Lecture 19 (Dec. 7)

- **Removal of resonances**

We now consider how to deal with resonances arising in canonical perturbation theory. We start with the periodic time-dependent Hamiltonian,

$$H(\phi, J, t) = H_0(J) + \epsilon V(\phi, J, t)$$

where

$$V(\phi, J, t) = V(\phi + 2\pi, J, t) = V(\phi, J, t+T)$$

This is identified as $n = \frac{3}{2}$ degrees of freedom, since it is equivalent to a dynamical system of dimension $2n = 3$.

The double periodicity of $V(\phi, J, t)$ entails that it may be expressed as a double Fourier sum, viz.

$$V(\phi, J, t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \hat{V}_{k,l}(J)\, e^{ik\phi}\, e^{-il\Omega t}$$

$$\left(\hat{V}_{-k,-l} = \hat{V}^*_{k,l} \right)$$

where $\Omega = 2\pi/T$. Hamilton's equations are then

$$\dot{J} = -\frac{\partial H}{\partial \phi} = -\epsilon \frac{\partial V}{\partial \phi} = -i\epsilon \sum_{k,l} k\, \hat{V}_{k,l}(J)\, e^{i(k\phi - l\Omega t)}$$

$$\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0(J) + \epsilon \sum_{k,l} \frac{\partial \hat{V}_{k,l}(J)}{\partial J}\, e^{i(k\phi - l\Omega t)}$$

where $\omega_0(J) = \partial H_0/\partial J$. The resonance condition follows from inserting the $\mathcal{O}(\epsilon^0)$ solution $\phi(t) = \omega_0(J)\,t$, yielding

$$k\,\omega_0(J) - l\,\Omega = 0$$

When this condition is satisfied, secular forcing results in a linear increase of J with time. To do better, let's focus on a particular resonance $(k,l) = (k_0, l_0)$. The resonance condition $k_0\,\omega_0(J) = l_0\,\Omega$ fixes the action J. There is still an infinite set of possible (k,l) values leading to resonance at the same value of J, i.e. $(k,l) = (pk_0, pl_0)$ for all $p \in \mathbb{Z}$. But the Fourier amplitudes $\hat{V}_{pk_0,pl_0}(J)$ decrease in magnitude, typically exponentially in $|p|$. So we will assume k_0 and l_0 are relatively prime, and consider $p \in \{-1, 0, +1\}$. We define

$$\hat{V}_{0,0}(J) \equiv \hat{V}_0(J)\,, \quad \hat{V}_{k_0,l_0}(J) = \hat{V}^*_{-k_0,-l_0}(J) \equiv \hat{V}_1(J)\,e^{i\delta}$$

and obtain

$$\dot{J} = 2\epsilon k_0\,\hat{V}_1(J)\sin(k_0\phi - l_0\Omega t + \delta)$$

$$\dot{\phi} = \omega_0(J) + \epsilon\,\frac{\partial \hat{V}_0(J)}{\partial J} + 2\epsilon\,\frac{\partial \hat{V}_1(J)}{\partial J}\cos(k_0\phi - l_0\Omega t + \delta)$$

Now let's expand, writing $J = J_0 + \Delta J$ and

$$\psi = k_0\phi - l_0\Omega t + \delta + \begin{cases} 0 & \text{if } \epsilon > 0 \\ \pi & \text{if } \epsilon < 0 \end{cases}$$

resulting in (assume wolog $\epsilon > 0$)

$$\dot{\Delta J} = -2\epsilon\, k_0\, \hat{V}_1(J_0) \sin\psi$$

$$\dot{\psi} = k_0\, \omega_0'(J_0)\, \Delta J + \epsilon\, k_0\, \hat{V}_0'(J_0) - 2\epsilon\, k_0\, \hat{V}_1'(J_0) \cos\psi$$

