order terms)}$$

Now we expand $\tilde{H}(\phi_0, J_0) = \tilde{H}(\phi_0, J + \underbrace{(J_0 - J)}_{\delta J})$ in powers of $(J_0 - J)$:

$$\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \frac{\partial \tilde{H}_0}{\partial J}(J_0 - J) + \frac{1}{2}\frac{\partial^2 \tilde{H}_0}{\partial J^2}(J_0 - J)^2$$
$$+ \epsilon \tilde{H}_1(\phi_0, J) + \epsilon \left.\frac{\partial \tilde{H}_1}{\partial J}\right|_{\phi_0}(J_0 - J) + \dots$$

Substitute

$$J_0 - J = \epsilon \frac{\partial S_1}{\partial \phi_0} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0} + \dots$$

and collect terms to obtain

$$\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \left(\tilde{H}_1 + \frac{\partial \tilde{H}_0}{\partial J}\frac{\partial S_1}{\partial \phi_0}\right)\epsilon$$
$$+ \left(\frac{\partial \tilde{H}_0}{\partial J}\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\frac{\partial^2 \tilde{H}_0}{\partial J^2}\left(\frac{\partial S_1}{\partial \phi_0}\right)^2 + \frac{\partial \tilde{H}_1}{\partial J}\frac{\partial S_1}{\partial \phi_0}\right)\epsilon^2 + \dots$$

where all terms on the RHS are expressed in terms of ϕ_0 and J. We may now read off

(0) $E_0(J) = \tilde{H}_0(J)$

(1) $E_1(J) = \tilde{H}_1(\phi_0, J) + \dfrac{\partial \tilde{H}_0}{\partial J}\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}$

(2) $E_2(J) = \dfrac{\partial \tilde{H}_0}{\partial J}\dfrac{\partial S_2(\phi_0, J)}{\partial \phi_0} + \dfrac{1}{2}\dfrac{\partial^2 H_0}{\partial J^2}\left(\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}\right)^2 + \dfrac{\partial \tilde{H}_1(\phi_0, J)}{\partial J}\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}$

But the RHS should be independent of ϕ_0! How can this be? **We use the freedom in the functions $S_k(\phi_0, J)$ to make it so.** Let's see just how this works.

Each of the expressions on the RHSs must be equal to its average over ϕ_0 if it is to be independent of ϕ_0:

$$\langle f(\phi_0)\rangle = \int_0^{2\pi} \frac{d\phi_0}{2\pi}\, f(\phi_0)$$

The averages $\langle RHS(\phi_0, J)\rangle$ are taken **at fixed J** and **not** at fixed J_0. We must have that

$$S_k(\phi_0, J) = \sum_{\ell=-\infty}^{\infty} S_{k,\ell}(J)\, e^{i\ell\phi_0}$$

Thus

$$\left\langle \frac{\partial S_k}{\partial \phi_0} \right\rangle = \frac{1}{2\pi}\Big\{ S_k(2\pi, J) - S_k(0, J)\Big\} = 0$$

Now let's implement this in our hierarchy. Consider the level (1) equation,

$$E_1(J) = \tilde{H}_1(\phi_0, J) + \underbrace{\frac{\partial \tilde{H}_0}{\partial J}}_{\nu_0(J)} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0}$$

Taking the average,

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J)\rangle + \frac{\partial \tilde{H}_0}{\partial J} \underbrace{\left\langle \frac{\partial S_1(\phi_0, J)}{\partial \phi_0}\right\rangle}_{\text{this vanishes}}$$

$$= \langle \tilde{H}_1 \rangle$$

Thus,

$$\langle \tilde{H}_1 \rangle = \tilde{H}_1 + \nu_0(J)\frac{\partial S_1}{\partial \phi_0} \Rightarrow \frac{\partial S_1(\phi_0,J)}{\partial \phi_0} = \frac{\langle \tilde{H}_1 \rangle_J - \tilde{H}_1(\phi_0,J)}{\nu_0(J)}$$

If we Fourier decompose

$$\tilde{H}_1(\phi_0,J) = \sum_{l=-\infty}^{\infty} \tilde{H}_{1,l}(J)\, e^{il\phi_0}$$

then we obtain

$$l \neq 0:\ il\, S_{1,l}(J) = \tilde{H}_{1,l}(J) \times (1-\delta_{l,0}) \Rightarrow S_{1,l}(J) = -\frac{i}{l}\tilde{H}_{1,l}(J)$$

We are free to set $S_{1,0}(J) \equiv 0$ (why?).

Now that we've got the hang of the logic here, let's go to second order:

$$E_2(J) = \underbrace{\frac{\partial \tilde{H}_0}{\partial J}}_{\nu_0(J)} \underbrace{\frac{\partial S_2(\phi_0,J)}{\partial \phi_0}}_{\text{averages to zero}} + \frac{1}{2}\underbrace{\frac{\partial^2 H_0}{\partial J^2}}_{\partial\nu_0/\partial J}\Big(\underbrace{\frac{\partial S_1(\phi_0,J)}{\partial \phi_0}}_{\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}}\Big)^2 + \frac{\partial \tilde{H}_1(\phi_0,J)}{\partial J}\underbrace{\frac{\partial S_1(\phi_0,J)}{\partial \phi_0}}_{\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}}$$

Taking the average,

$$E_2 = \frac{1}{2}\frac{\partial \nu_0}{\partial J}\left\langle \left(\frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}\right)^2\right\rangle + \left\langle \frac{\partial \tilde{H}_1}{\partial J}\left(\frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}\right)\right\rangle$$

which yields, after some work,

$$\frac{\partial S_2}{\partial \phi_0} = \frac{1}{\nu_0^2}\Bigg\{\left\langle \frac{\partial \tilde{H}_1}{\partial J}\right\rangle \langle \tilde{H}_1\rangle - \left\langle \frac{\partial \tilde{H}_1}{\partial J}\tilde{H}_1\right\rangle - \frac{\partial \tilde{H}_1}{\partial J}\langle \tilde{H}_1\rangle + \frac{\partial \tilde{H}_1}{\partial J}\tilde{H}_1$$
$$+ \frac{1}{2}\frac{\partial \ln \nu_0}{\partial J}\left(\langle \tilde{H}_1^2\rangle - 2\langle \tilde{H}_1\rangle^2 + 2\langle \tilde{H}_1\rangle \tilde{H}_1 - \tilde{H}_1^2\right)\Bigg\}$$

and the energy to second order is

$$E(J) = \tilde{H}_0 + \epsilon \langle \tilde{H}_1 \rangle + \frac{\epsilon^2}{\nu_0} \left\{ \left\langle \frac{\partial \tilde{H}_1}{\partial J} \right\rangle \langle \tilde{H}_1 \rangle - \left\langle \frac{\partial \tilde{H}_1}{\partial J} \tilde{H}_1 \right\rangle + \frac{1}{2} \frac{\partial \ln \nu_0}{\partial J} \left(\langle \tilde{H}_1^2 \rangle - \langle \tilde{H}_1 \rangle^2 \right) \right\} + \mathcal{O}(\epsilon^3)$$

Note that we don't need $S(\phi_0, J)$ to obtain $E(J)$, though of course we do need it to obtain (ϕ_0, J_0) in terms of (ϕ, J). The perturbed frequencies are $\nu(J) = \partial E/\partial J$. For the full motion, we need

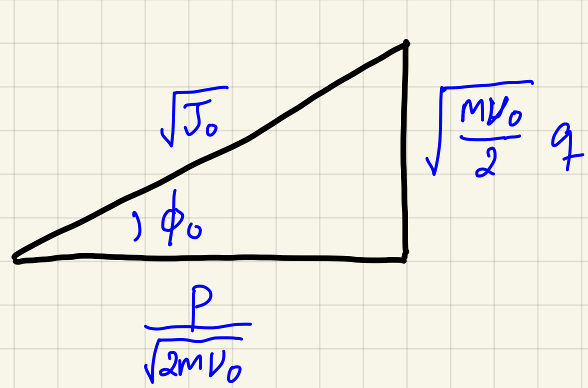
$$(\phi, J) \longrightarrow (\phi_0, J_0) \longrightarrow (q, p)$$

- **Example : quartic oscillator**

The Hamiltonian is

$$H(q,p) = \underbrace{\frac{p^2}{2m} + \frac{1}{2} m \nu_0^2 q^2}_{H_0} + \underbrace{\frac{\alpha}{4} \epsilon q^4}_{\epsilon H_1}$$

Recall the AAV for the SHO :



$$J_0 = \frac{p^2}{2m\nu_0} + \frac{1}{2} m \nu_0 q^2 = \frac{H_0}{\nu_0}$$

$$\phi_0 = \tan^{-1}\left(\frac{m \nu_0 q}{p} \right)$$

$$q = \left(\frac{2J_0}{m\nu_0} \right)^{1/2} \sin \phi_0$$

$$p = \sqrt{2 J_0 m \nu_0} \cos \phi_0$$

Thus, we have

$$\tilde{H}(\phi_0, J_0) = \nu_0 J_0 + \frac{\alpha}{4}\epsilon\left(\sqrt{\frac{2J_0}{m\nu_0}}\,\sin\phi_0\right)^4$$

$$= \underbrace{\nu_0 J_0}_{\tilde{H}_0(J_0)} + \underbrace{\epsilon\left(\frac{\alpha}{m^2\nu_0^2}\right) J_0^{\,2}\sin^4\phi_0}_{\tilde{H}_1(\phi_0, J_0)}$$

We therefore have

$\langle \sin^4\phi_0 \rangle = \frac{3}{8}$

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle$$

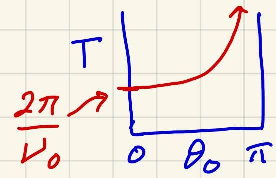
$$= \frac{\alpha J^2}{m^2\nu_0^2}\int_0^{2\pi}\frac{d\phi_0}{2\pi}\sin^4\phi_0 = \frac{3\alpha J^2}{8m^2\nu_0^2}$$

The frequency, to order ϵ, is then

$$\nu(J) = \frac{\partial}{\partial J}(E_0 + \epsilon E_1) = \nu_0 + \frac{3\epsilon\alpha J}{4m^2\nu_0^2} + \mathcal{O}(\epsilon^2)$$

To this order, we may replace J above by $J_0 = \frac{1}{2}m\nu_0 A^2$, where A = amplitude of oscillations. Thus,

pendulum:



$$\nu(A) = \nu_0 + \frac{3\epsilon\alpha A^2}{8m\nu^2} + \mathcal{O}(\epsilon^2)$$

Only for the linear oscillator $\ddot{q} = -\nu_0^{\,2} q$ is the oscillation frequency independent of the amplitude.

Next, let's work through the CT $(\phi_0, J_0) \to (\phi, J)$.

We have

$$v_0 \frac{\partial S_1}{\partial \phi_0} = \frac{\alpha J^2}{m^2 v_0^2}\left(\frac{3}{8} - \sin^4\phi_0\right)$$

$$\Rightarrow \quad S_1(\phi_0, J) = \frac{\alpha J^2}{8m^2 v_0^3}\left(3 + 2\sin^2\phi_0\right)\sin\phi_0\cos\phi_0$$

and

$$\phi = \phi_0 + \epsilon\frac{\partial S_1}{\partial J} + \mathcal{O}(\epsilon^2)$$

$$= \phi_0 + \frac{\epsilon\alpha J}{4m^2 v_0^3}\left(3 + 2\sin^2\phi_0\right)\sin\phi_0\cos\phi_0 + \mathcal{O}(\epsilon^2)$$

$$J_0 = J + \epsilon\frac{\partial S_1}{\partial \phi_0}$$

$$= J + \frac{\epsilon\alpha J^2}{8m^2 v_0^3}\left(4\cos(2\phi_0) - \cos(4\phi_0)\right) + \mathcal{O}(\epsilon^2)$$

To lowest nontrivial order we may invert to obtain

$$J = J_0 - \frac{\epsilon\alpha J_0^2}{8m^2 v_0^3}\left(4\cos(2\phi_0) - \cos(4\phi_0)\right) + \mathcal{O}(\epsilon^2)$$

With $q = (2J_0/mv_0)^{1/2}\sin\phi_0$ and $p = (2mv_0 J_0)^{1/2}\cos\phi_0$, we can obtain (q,p) in terms of (ϕ, J).