To lowest nontrivial order in ϵ, we may drop the $\mathcal{O}(\epsilon)$ terms in the second equation, and write

$$\frac{d\Delta J}{dt} = -\frac{\partial K}{\partial \psi} \quad , \quad \frac{d\psi}{dt} = \frac{\partial K}{\partial \Delta J}$$

with

$$K(\psi, \Delta J) = \tfrac{1}{2} k_0\, \omega_0'(J_0)\, (\Delta J)^2 - 2\epsilon\, k_0\, \hat{V}_1'(J_0) \cos\psi$$

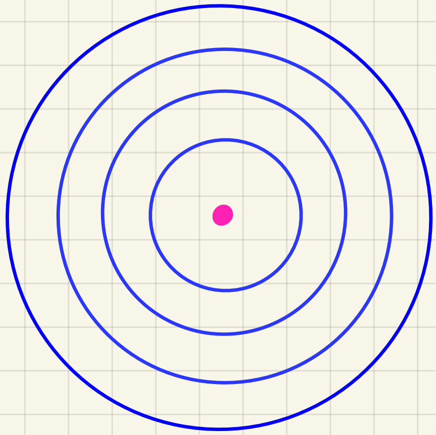
which is the Hamiltonian for a simple pendulum! The resulting equations of motion yield $\ddot{\psi} + \gamma^2 \sin\psi = 0$, with $\gamma^2 = 2\epsilon\, k_0^2\, \omega_0'(J_0)\, \hat{V}_1(J_0)$.

So what do we conclude from this analysis? The original 1-torus (i.e. circle S^1), with

$$J(t) = J_0 \quad , \quad \phi(t) = \omega_0(J_0)\, t + \phi(0)$$

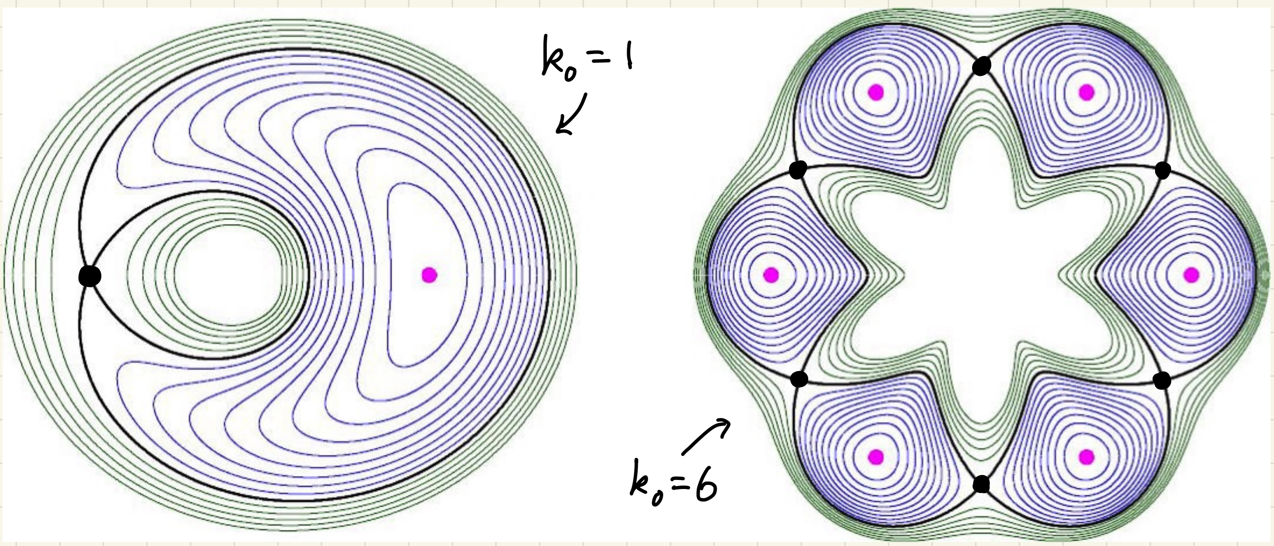
is destroyed. Both it and its neigboring 1-tori are replaced by a separatrix and surrounding libration and rotation phase curves (see figure). The amplitude

Unperturbed ($\epsilon = 0$) : $H_0(q,p) = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2$



- librations only
- no separatrix
- elliptic fixed point

Perturbed ($\epsilon > 0$) :



Librations (blue), rotations (green), and separatrices (black) for $k_0 = 1$ (left) and $k_0 = 6$ (right), plotted in (q,p) plane. Elliptic fixed points are shown as magenta dots. Hyperbolic (black) fixed points lie at the self-intersections of the separatrices.

of the separatrix is $(8\epsilon \hat{V}_1(J_0)/\omega'(J_0))^{1/2}$. This analysis is justified provided $(\Delta J)_{max} \ll J_0$ and $\gamma \ll \omega_0$, or

$$\epsilon \ll \frac{d \ln \omega_0}{d \ln J_0} \ll \frac{1}{\epsilon}$$

- $n=2$ systems

We now consider the Hamiltonian $H(\vec{\phi},\vec{J}) = H_0(\vec{\phi}) + \epsilon H_1(\vec{\phi},\vec{J})$ with $\vec{\phi} = (\phi_1, \phi_2)$ and $\vec{J} = (J_1, J_2)$. We write

$$H_1(\vec{\phi},\vec{J}) = \sum_{\vec{l}\in\mathbb{Z}^2} \hat{V}_{\vec{l}}(\vec{J})\, e^{i\vec{l}\cdot\vec{\phi}}$$

with $\vec{l} = (l_1, l_2)$ and $\hat{V}_{-\vec{l}}(\vec{J}) = \hat{V}^*_{\vec{l}}(\vec{J})$ since $H_1(\vec{\phi},\vec{J}) \in \mathbb{R}$. Resonances exist whenever $r\omega_1(\vec{J}) = s\omega_2(\vec{J})$, where

$$\omega_{1,2}(\vec{J}) = \frac{\partial H_0}{\partial J_{1,2}}$$

We eliminate the resonance in two steps:

(1) Invoke a CT $(\vec{\phi},\vec{J}) \to (\vec{\varphi},\vec{\mathcal{J}})$ generated by

$$F_2(\vec{\phi},\vec{\mathcal{J}}) = (r\phi_1 - s\phi_2)\mathcal{J}_1 + \phi_2\mathcal{J}_2$$

This yields

$$J_1 = \frac{\partial F_2}{\partial \phi_1} = r\mathcal{J}_1 \qquad \varphi_1 = \frac{\partial F_2}{\partial \mathcal{J}_1} = r\phi_1 - s\phi_2$$

$$J_2 = \frac{\partial F_2}{\partial \phi_2} = \mathcal{J}_2 - s\mathcal{J}_1 \qquad \varphi_2 = \frac{\partial F_2}{\partial \mathcal{J}_2} = \phi_2$$

Why did we do this? We did so in order to transform to a rotating frame where $\varphi_1 = r\phi_1 - s\phi_2$ is slowly varying, i.e. $\dot{\varphi}_1 = r\dot{\phi}_1 - s\dot{\phi}_2 \approx r\omega_1 - s\omega_2 = 0$. We also have $\dot{\varphi}_2 = \dot{\phi}_2 \approx \omega_2$. Now we could instead have used the generator

$$F_2 = \phi_1 \mathcal{J}_1 + (r\phi_1 - s\phi_2)\mathcal{J}_2$$

resulting in $\varphi_1 = \phi_1$ and $\varphi_2 = r\phi_1 - s\phi_2$. Here φ_2 is the slow variable while φ_1 oscillates with frequency $\approx \omega_1$. Which should we choose? We will wind up averaging over the faster of $\varphi_{1,2}$, and we want the fast frequency itself to be as slow as possible, for reasons which have to do with the removal of higher order resonances. (More on this further on below.) We'll assume wolog that $\omega_1 > \omega_2$.