- **$n > 1$: degeneracies and resonances**

Generalizing the CPT formalism to $n > 1$ is straightforward. We have $S = S(\vec{\phi}_0, \vec{J})$, so with $\alpha \in \{1, \ldots, n\}$,

$$J_0^\alpha = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \epsilon \frac{\partial S_1}{\partial \phi_0^\alpha} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0^\alpha} + \dots$$

$$\phi^\alpha = \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \epsilon \frac{\partial S_1}{\partial J^\alpha} + \epsilon^2 \frac{\partial S_2}{\partial J^\alpha} + \dots$$

and

$$E_0(\vec{J}) = \tilde{H}_0(\vec{J})$$

$$E_1(\vec{J}) = \tilde{H}_1(\vec{\phi}_0, \vec{J}) + \nu_0^\alpha(\vec{J}) \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha}$$

$$E_2(\vec{J}) = \nu_0^\alpha(\vec{J}) \frac{\partial S_2(\phi_0, \vec{J})}{\partial \phi_0^\alpha} + \frac{1}{2} \frac{\partial \nu_0^\alpha(\vec{J})}{\partial J^\beta} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\beta} + \frac{\partial \tilde{H}_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha}$$

where $\nu_0^\alpha(\vec{J}) = \partial \tilde{H}_0(\vec{J}) / \partial J^\alpha$. Now we average:

$$\langle f(\vec{\phi}_0, \vec{J}) \rangle = \int_0^{2\pi} \frac{d\phi_0^1}{2\pi} \dots \int_0^{2\pi} \frac{d\phi_0^n}{2\pi} f(\vec{\phi}_0, \vec{J})$$

The equation for $S_1(\vec{\phi}_0, \vec{J})$ is

$$\nu_0^\alpha \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha} = \langle \tilde{H}_1(\vec{\phi}_0, \vec{J}) \rangle - \tilde{H}_1(\vec{\phi}_0, \vec{J})$$

$$= -\sum_{\vec{l} \in \mathbb{Z}^n}{}' V_{\vec{l}}(\vec{J}) e^{i \vec{l} \cdot \vec{\phi}_0}$$

where $V_{\vec{l}}(\vec{J}) = \tilde{H}_{1,\vec{l}}(\vec{J})$, i.e. $\tilde{H}_1(\vec{\phi}_0, \vec{J}) = \sum_{\vec{l}} V_{\vec{l}}(\vec{J}) e^{i \vec{l} \cdot \vec{\phi}_0}$

The prime on the sum means $\vec{\ell} = (0, 0, \ldots, 0)$ is excluded. The solution is

$$S_1(\vec{\phi}_0, \vec{J}) = -i \sum_{\vec{\ell} \in \mathbb{Z}^n}{}' \frac{V_{\vec{\ell}}(\vec{J})}{\vec{\ell} \cdot \vec{\nu}_0(\vec{J})} \, e^{i \vec{\ell} \cdot \vec{\phi}_0}$$

When the **resonance condition**

$$\vec{\ell} \cdot \vec{\nu}_0(\vec{J}) = 0$$

pertains (with $\vec{\ell} \neq 0$), the denominator vanishes and CPT breaks down. One can always find such an $\vec{\ell}$ whenever two or more of the frequencies $\nu_0^\alpha(\vec{J})$ have a rational ratio. Suppose for example that $\nu_0^2(\vec{J}) / \nu_0^1(\vec{J}) = r/s$ with $r, s \in \mathbb{Z}$ relatively prime. Then $r\nu_0^1 = s\nu_0^2$ and with $\vec{\ell} = (r, -s, 0, \ldots, 0)$, we have $\vec{\ell} \cdot \vec{\nu}_0 = 0$. Even if all the frequency ratios are irrational, for large enough $|\vec{\ell}|$ we can make $|\vec{\ell} \cdot \vec{\nu}_0|$ as small (but finite) as we please. In §15.9, we'll see how any given resonance may be **removed** canonically. We're just looking at things the wrong way at the moment.

Lecture 19 (Dec. 7)

- **Removal of resonances**

We now consider how to deal with resonances arising in canonical perturbation theory. We start with the periodic time-dependent Hamiltonian,

$$H(\phi, J, t) = H_0(J) + \epsilon V(\phi, J, t)$$

where

$$V(\phi, J, t) = V(\phi + 2\pi, J, t) = V(\phi, J, t + T)$$

This is identified as $n = \frac{3}{2}$ degrees of freedom, since it is equivalent to a dynamical system of dimension $2n = 3$.

The double periodicity of $V(\phi, J, t)$ entails that it may be expressed as a double Fourier sum, viz.

$$V(\phi, J, t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{V}_{k,l}(J)\, e^{ik\phi}\, e^{-il\Omega t}$$

$$\left(\hat{V}_{-k,-l} = \hat{V}^*_{k,l}\right)$$

where $\Omega = 2\pi/T$. Hamilton's equations are then

$$\dot{J} = -\frac{\partial H}{\partial \phi} = -\epsilon \frac{\partial V}{\partial \phi} = -i\epsilon \sum_{k,l} k\, \hat{V}_{k,l}(J)\, e^{i(k\phi - l\Omega t)}$$

$$\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0(J) + \epsilon \sum_{k,l} \frac{\partial \hat{V}_{k,l}(J)}{\partial J}\, e^{i(k\phi - l\Omega t)}$$

where $\omega_0(J) = \partial H_0/\partial J$. The resonance condition follows from inserting the $\mathcal{O}(\epsilon^0)$ solution $\phi(t) = \omega_0(J)\,t$, yielding

$$k\,\omega_0(J) - l\,\Omega = 0$$

When this condition is satisfied, secular forcing results in a linear increase of J with time. To do better, let's focus on a particular resonance $(k,l) = (k_0, l_0)$. The resonance condition $k_0\,\omega_0(J) = l_0\,\Omega$ fixes the action J. There is still an infinite set of possible (k,l) values leading to resonance at the same value of J, i.e. $(k,l) = (pk_0, pl_0)$ for all $p \in \mathbb{Z}$. But the Fourier amplitudes $\hat{V}_{pk_0,pl_0}(J)$ decrease in magnitude, typically exponentially in $|p|$. So we will assume k_0 and l_0 are relatively prime, and consider $p \in \{-1, 0, +1\}$. We define

$$\hat{V}_{0,0}(J) \equiv \hat{V}_0(J)\,, \quad \hat{V}_{k_0,l_0}(J) = \hat{V}^{*}_{-k_0,-l_0}(J) \equiv \hat{V}_1(J)\,e^{i\delta}$$

and obtain

$$\dot{J} = 2\epsilon k_0\,\hat{V}_1(J)\,\sin(k_0\phi - l_0\Omega t + \delta)$$

$$\dot{\phi} = \omega_0(J) + \epsilon\,\frac{\partial \hat{V}_0(J)}{\partial J} + 2\epsilon\,\frac{\partial \hat{V}_1(J)}{\partial J}\,\cos(k_0\phi - l_0\Omega t + \delta)$$

Now let's expand, writing $J = J_0 + \Delta J$ and

$$\psi = k_0\phi - l_0\Omega t + \delta + \begin{cases} 0 & \text{if } \epsilon > 0 \\ \pi & \text{if } \epsilon < 0 \end{cases}$$

resulting in (assume wolog $\epsilon > 0$)

$$\dot{\Delta J} = -2\epsilon\, k_0\, \hat{V}_1(J_0)\sin\psi$$

$$\dot{\psi} = k_0\, \omega_0'(J_0)\,\Delta J + \epsilon\, k_0\, \hat{V}_0'(J_0) - 2\epsilon\, k_0\, \hat{V}_1'(J_0)\cos\psi$$

To lowest nontrivial order in ϵ, we may drop the $\mathcal{O}(\epsilon)$ terms in the second equation, and write

$$\frac{d\Delta J}{dt} = -\frac{\partial K}{\partial \psi} \quad , \quad \frac{d\psi}{dt} = \frac{\partial K}{\partial \Delta J}$$

with

$$K(\psi, \Delta J) = \tfrac{1}{2} k_0\, \omega_0'(J_0)\,(\Delta J)^2 - 2\epsilon\, k_0\, \hat{V}_1'(J_0)\cos\psi$$

which is the Hamiltonian for a simple pendulum! The resulting equations of motion yield $\ddot{\psi} + \gamma^2 \sin\psi = 0$, with $\gamma^2 = 2\epsilon\, k_0^2\, \omega_0'(J_0)\, \hat{V}_1(J_0)$.

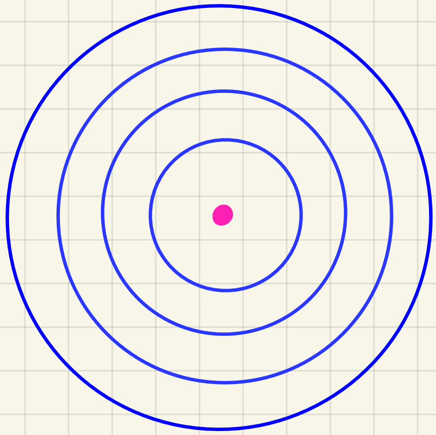
So what do we conclude from this analysis? The original 1-torus (i.e. circle S^1), with

$$J(t) = J_0 \quad , \quad \phi(t) = \omega_0(J_0)\, t + \phi(0)$$

is destroyed. Both it and its neigboring 1-tori are replaced by a separatrix and surrounding libration and rotation phase curves (see figure). The amplitude

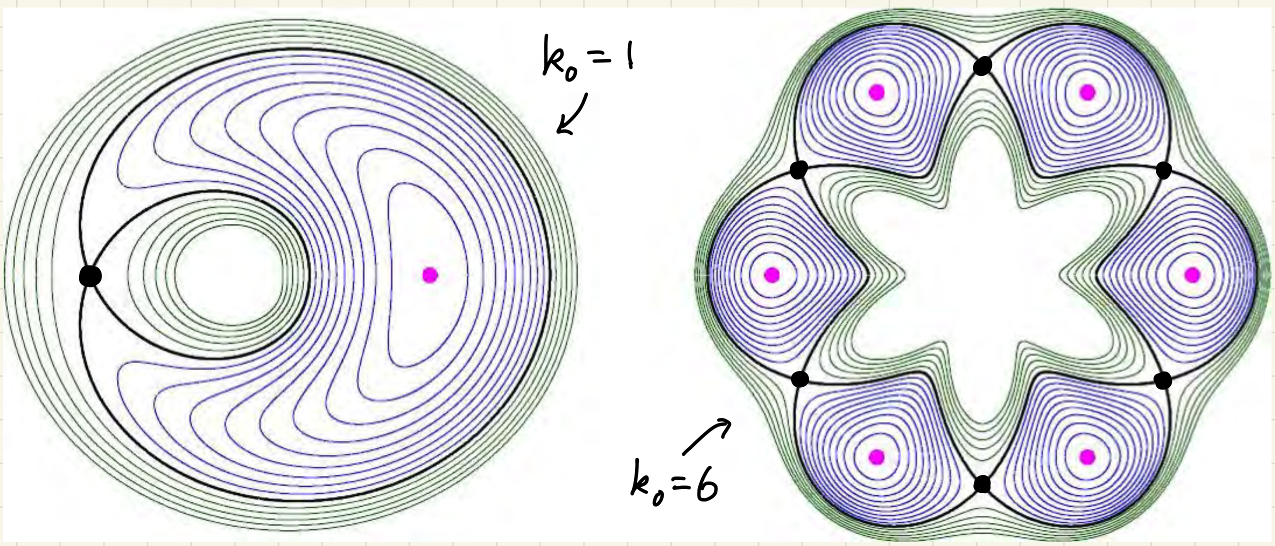
Unperturbed ($\epsilon = 0$) :

$$H_0(q,p) = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2$$



- librations only
- no separatrix
- elliptic fixed point

Perturbed ($\epsilon > 0$) :