Inverting to find $\vec{\phi}(\vec{\varphi})$, we have

$$\phi_1 = \frac{1}{r}\varphi_1 + \frac{s}{r}\varphi_2 \qquad , \qquad \phi_2 = \varphi_2$$

so we have

$$\begin{aligned}\tilde{H}(\vec{\varphi}, \vec{\mathcal{J}}) &= H_0(\vec{J}(\vec{\mathcal{J}})) + \epsilon H_1(\vec{\phi}(\vec{\varphi}), \vec{J}(\vec{\mathcal{J}})) \\ &\equiv \tilde{H}_0(\vec{\mathcal{J}}) + \epsilon \underbrace{\sum_{\vec{\ell}} \tilde{\tilde{V}}_{\vec{\ell}}(\vec{\mathcal{J}}) \exp\left\{\frac{i\ell_1}{r}\varphi_1 + i\left(\frac{\ell_1 s}{r} + \ell_2\right)\varphi_2\right\}}_{\tilde{H}_1(\vec{\varphi}, \vec{\mathcal{J}})}\end{aligned}$$

We now average over the fast variable φ_2. This yields the constraint $s\ell_1 + r\ell_2 = 0$, which we solve by writing $(\ell_1, \ell_2) = (pr, -ps)$ for $p \in \mathbb{Z}$. We then have

$$\langle \tilde{H}_1(\vec{\varphi}, \vec{\mathcal{J}}) \rangle = \sum_p \hat{\tilde{V}}_{pr,-ps}(\vec{\mathcal{J}})\, e^{ips}$$

The averaging procedure is justified close to a resonance, where $|\dot{\varphi}_2| \gg |\dot{\varphi}_1|$. Note that $\mathcal{J}_2$ now is conserved, i.e. $\dot{\mathcal{J}}_2 = 0$. Thus $\mathcal{J}_2 = \frac{s}{r} J_1 + J_2$ is a new invariant.

At this point, only the $(\varphi_1, \mathcal{J}_1)$ variables are dynamical. φ_2 has been averaged out and $\mathcal{J}_2$ is constant. Since the Fourier amplitudes $\hat{\tilde{V}}_{pr,-ps}(\vec{\mathcal{J}})$ are assumed to decay rapidly with increasing $|p|$, we consider only $p \in \{-1, 0, +1\}$ as we did in the $n = \frac{3}{2}$ case. We thereby obtain the effective Hamiltonian

$$K(\varphi_1, \mathcal{J}_1, \mathcal{J}_2) \approx \tilde{H}_0(\mathcal{J}_1, \mathcal{J}_2) + \epsilon \hat{\tilde{V}}_{0,0}(\mathcal{J}_1, \mathcal{J}_2) + 2\epsilon \hat{\tilde{V}}_{r,-s}(\mathcal{J}_1, \mathcal{J}_2) \cos \varphi_1$$

where we have absorbed any phase in $\hat{\tilde{V}}_{r,-s}(\vec{\mathcal{J}})$ into a shift of φ_1, so we may consider $\hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}})$ and $\hat{\tilde{V}}_{r,s}(\vec{\mathcal{J}})$ to be real functions of $\vec{\mathcal{J}} = (\mathcal{J}_1, \mathcal{J}_2)$. The fixed points of the dynamics then satisfy

$$\dot{\varphi}_1 = \frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1} + \epsilon \frac{\partial \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1} + 2\epsilon \frac{\partial \hat{\tilde{V}}_{r,-s}}{\partial \mathcal{J}_1} \cos\varphi_1 = 0$$

$$\dot{\mathcal{J}}_1 = -2\epsilon \hat{\tilde{V}}_{r,-s} \sin\varphi_1 = 0$$

Note that a stationary solution here corresponds to a periodic solution in our original variables, since we have shifted to a rotating frame. Thus $\varphi_1 = 0$ or $\varphi_1 = \pi$, and