Librations (blue), rotations (green), and separatrices (black) for $k_0 = 1$ (left) and $k_0 = 6$ (right), plotted in (q,p) plane. Elliptic fixed points are shown as magenta dots. Hyperbolic (black) fixed points lie at the self-intersections of the separatrices.

of the separatrix is $\left(8\epsilon \hat{V}_1(J_0)/\omega'(J_0)\right)^{1/2}$. This analysis is justified provided $(\Delta J)_{max} \ll J_0$ and $\gamma \ll \omega_0$, or

$$\epsilon \ll \frac{d\ln \omega_0}{d\ln J_0} \ll \frac{1}{\epsilon}$$

- $n=2$ systems

We now consider the Hamiltonian $H(\vec{\phi},\vec{J}) = H_0(\vec{\phi}) + \epsilon H_1(\vec{\phi},\vec{J})$ with $\vec{\phi} = (\phi_1, \phi_2)$ and $\vec{J} = (J_1, J_2)$. We write

$$H_1(\vec{\phi},\vec{J}) = \sum_{\vec{l}\in\mathbb{Z}^2} \hat{V}_{\vec{l}}(\vec{J})\, e^{i\vec{l}\cdot\vec{\phi}}$$

with $\vec{l} = (l_1, l_2)$ and $\hat{V}_{-\vec{l}}(\vec{J}) = \hat{V}^*_{\vec{l}}(\vec{J})$ since $H_1(\vec{\phi},\vec{J}) \in \mathbb{R}$.
Resonances exist whenever $r\omega_1(\vec{J}) = s\omega_2(\vec{J})$, where

$$\omega_{1,2}(\vec{J}) = \frac{\partial H_0}{\partial J_{1,2}}$$

We eliminate the resonance in two steps:

(1) Invoke a CT $(\vec{\phi},\vec{J}) \to (\vec{\varphi},\vec{\mathcal{J}})$ generated by

$$F_2(\vec{\phi},\vec{\mathcal{J}}) = (r\phi_1 - s\phi_2)\mathcal{J}_1 + \phi_2\mathcal{J}_2$$

This yields

$$J_1 = \frac{\partial F_2}{\partial \phi_1} = r\mathcal{J}_1 \qquad \varphi_1 = \frac{\partial F_2}{\partial J_1} = r\phi_1 - s\phi_2$$

$$J_2 = \frac{\partial F_2}{\partial \phi_2} = \mathcal{J}_2 - s\mathcal{J}_1 \qquad \varphi_2 = \frac{\partial F_2}{\partial \mathcal{J}_2} = \phi_2$$

Why did we do this? We did so in order to transform to a rotating frame where $\varphi_1 = r\phi_1 - s\phi_2$ is slowly varying, i.e. $\dot{\varphi}_1 = r\dot{\phi}_1 - s\dot{\phi}_2 \approx r\omega_1 - s\omega_2 = 0$. We also have $\dot{\varphi}_2 = \dot{\phi}_2 \approx \omega_2$. Now we could instead have used the generator

$$F_2 = \phi_1 \mathcal{J}_1 + (r\phi_1 - s\phi_2)\mathcal{J}_2$$

resulting in $\varphi_1 = \phi_1$ and $\varphi_2 = r\phi_1 - s\phi_2$. Here φ_2 is the slow variable while φ_1 oscillates with frequency $\approx \omega_1$. Which should we choose? We will wind up averaging over the faster of $\varphi_{1,2}$, and we want the fast frequency itself to be as slow as possible, for reasons which have to do with the removal of higher order resonances. (More on this further on below.) We'll assume wolog that $\omega_1 > \omega_2$.

Inverting to find $\vec{\phi}(\vec{\varphi})$, we have

$$\phi_1 = \frac{1}{r}\varphi_1 + \frac{s}{r}\varphi_2 \quad , \quad \phi_2 = \varphi_2$$

so we have

$$\begin{aligned}\tilde{H}(\vec{\varphi},\vec{\mathcal{J}}) &= H_0(\vec{J}(\vec{\mathcal{J}})) + \epsilon H_1(\vec{\phi}(\vec{\varphi}), \vec{J}(\vec{\mathcal{J}})) \\ &\equiv \tilde{H}_0(\vec{\mathcal{J}}) + \epsilon \underbrace{\sum_{\vec{\ell}} \tilde{\hat{V}}_{\vec{\ell}}(\vec{\mathcal{J}}) \exp\left\{\frac{i\ell_1}{r}\varphi_1 + i\left(\frac{\ell_1 s}{r} + \ell_2\right)\varphi_2\right\}}_{\tilde{H}_1(\vec{\varphi},\vec{\mathcal{J}})}\end{aligned}$$

We now average over the fast variable φ_2. This yields the constraint $s\ell_1 + r\ell_2 = 0$, which we solve by writing $(\ell_1, \ell_2) = (pr, -ps)$ for $p \in \mathbb{Z}$. We then have

$$\langle \tilde{H}_1(\vec{\varphi}, \vec{\mathcal{J}}) \rangle = \sum_p \hat{\tilde{V}}_{pr,-ps}(\vec{\mathcal{J}}) e^{ip\varphi_1}$$

The averaging procedure is justified close to a resonance, where $|\dot{\varphi}_2| \gg |\dot{\varphi}_1|$. Note that $\mathcal{J}_2$ now is conserved, i.e. $\dot{\mathcal{J}}_2 = 0$. Thus $\mathcal{J}_2 = \frac{s}{r} J_1 + J_2$ is a new invariant.

At this point, only the $(\varphi_1, \mathcal{J}_1)$ variables are dynamical. φ_2 has been averaged out and $\mathcal{J}_2$ is constant. Since the Fourier amplitudes $\hat{\tilde{V}}_{pr,-ps}(\vec{\mathcal{J}})$ are assumed to decay rapidly with increasing $|p|$, we consider only $p \in \{-1, 0, +1\}$ as we did in the $n = \frac{3}{2}$ case. We thereby obtain the effective Hamiltonian

$$\begin{aligned} K(\varphi_1, \mathcal{J}_1, \mathcal{J}_2) \approx \tilde{H}_0(\mathcal{J}_1, \mathcal{J}_2) &+ \epsilon \hat{\tilde{V}}_{0,0}(\mathcal{J}_1, \mathcal{J}_2) \\ &+ 2\epsilon \hat{\tilde{V}}_{r,-s}(\mathcal{J}_1, \mathcal{J}_2) \cos \varphi_1 \end{aligned}$$

where we have absorbed any phase in $\hat{\tilde{V}}_{r,-s}(\vec{\mathcal{J}})$ into a shift of φ_1, so we may consider $\hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}})$ and $\hat{\tilde{V}}_{r,s}(\vec{\mathcal{J}})$ to be real functions of $\vec{\mathcal{J}} = (\mathcal{J}_1, \mathcal{J}_2)$. The fixed points of the dynamics then satisfy

$$\dot{\varphi}_1 = \frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1} + \epsilon \frac{\partial \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1} + 2\epsilon \frac{\partial \hat{\tilde{V}}_{r,-s}}{\partial \mathcal{J}_1} \cos\varphi_1 = 0$$

$$\dot{\mathcal{J}}_1 = -2\epsilon \hat{\tilde{V}}_{r,-s} \sin\varphi_1 = 0$$

Note that a stationary solution here corresponds to a periodic solution in our original variables, since we have shifted to a rotating frame. Thus $\varphi_1 = 0$ or $\varphi_1 = \pi$, and

$$\frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1} = \frac{\partial H_0}{\partial J_1}\frac{\partial J_1}{\partial \mathcal{J}_1} + \frac{\partial H_0}{\partial J_2}\frac{\partial J_2}{\partial \mathcal{J}_1}$$

$$= r w_1 - s w_2 = 0$$

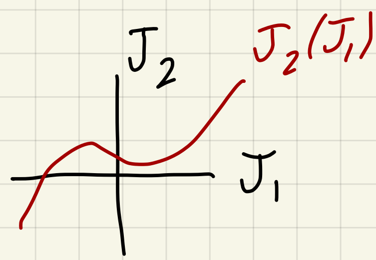
Thus fixed points occur for

$$\frac{\partial \hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}})}{\partial \mathcal{J}_1} \pm 2 \frac{\partial \hat{\tilde{V}}_{r,-s}(\vec{\mathcal{J}})}{\partial \mathcal{J}_1} = 0 \qquad \begin{pmatrix} \varphi_1 = 0 \\ \varphi_1 = \pi \end{pmatrix}$$

There are two cases to consider:

- **accidental degeneracy**

In this case, the degeneracy condition



$$r w_1(J_1, J_2) = s w_2(J_1, J_2)$$

Thus, we have $J_2 = J_2(J_1)$. This is the case when $H_0(J_1, J_2)$ is a generic function of its arguments. The excursions of $\mathcal{J}_1$ relative to its fixed point value $\mathcal{J}_1^{(0)}$ are then on the order of $\epsilon \hat{\tilde{V}}_{r_1,-s}(\mathcal{J}_1^{(0)}, \mathcal{J}_2)$, and we may expand

$$\tilde{H}_0(\mathcal{J}_1, \mathcal{J}_2) = \tilde{H}_0(\mathcal{J}_1^{(0)}, \mathcal{J}_2) + \frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1} \Delta \mathcal{J}_1 + \frac{1}{2} \frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2} (\Delta \mathcal{J}_1)^2 + \ldots$$

where derivatives are evaluated at $(\mathcal{J}_1^{(0)}, \mathcal{J}_2)$. We thus arrive at the **standard Hamiltonian**,

$$K(\varphi_1, \Delta \mathcal{J}_1) = \frac{1}{2} G (\Delta \mathcal{J}_1)^2 - F \cos \varphi_1$$

where

$$G(\mathcal{J}_2) = \frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2}\bigg|_{(\mathcal{J}_1^{(0)}, \mathcal{J}_2)} \quad , \quad F(\mathcal{J}_2) = -2\epsilon \hat{\tilde{V}}_{r_1,-s}(\mathcal{J}_1^{(0)}, \mathcal{J}_2)$$

Thus, **the motion in the vicinity of every resonance is like that of a pendulum**. F is the amplitude of the first ($|p| = 1$) Fourier mode of the resonant perturbation, and G is the "nonlinearity parameter". For $FG > 0$, the elliptic fixed point (EFP) at $\varphi_1 = 0$ and the hyperbolic fixed point (HFP) is at $\varphi_1 = \pi$. For $FG < 0$, their locations are switched. The libration frequency about the EFP is $\nu_1 = \sqrt{FG} = \mathcal{O}\left(\sqrt{\epsilon \hat{\tilde{V}}_{r_1,-s}}\right)$, which decreases to zero as the separatrix is approached. The maximum excursion of $\Delta \mathcal{J}_1$ along the separatrix is $(\Delta \mathcal{J}_1)_{max} = 2\sqrt{F/G}$ which is also $\mathcal{O}(\sqrt{\epsilon \hat{\tilde{V}}_{r_1,-s}})$.

- **intrinsic degeneracy**

In this case, $H_0(J_1, J_2)$ is only a function of the action $\mathcal{J}_2 = (s/r)J_1 + J_2$. Then

$$K(\varphi_1, \vec{\mathcal{J}}) = \tilde{H}_0(\mathcal{J}_2) + \epsilon \hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}}) + 2\epsilon \hat{\tilde{V}}_{r_1 s}(\vec{\mathcal{J}}) \cos\varphi_1$$

Since both $\Delta \mathcal{J}_1$ and $\Delta \varphi_1$ vary on the same $\mathcal{O}(\epsilon \hat{\tilde{V}}_{\cdot,\cdot})$, we can't expand in $\Delta \mathcal{J}_1$. However, in the vicinity of an EFP we may expand in both $\Delta\varphi_1$ and $\Delta \mathcal{J}_1$ to get

$$K(\Delta\varphi_1, \Delta \mathcal{J}_1) = \frac{1}{2} G (\Delta \mathcal{J}_1)^2 + \frac{1}{2} F (\Delta \varphi_1)^2$$

with

$$G(\mathcal{J}_2) = \left[\frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2} + \epsilon \frac{\partial^2 \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1^2} + 2\epsilon \frac{\partial^2 \hat{\tilde{V}}_{r_1,-s}}{\partial \mathcal{J}_1^2} \right]_{(\mathcal{J}_1^{(0)}, \mathcal{J}_2)}$$

$$F(\mathcal{J}_2) = -2\epsilon \hat{\tilde{V}}_{r_1-s}(\mathcal{J}_1^{(0)}, \mathcal{J}_2)$$

This expansion is general, but for intrinsic case $\frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2} = 0$. Thus both F and G are $\mathcal{O}(\epsilon \hat{\tilde{V}}_{\cdot,\cdot})$ and $\nu_1 = \sqrt{FG} = \mathcal{O}(\epsilon)$ and the ratio of semimajor to seminimor axis lengths is

$$\frac{(\Delta \mathcal{J}_1)_{max}}{(\Delta \varphi_1)_{max}} = \sqrt{\frac{F}{G}} = \mathcal{O}(1)$$

(2) Secondary resonances

Details to be found in §15.9.3. Here just a sketch:

- CT $(\Delta\varphi_1, \Delta J_1) \to (I_1, \chi_1)$, given by

$$\Delta\varphi_1 = \left(2\sqrt{G/F}\, I_1\right)^{1/2} \sin\chi_1 \qquad \Delta J_1 = \left(2\sqrt{F/G}\, I_1\right)^{1/2} \cos\chi_1$$

- Define $\chi_2 \equiv \varphi_2$ and $I_2 \equiv J_2$. Then

$$K_0(\vec{\varphi}, \vec{J}) \to \tilde{K}_0(\vec{I}) = \tilde{H}_0(J_1^{(0)}, I_2) + \nu_1(I_2)\, I_1 - \tfrac{1}{16} G(I_2)\, I_1^2 + \dots$$

- To this we add back the terms with $s\ell_1 + r\ell_2 \neq 0$ which we previously dropped:

$$\tilde{K}_1(\vec{\chi}, \vec{I}) = \sum_{\vec{\ell}} \sum_n W_{\vec{\ell},n}(\vec{I})\, e^{in\chi_1} e^{i(s\ell_1 + r\ell_2)\chi_2/r}$$

where

$$W_{\vec{\ell},n}(\vec{I}) = \hat{\tilde{V}}_{\vec{\ell}}(J_1^{(0)}, I_2)\, J_n\!\left(\frac{\ell_1}{r} \sqrt[4]{\frac{G}{F}} \sqrt{2I_1}\right)$$

Bessel function

- We now have $\tilde{K}(\vec{\chi}, \vec{I}) = \tilde{K}_0(\vec{I}) + \epsilon' \tilde{K}_1(\vec{\chi}, \vec{I})$

Note that ϵ also appears within $\tilde{K}_0$, and $\epsilon' = \epsilon$.

- A secondary resonance occurs if $r'\nu_1 = s'\nu_2$, where

$$\nu_{1,2}(\vec{I}) = \frac{\partial \tilde{K}_0(\vec{I})}{\partial I_{1,2}}$$

- Do as we did before: CT $(\vec{\chi},\vec{I}) \to (\vec{\psi},\vec{M})$ via

$$F_2'(\vec{\chi},\vec{M}) = (r'\chi_1 - s'\chi_2)M_1 + \chi_2 M_2$$

Then

$$n\chi_1 + \left(\frac{s}{r}l_1 + l_2\right)\chi_2 = \frac{n}{r'}\psi_1 + \left(\frac{ns'}{r'} + \frac{s}{r}l_1 + l_2\right)\psi_2$$

and averaging over ψ_2 yields $nrs' + sr'l_1 + rr'l_2 = 0$, which entails

$$n = jr' \quad , \quad l_1 = kr \quad , \quad l_2 = -js' - ks$$

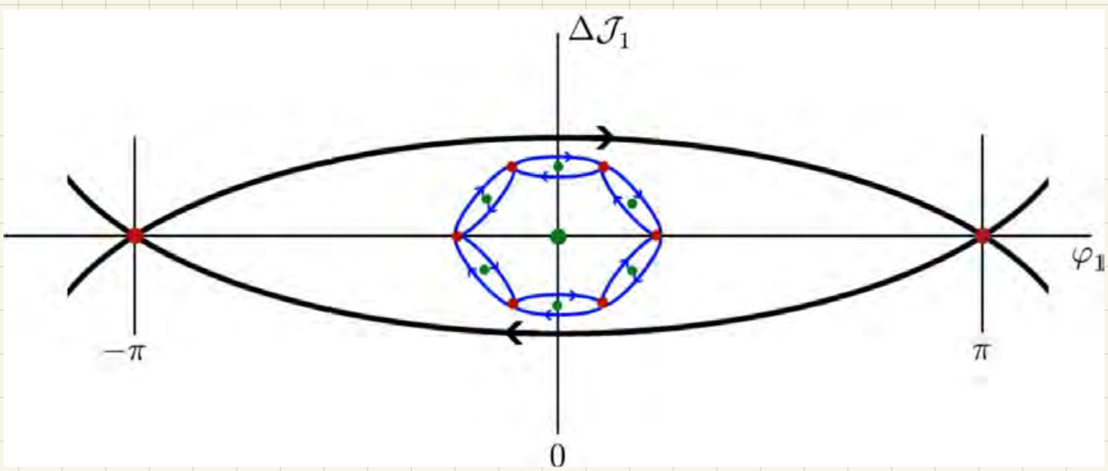
with $j,k \in \mathbb{Z}$.

- Averaging results in

see eqn. 15.304

$$\langle \tilde{K} \rangle_{\psi_2} = \tilde{\tilde{K}}_0(\vec{M}) + \epsilon' \sum_j \Gamma_{jr',-js'}(\vec{M})\, e^{-ij\psi_1}$$

- $M_2 = (s'/r')I_1 + I_2$ is the adiabatic invariant for the new oscillation



Motion in the vicinity of a secondary resonance with $r' = 6$ and $s' = 1$. EFPs in green, HFPs in red. Separatrices in black and blue. Note self-similarity.

Lecture 20 (Dec. 9) : MAPS $(\vec{\varphi}_{n+1} = \hat{T}\vec{\varphi}_n)$

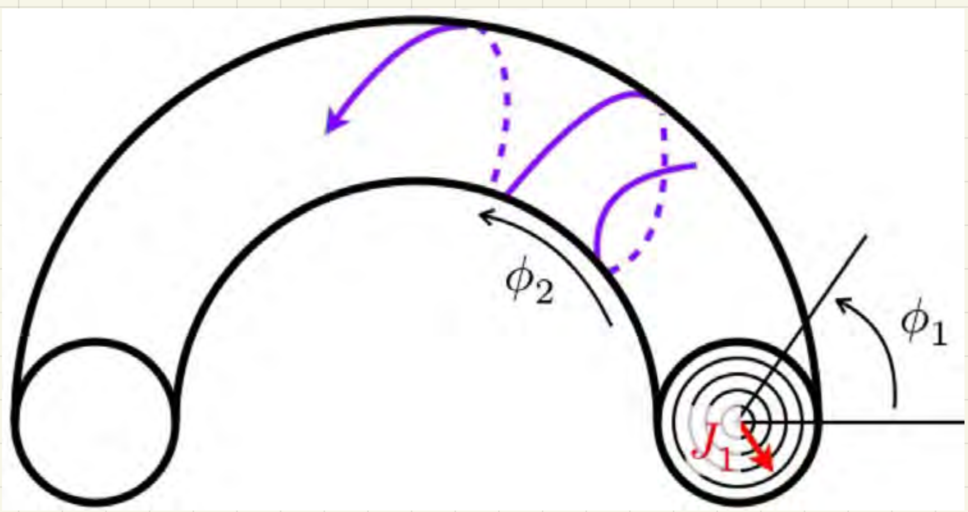
- **Motion on resonant tori**

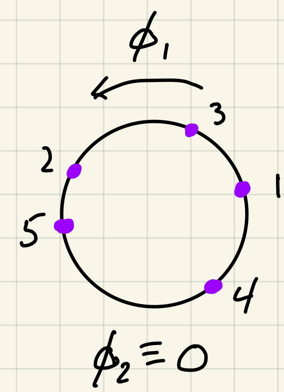
Consider the motion on a resonant torus in terms of the AAV:

$$\vec{\phi}(t) = \vec{\omega}(\vec{J})t + \vec{\phi}(0)$$

Resonance means that there exist some n-tuples $\vec{l} = \{l_1, \ldots, l_n\}$ for which $\vec{l}\cdot\vec{\omega} = 0$. If the motion is periodic, so that $\omega_j = k_j \omega_0$ with $k_j \in \mathbb{Z}$ for each $j \in \{1, \ldots, n\}$, then all of the frequencies are in resonance.

Let's consider the case $n=2$. Dynamics sketched below:





Since the energy E is fixed, we can regard $J_2 = J_2(J_1, E)$ and the motion as occurring in the 3-dim! space (ϕ_1, ϕ_2, J_1). Suppose we plot the consecutive intersections of the system's motion with the two-dim! subspace defined by fixing E and also ϕ_2 (say $\phi_2 \equiv 0$). Let's write $\phi \equiv \phi_1$ and $J \equiv J_1$, and define (ϕ_k, J_k) to be the values of (ϕ, J) at the k^{th} consecutive intersection of the system's motion with the subspace $(\phi_2 = 0,$ E fixed). The 2d space (ϕ_2, J_2) is called the **surface of section**. Since $\dot{\phi}_2 = \omega_2$, we have

(or E)

$$\phi_{k+1} - \phi_k = \omega_1 \cdot \frac{2\pi}{\omega_2} \equiv 2\pi\alpha$$

$\alpha(J) \equiv \dfrac{\omega_1(J)}{\omega_2(J)}$ (E suppressed)

and therefore

$$\phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1})$$

$$J_{k+1} = J_k$$

"twist map"

Note that we've written here $\alpha(J_{n+1})$ in the first equation. [Since $J_{k+1} = J_k$, it doesn't matter since J never changes for these dynamics. But writing the equations this way is more convenient.] Note that $(\phi_n, J_n) \to (\phi_{n+1}, J_{n+1})$ is canonical:

$$\{\phi_{k+1}, J_{k+1}\}_{(\phi_k, J_k)} = \det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)}$$

$$= \frac{\partial \phi_{k+1}}{\partial \phi_k}\frac{\partial J_{k+1}}{\partial J_k} - \frac{\partial \phi_{k+1}}{\partial J_k}\frac{\partial J_{k+1}}{\partial \phi_k} = 1 \cdot 1 - 0 \cdot 0 = 1$$

Formally, we may write this map as

$$\vec{\varphi}_{k+1} = \hat{T}\vec{\varphi}_k$$

where $\vec{\varphi}_k = (\phi_k, J_k)$ and $\hat{T}$ is the map. Note that if

$\alpha = \frac{r}{s} \in \mathbb{Q}$, then $\hat{T}^s$ acts as the identity, leaving every point in the (ϕ, J) plane fixed.

For systems with n degrees of freedom, and with the surface of section fixed by (ϕ_n, J_n) or (ϕ_n, E), define $\vec{\varphi} \equiv (\phi_1, \dots, \phi_{n-1})$ and $\vec{J} \equiv (J_1, \dots, J_{n-1})$. Then with $\vec{\alpha} \equiv \left(\frac{\omega_1}{\omega_n}, \dots, \frac{\omega_{n-1}}{\omega_n}\right)$,

$$\vec{\varphi}_{k+1} = \vec{\varphi}_k + 2\pi \vec{\alpha}(\vec{J}_{k+1})$$
$$\vec{J}_{k+1} = \vec{J}_k$$

which is canonical. Note $\vec{\varphi}_k = (\varphi_{1,k}, \dots, \varphi_{n-1,k})$ where $\varphi_{j,k}$ is the value of φ_j the k^{th} time the motion passes through the SOS. We call this map the **twist map.**

Perturbed twist map: Now consider a Hamiltonian $H(\vec{\phi}, \vec{J}) = H_0(\vec{J}) + \epsilon H_1(\vec{\phi}, \vec{J})$. Again we will take $n=2$. We expect the resulting map on the SOS to be given by

$$\hat{T}_\epsilon \vec{\varphi}_k = \varphi_{k+1} : \begin{cases} \phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1}) + \epsilon f(\phi_k, J_{k+1}) + \dots \\ J_{k+1} = J_k + \epsilon g(\phi_k, J_{k+1}) + \dots \end{cases}$$

Is this map canonical? Let's check that $\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = 1$:

$$d\phi_{k+1} = d\phi_k + 2\pi\alpha'(J_{k+1})\, dJ_{k+1} + \epsilon \frac{\partial f}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial f}{\partial J_{k+1}} dJ_{k+1}$$

$$dJ_{k+1} = dJ_k + \epsilon \frac{\partial g}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial g}{\partial J_{k+1}} dJ_{k+1}$$

Now bring $d\phi_{k+1}$ and dJ_{k+1} to the LHS of each eqn and bring $d\phi_k$ and dJ_k to the RHS. We obtain

$$\underbrace{\begin{pmatrix} 1 & -2\pi\alpha'(J_{k+1}) - \epsilon\dfrac{\partial f}{\partial J_{k+1}} \\ 0 & 1-\epsilon\dfrac{\partial g}{\partial J_{k+1}} \end{pmatrix}}_{A_{k+1}} \begin{pmatrix} d\phi_{k+1} \\ dJ_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\epsilon\dfrac{\partial f}{\partial \phi_k} & 0 \\ \epsilon\dfrac{\partial g}{\partial \phi_k} & 1 \end{pmatrix}}_{B_k} \begin{pmatrix} d\phi_k \\ dJ_k \end{pmatrix}$$

Thus

$$\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = \frac{\det B_k}{\det A_{k+1}} = \frac{1+\epsilon\dfrac{\partial f}{\partial \phi_k}}{1-\epsilon\dfrac{\partial g}{\partial J_{k+1}}} \equiv 1$$

and we conclude the necessary condition is $\dfrac{\partial f}{\partial \phi_k} = \dfrac{\partial g}{\partial J_{k+1}}$. This guarantees the map $\hat{T}_\epsilon$ is canonical.

If we restrict to $g = g(\phi)$, then we have $f = f(J)$. We may then write $2\pi\alpha(J_{k+1}) + \epsilon f(J_{k+1}) \equiv 2\pi\alpha_\epsilon(J_{k+1})$. (We'll drop the ϵ subscript on α.) Thus, our perturbed twist map is given by

$$\phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1})$$
$$J_{k+1} = J_k + \epsilon\, g(\phi_k)$$

For $\alpha(J) = J$ and $g(\phi) = -\sin\phi$, we obtain the **standard map**

$$\phi_{k+1} = \phi_k + 2\pi J_{k+1} \quad , \quad J_{k+1} = J_k - \epsilon \sin\phi_k$$

- Maps from time-dependent Hamiltonians

– **Parametric oscillator**, e.g. pendulum with time-dependent length $\ell(t)$: $\ddot{x} + \omega_0^2(t)\,x = 0$ with $\omega_0(t) = \sqrt{g/\ell(t)}$. This describes pumping a swing by periodically extending and withdrawing one's legs. We have

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix}}_{\dot{\vec{\varphi}}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}}_{A(t)} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\vec{\varphi}(t)} \qquad (v = \dot{x})$$

The formal sol$^{\underline{n}}$ to $\dot{\vec{\varphi}}(t) = A(t)\,\vec{\varphi}(t)$ is

$$\vec{\varphi}(t) = T \exp\left\{ \int_0^t dt'\, A(t') \right\} \vec{\varphi}(0)$$

where T is the **time ordering operator** which puts earlier times to the right. Thus

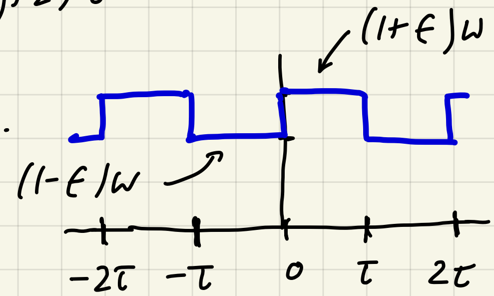
$$T \exp\left\{ \int_0^t dt'\, A(t') \right\} = \lim_{N\to\infty} \big(1 + A(t_{N-1})\,\delta\big) \cdots \big(1 + A(0)\,\delta\big)$$

where $t_j = j\delta$ with $\delta \equiv t/N$. Note if $A(t)$ is time independent then

$$T \exp\left\{ \int_0^t dt'\, A(t') \right\} = e^{At} = \lim_{N\to\infty} \left(1 + \frac{At}{N}\right)^N$$

There are no general methods for analytically evaluating time-ordered exponentials as we have here. But one tractable case is where the matrix $A(t)$ oscillates as a square wave:

$$\omega(t) = \begin{cases} (1+\epsilon)\,\omega_0 & \text{if } 2j\tau \leq t < (2j+1)\tau \\ (1-\epsilon)\,\omega_0 & \text{if } (2j+1)\tau \leq t < (2j+2)\tau \end{cases} \qquad (\text{for } j \in \mathbb{Z})$$



The period is 2τ. Define $\vec{\varphi}_n = \vec{\varphi}(t = 2n\tau)$. Then we have

$$\vec{\varphi}_{n+1} = e^{A_-\tau}\, e^{A_+\tau}\, \vec{\varphi}_n$$

NB: $e^{A_-\tau} e^{A_+\tau} \neq e^{(A_- + A_+)\tau}$

with

$$\vec{A}_\pm = \begin{pmatrix} 0 & 1 \\ -\omega_\pm^2 & 0 \end{pmatrix}, \qquad \omega_\pm \equiv (1 \pm \epsilon)\,\omega_0$$

Note that $A_\pm^2 = -\omega_\pm^2 \mathbb{1}$ and that

$$\begin{aligned} U_\pm \equiv e^{A_\pm \tau} &= \mathbb{1} + A_\pm \tau + \frac{1}{2!} A_\pm^2 \tau^2 + \frac{1}{3!} A_\pm^3 \tau^3 + \dots \\ &= \left(1 - \frac{1}{2!}\omega_\pm^2 \tau^2 + \frac{1}{4!}\omega_\pm^4 \tau^4 + \dots\right)\mathbb{1} \\ &\quad + \left(\tau - \frac{1}{3!}\omega_\pm^2 \tau^3 + \frac{1}{5!}\omega_\pm^4 \tau^5 - \dots\right) A_\pm \\ &= \cos(\omega_\pm \tau)\,\mathbb{1} + \omega_\pm^{-1} \sin(\omega_\pm \tau)\, A_\pm \\ &= \begin{pmatrix} \cos(\omega_\pm \tau) & \omega_\pm^{-1} \sin(\omega_\pm \tau) \\ -\omega_\pm \sin(\omega_\pm \tau) & \cos(\omega_\pm \tau) \end{pmatrix} \end{aligned}$$

Note also that $\det \mathcal{U}_\pm = 1$, since $\mathcal{U}_\pm$ is simply Hamiltonian evolution over half a period, and it must be canonical.

Now we need

$$\mathcal{U} = \hat{T} \exp\left\{ \int_0^{2\tau} dt\, A(t) \right\} = \mathcal{U}_- \mathcal{U}_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(real, not symmetric)

$$a = \cos(\omega_- \tau)\cos(\omega_+ \tau) - \omega_-^{-1}\omega_+ \sin(\omega_- \tau)\sin(\omega_+ \tau)$$
$$b = \omega_+^{-1} \cos(\omega_- \tau)\sin(\omega_+ \tau) + \omega_-^{-1} \sin(\omega_- \tau)\cos(\omega_+ \tau)$$
$$c = -\omega_+ \cos(\omega_- \tau)\sin(\omega_+ \tau) - \omega_- \sin(\omega_- \tau)\cos(\omega_+ \tau)$$
$$d = \cos(\omega_- \tau)\cos(\omega_+ \tau) - \omega_+^{-1}\omega_- \sin(\omega_- \tau)\sin(\omega_+ \tau)$$

It follows from $\mathcal{U} = \mathcal{U}_- \mathcal{U}_+$ that $\mathcal{U}$ is also canonical (i.e. $\vec{\varphi}_{n+1} = \mathcal{U}\vec{\varphi}_n$ is a canonical transformation).

The eigenvalues $\lambda_\pm$ of $\mathcal{U}$ thus satisfy $\lambda_+ \lambda_- = 1$. For a 2×2 matrix $\mathcal{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is

$$P(\lambda) = \det(\lambda \mathbb{1} - \mathcal{U}) = \lambda^2 - T\lambda + \Delta$$

where $T = \mathrm{tr}\,\mathcal{U} = a + d$ and $\Delta = \det \mathcal{U} = ad - bc$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2} T \pm \tfrac{1}{2}\sqrt{T^2 - 4\Delta}$$

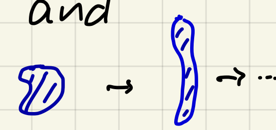
But in our case $\mathcal{U}$ is special, and $\det \mathcal{U} = 1$, so

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2-4}$$

We therefore have:

$$|T|<2 \; : \; \lambda_+ = \lambda_-^* = e^{i\delta} \quad \text{with } \delta = \cos^{-1}\!\left(\tfrac{1}{2}T\right)$$

$$|T|>2 \; : \; \lambda_+ = \lambda_-^{-1} = e^{\mu}\,\mathrm{sgn}(T) \quad \text{with } \mu = \cosh^{-1}\!\left(\tfrac{1}{2}|T|\right)$$

Note $\lambda_+\lambda_- = \det \mathcal{U} = 1$ always. Thus, for $|T|<2$, the motion is bounded, but for $|T|>2$ we have that $|\vec{\varphi}|$ increases exponentially with time, even though phase space volumes are preserved by the dynamics. I.e. we have exponential stretching along the eigenvector $\vec{\psi}_+$ and exponential squeezing along the eigenvector $\vec{\psi}_-$. 

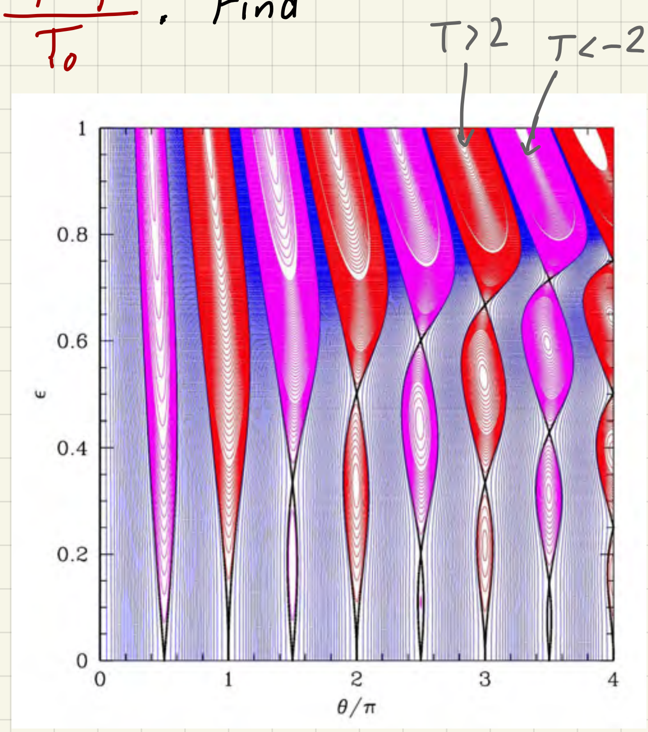
Let's set $\theta = \omega_0 \tau = 2\pi\tau/T_0$ where T_0 is the natural oscillation period when $\epsilon = 0$. Since the period of the pumping is $T_{pump} = 2\tau$, we have $\frac{\theta}{\pi} = \frac{T_{pump}}{T_0}$. Find

$$\mathrm{Tr}\,\mathcal{U} = \frac{2\cos(2\theta) - 2\epsilon^2\cos(2\epsilon\theta)}{1-\epsilon^2}$$

$$T=+2 : \; \theta = n\pi + \delta \,, \; \epsilon = \pm\left|\frac{\delta}{n\pi}\right|^{1/2}$$

$$T=-2 : \; \theta = (n+\tfrac{1}{2})\pi + \delta \,, \; \epsilon = \pm\,\delta$$

The phase diagram in (θ,ϵ) space is shown at the right.



Kicked dynamics: Let $H(t) = T(p) + V(q)K(t)$, where

$$K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau)$$

"Dirac comb"

As $\tau \to 0$, $K(t) \to 1$ (constant).

Equations of motion:

$$\dot{q} = T'(p) \qquad , \qquad \dot{p} = -V'(q)\,K(t)$$

Define $q_n \equiv q(t = n\tau^+)$ and $p_n = p(t = n\tau^+)$ and integrate from $t = n\tau^+$ to $t = (n+1)\tau^+$:

$$q_{n+1} = q_n + \tau T'(p_n)$$
$$p_{n+1} = p_n - \tau V'(q_{n+1})$$

This is our map $\vec{\varphi}_{n+1} = \hat{T}\vec{\varphi}_n$. Note that it is q_{n+1} which appears as the argument of V' in the second equation. This is crucial in order that $\hat{T}$ be canonical:

$$dq_{n+1} = dq_n + \tau T''(p_n)\,dp_n$$
$$dp_{n+1} = dp_n - \tau V''(q_{n+1})\,dq_{n+1}$$

$$\begin{pmatrix} 1 & 0 \\ \tau V''(q_{n+1}) & 1 \end{pmatrix}\begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

$$\begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ -\tau V''(q_{n+1}) & 1 - \tau^2 T''(p_n) V''(q_{n+1}) \end{pmatrix}\begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

and thus

$$\det \frac{\partial(q_n, p_n)}{\partial(q_{n+1}, p_{n+1})} = 1$$

The **standard map** is obtained from

$$H(t) = \frac{L^2}{2I} - V\cos\phi \, K(t)$$

resulting in

$$\begin{aligned} \phi_{n+1} &= \phi_n + \frac{\tau}{I} L_n \\ L_{n+1} &= L_n - \tau V \sin\phi_{n+1} \end{aligned}$$

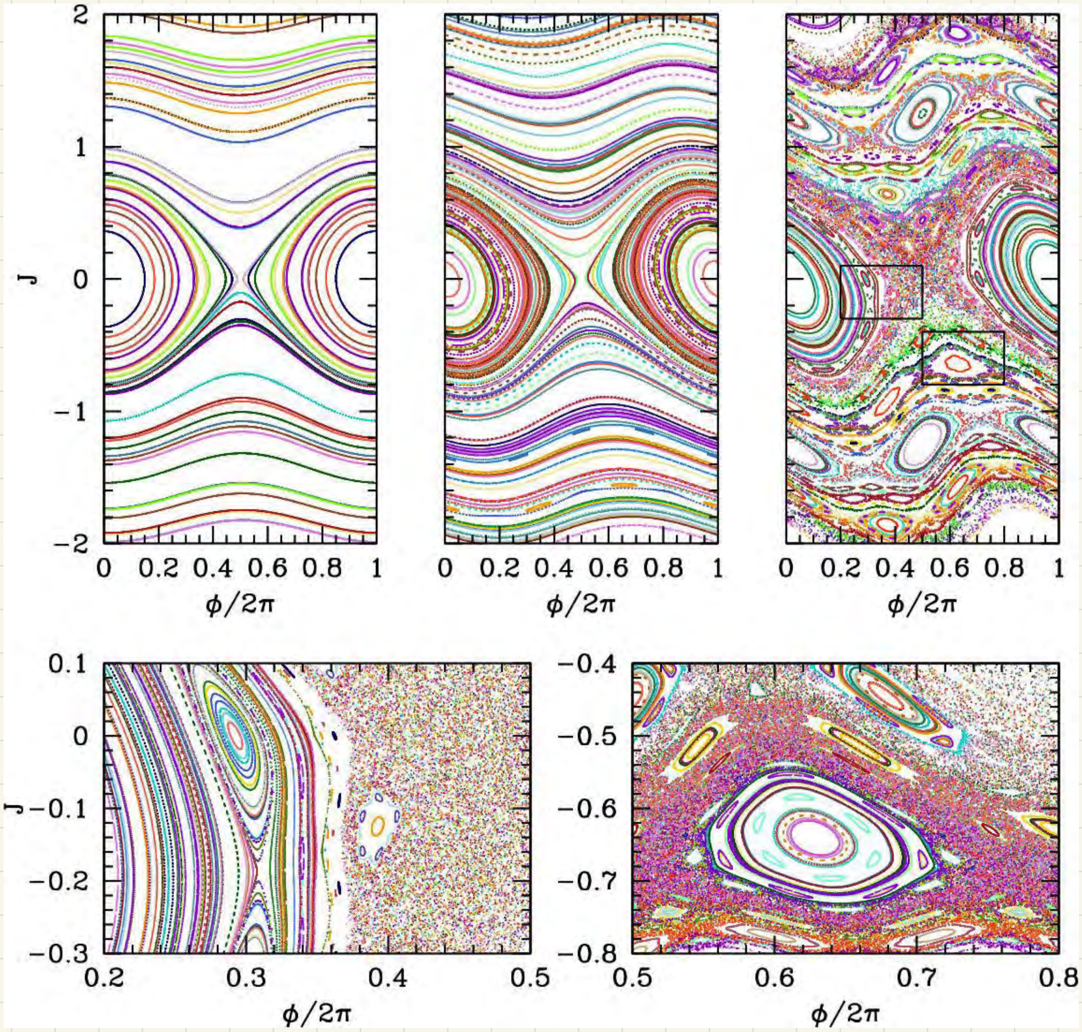
Defining $J_n \equiv L_n / \sqrt{2\pi I V}$ and $\epsilon \equiv \tau\sqrt{V/2\pi I}$ we arrive at

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\epsilon J_n \\ J_{n+1} &= J_n - \epsilon \sin\phi_{n+1} \end{aligned}$$

The phase space (ϕ, J) is thus a cylinder. As $\epsilon \to 0$,

$$\left.\begin{aligned} \frac{\phi_{n+1}-\phi_n}{\epsilon} &\to \frac{d\phi}{ds} = 2\pi J \\ \frac{J_{n+1}-J_n}{\epsilon} &\to \frac{dJ}{ds} = -\sin\phi \end{aligned}\right\} \Rightarrow \begin{array}{l} E = \pi J^2 - \cos\phi \\ \text{is preserved} \end{array}$$

This is because $\epsilon \to 0$ means $\tau \to 0$ hence $K(t) \to 1$, which is the simple pendulum. There is a separatrix at $E = 1$, along which $J(\phi) = \pm \frac{2}{\pi} \left|\cos(\phi/2)\right|$.



Top: $\epsilon = 0.01$ (left), $\epsilon = 0.2$ (center), $\epsilon = 0.4$ (right)
Bottom: details from $\epsilon = 0.4$ (upper right)

Another example is the **kicked Harper map**, when

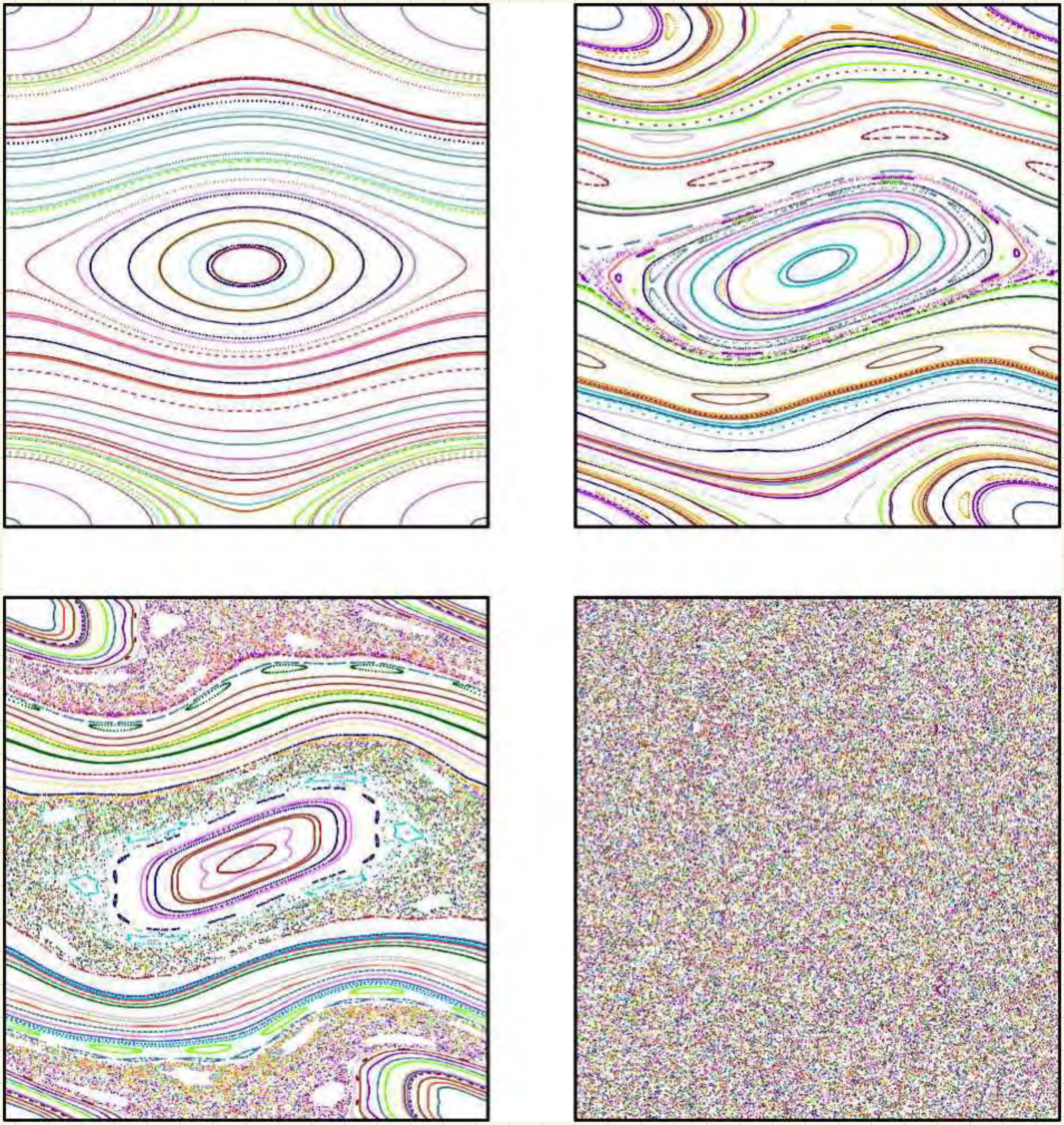
$$H(t) = -V_1 \cos\left(\frac{2\pi p}{P}\right) - V_2 \cos\left(\frac{2\pi q}{Q}\right) K(t)$$

This generates the map

$$x_{n+1} = x_n + \alpha \epsilon \sin(2\pi y_n) \qquad x \equiv q/Q \quad \alpha = \sqrt{V_1/V_2}$$

$$y_{n+1} = y_n - \alpha^{-1} \epsilon \sin(2\pi x_{n+1}) \qquad y \equiv p/P \quad \epsilon = \frac{2\pi\tau\sqrt{V_1 V_2}}{PQ}$$

on the torus $T^2 = [0,1] \times [0,1]$ with $x = 0,1$ identified and $y = 0,1$ identified.



Kicked Harper map with $\alpha = 2$ and $\epsilon = 0.01$ (UL), $\epsilon = 0.125$ (UR), $\epsilon = 0.2$ (LL), and $\epsilon = 5.0$ (LR).

Note PSF says $K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau) = \sum_{-\infty}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)$

and a kicked Hamiltonian may be written

$$H(J, \phi, t) = \underbrace{H_0(J) + V(\phi)}_{\text{integrable}} + \underbrace{2V(\phi) \sum_{m=1}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)}_{\text{resonances}}$$

Poincaré-Birkhoff Theorem

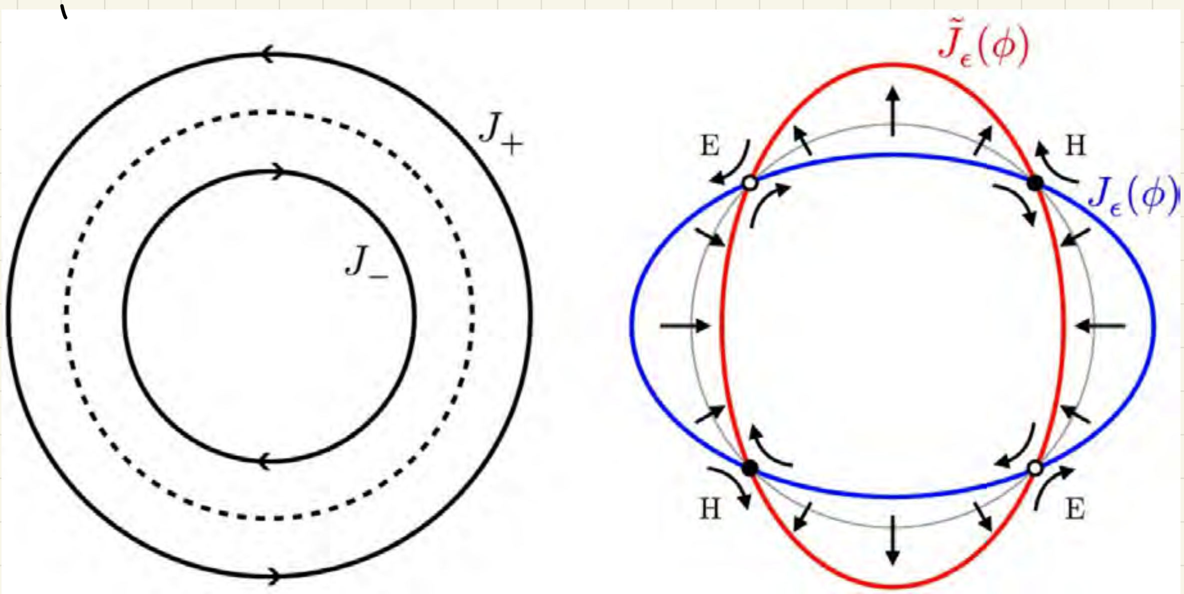
Back to our perturbed twist map, $\hat{T}_\epsilon$:

$$\phi_{n+1} = \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon f(\phi_n, J_{n+1})$$
$$J_{n+1} = J_n + \epsilon g(\phi_n, J_{n+1})$$

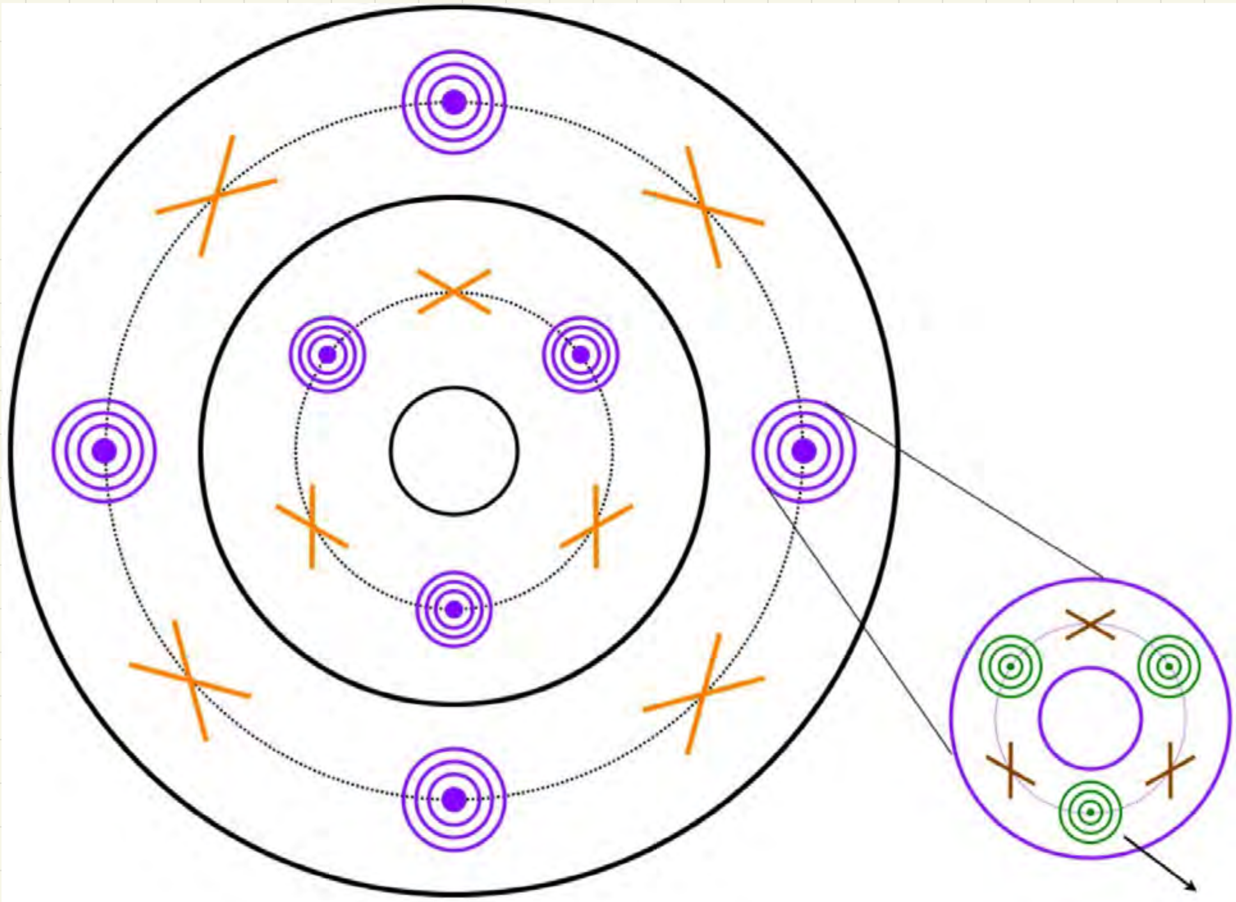
with

$$\frac{\partial f}{\partial \phi_n} + \frac{\partial g}{\partial J_{n+1}} = 0 \quad \Rightarrow \quad \hat{T}_\epsilon \text{ canonical}$$

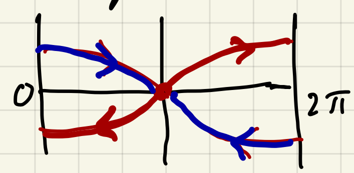
For $\epsilon = 0$, the map $\hat{T}_0$ leaves J invariant, and thus maps circles to circles. If $\alpha(J) \notin \mathbb{Q}$, the images of the iterated map $\hat{T}_0$ become dense on the circle. Suppose $\alpha(J) = \frac{r}{s} \in \mathbb{Q}$, and wolog assume $\alpha'(J) > 0$, so that on circles $J_\pm = J \pm \Delta J$ we have $\alpha(J_+) > r/s$ and $\alpha(J_-) < r/s$. Under $\hat{T}_0^{\,s}$, all points on the circle $C = C(J)$ are fixed. The circle $C_+ = C(J_+)$ rotates slightly counterclockwise while $C_- = C(J_-)$ rotates slightly clockwise. Now consider the action of $\hat{T}_\epsilon^{\,s}$, assuming that $\epsilon \ll \Delta J/J$. Acting on C_+, the result is still a net counterclockwise shift plus a small radial component of $\mathcal{O}(\epsilon)$. Similarly, C_- continues to rotate clockwise plus an $\mathcal{O}(\epsilon)$ radial component. By the Intermediate Value Theorem, for each value of ϕ there is some point $J = J_\epsilon(\phi)$ where the angular shift vanishes. Thus, along the curve $J_\epsilon(\phi)$ the action of $\hat{T}_\epsilon^s$ is **purely radial**. Next consider the curve $\tilde{J}_\epsilon(\phi) = \hat{T}_\epsilon^s J_\epsilon(\phi)$. Since $\hat{T}_\epsilon^s$ is volume-preserving, these curves must intersect at an even number of points.



The situation is depicted in the above figure. The intersections of $J_\epsilon(\phi)$ and $\tilde{J}_\epsilon(\phi)$ are thus **fixed points** of the map $\hat{T}_\epsilon^s$. We furthermore see that the intersection $J_\epsilon(\phi) \cap \tilde{J}_\epsilon(\phi)$ consists of an alternating sequence of elliptic and hyperbolic fixed points. This is the content of the PBT: a small perturbation of a resonant torus with $\alpha(J) = r/s$ results in an equal number of elliptic and hyperbolic fixed points for $\hat{T}_\epsilon^s$. Since T_ϵ has period s acting on these fixed points, the number of EFPs and HFPs must be equal and a multiple of s. **In the vicinity of each EFP, this structure repeats** (see the figure below).



Self-similar structures in the iterated twist map.



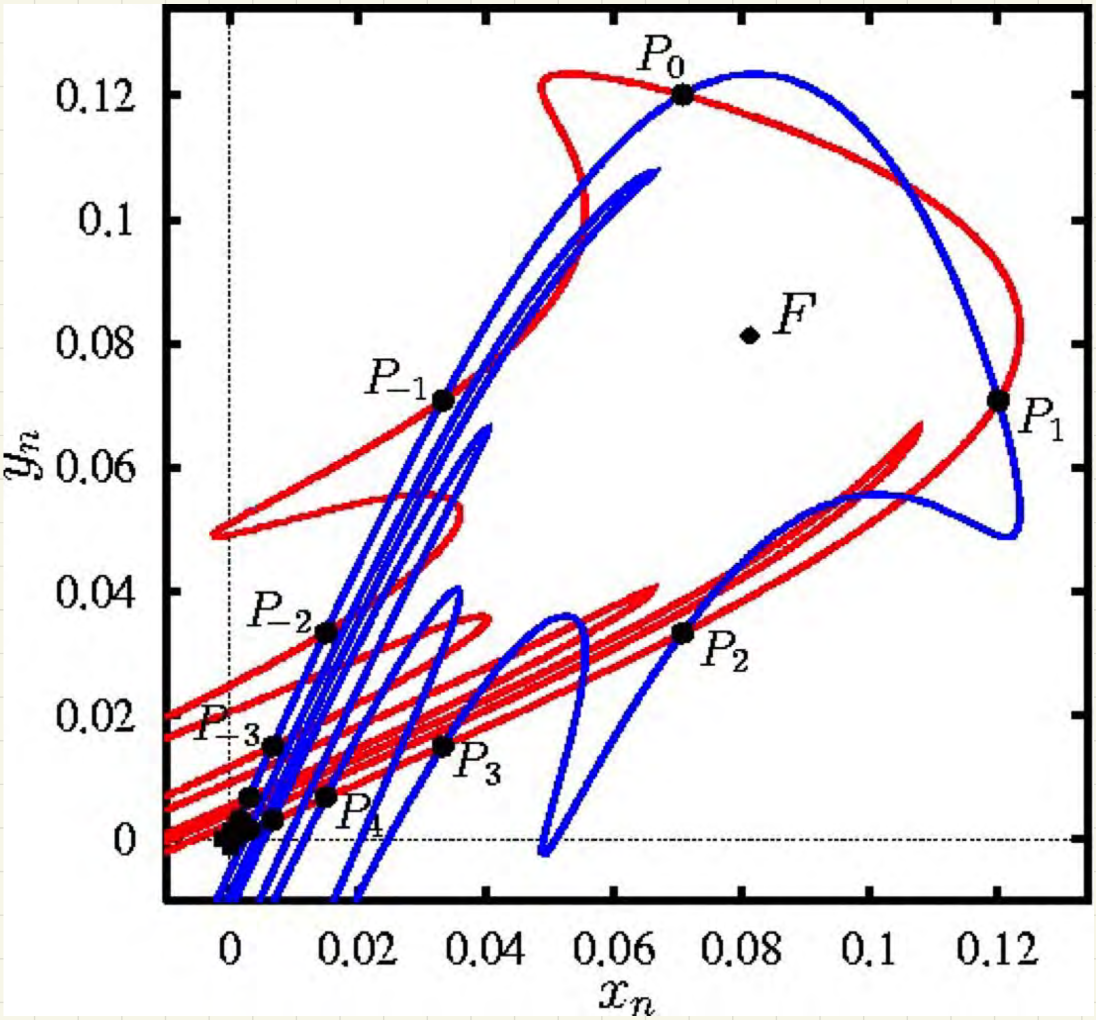
Stable and unstable manifolds

Emanating from each HFP are stable and unstable manifolds:

$$\vec{\varphi} \in \Sigma^{s}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_{\epsilon}^{ns}\vec{\varphi} = \vec{\varphi}^{*} \quad \text{(flows } \textbf{to } \vec{\varphi}^{*}\text{)}$$

$$\vec{\varphi} \in \Sigma^{u}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_{\epsilon}^{-ns}\vec{\varphi} = \vec{\varphi}^{*} \quad \text{(flows } \textbf{from } \vec{\varphi}^{*}\text{)}$$

Note $\Sigma^{s}(\vec{\varphi}_i^{*}) \cap \Sigma^{s}(\vec{\varphi}_j^{*}) = \phi$ and $\Sigma^{u}(\vec{\varphi}_i^{*}) \cap \Sigma^{u}(\vec{\varphi}_j^{*}) = \phi$ for $i \neq j$ (no s/s or u/u intersections). However, $\Sigma^{s}(\vec{\varphi}_i^{*})$ and $\Sigma^{u}(\vec{\varphi}_j^{*})$ **can** intersect. For $i=j$, this is called a **homoclinic point**. (On its way from $\vec{\varphi}_j^{*}$ to $\vec{\varphi}_i^{*}$.) For $i \neq j$, this is a **heteroclinic point**.



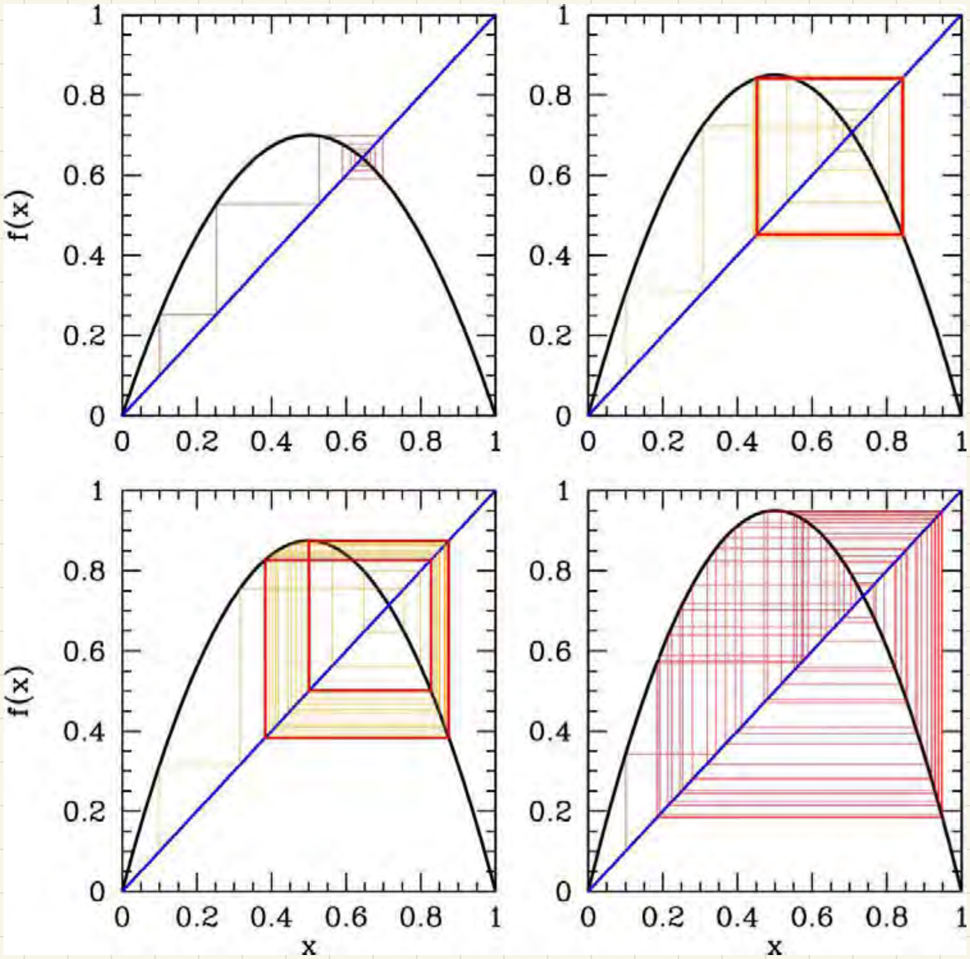
Homoclinic tangle for $x_{n+1} = y_n$ and $y_{n+1} = (a + b y_n^2) y_n - x_n$ with $a = 2.693$, $b = -104.888$. Blue curve is the stable manifold. Red curve is the unstable manifold. HFP at $(0,0)$. The fact that neither red nor blue curve can self intersect requires them to become increasingly tortured.

But since $\hat{T}^s_\epsilon$ is continuous and invertible, its action on a homoclinic (heteroclinic) point will produce a <u>new</u> homoclinic (heteroclinic) point, ad infinitum! For homoclinic intersections, the result is known as a **homoclinic tangle**.

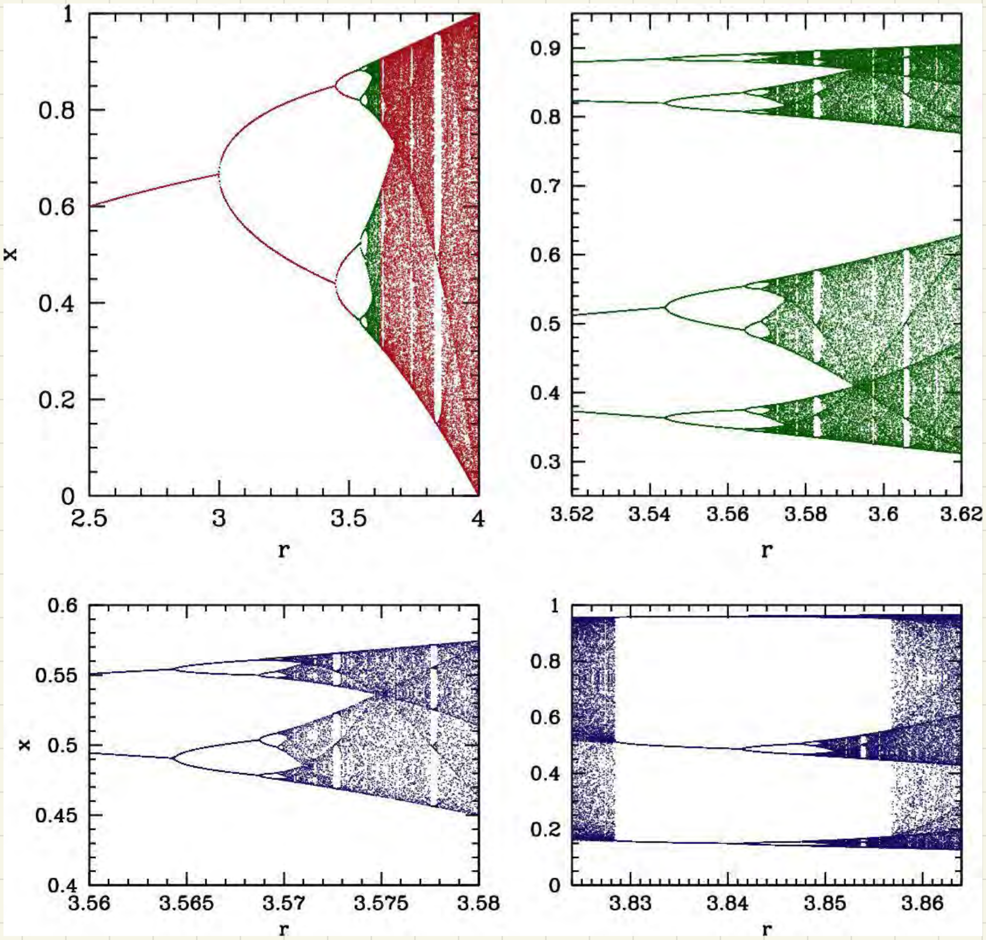
- **Maps in $d=1$**: $x_{n+1} = f(x_n)$; fixed point $x^* = f(x^*)$

  If $x = x^* + u$, then $u_{n+1} = f'(x^*)\, u_n + \mathcal{O}(u^2)$

  FP is **stable** if $|f'(x^*)| < 1$, **unstable** if $|f'(x^*)| > 1$.



Cobweb diagram for $f(x) = r\,x(1-x)$



Fixed points and cycles for $f(x) = rx(1-x)$