$$\frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1} = \frac{\partial H_0}{\partial J_1}\frac{\partial J_1}{\partial \mathcal{J}_1} + \frac{\partial H_0}{\partial J_2}\frac{\partial J_2}{\partial \mathcal{J}_1}$$

$$= r\omega_1 - s\omega_2 = 0$$

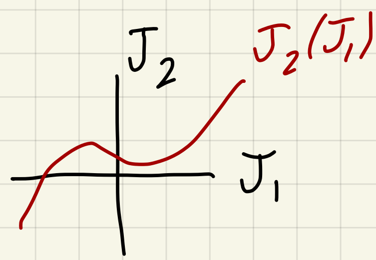
Thus fixed points occur for

$$\frac{\partial \hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}})}{\partial \mathcal{J}_1} \pm 2 \frac{\partial \hat{\tilde{V}}_{r,-s}(\vec{\mathcal{J}})}{\partial \mathcal{J}_1} = 0 \qquad \begin{pmatrix} \varphi_1 = 0 \\ \varphi_1 = \pi \end{pmatrix}$$

There are two cases to consider:

- **accidental degeneracy**

In this case, the degeneracy condition



$$r\omega_1(J_1, J_2) = s\omega_2(J_1, J_2)$$

Thus, we have $J_2 = J_2(J_1)$. This is the case when $H_0(J_1, J_2)$ is a generic function of its arguments. The excursions of $\mathcal{J}_1$ relative to its fixed point value $\mathcal{J}_1^{(0)}$ are then on the order of $\epsilon \hat{\tilde{V}}_{r_1,-s}(\mathcal{J}_1^{(0)},\mathcal{J}_2)$, and we may expand

$$\tilde{H}_0(\mathcal{J}_1,\mathcal{J}_2) = \tilde{H}_0(\mathcal{J}_1^{(0)},\mathcal{J}_2) + \frac{\partial \tilde{H}_0}{\partial \mathcal{J}_1}\Delta\mathcal{J}_1 + \frac{1}{2}\frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2}(\Delta\mathcal{J}_1)^2 + \ldots$$

where derivatives are evaluated at $(\mathcal{J}_1^{(0)},\mathcal{J}_2)$. We thus arrive at the **standard Hamiltonian**,

$$K(\varphi_1,\Delta\mathcal{J}_1) = \frac{1}{2}G(\Delta\mathcal{J}_1)^2 - F\cos\varphi_1$$

where

$$G(\mathcal{J}_2) = \frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2}\bigg|_{(\mathcal{J}_1^{(0)},\mathcal{J}_2)} \quad , \quad F(\mathcal{J}_2) = -2\epsilon\,\hat{\tilde{V}}_{r_1,-s}(\mathcal{J}_1^{(0)},\mathcal{J}_2)$$

Thus, **the motion in the vicinity of every resonance is like that of a pendulum**. F is the amplitude of the first ($|p|=1$) Fourier mode of the resonant perturbation, and G is the "nonlinearity parameter". For $FG>0$, the elliptic fixed point (EFP) at $\varphi_1=0$ and the hyperbolic fixed point (HFP) is at $\varphi_1=\pi$. For $FG<0$, their locations are switched. The libration frequency about the EFP is $\nu_1=\sqrt{FG}=\mathcal{O}(\sqrt{\epsilon\hat{\tilde{V}}_{r_1,-s}})$, which decreases to zero as the separatrix is approached. The maximum excursion of $\Delta \mathcal{J}_1$ along the separatrix is $(\Delta \mathcal{J}_1)_{max} = 2\sqrt{F/G}$ which is also $\mathcal{O}(\sqrt{\epsilon \hat{\tilde{V}}_{r_1 -s}})$.

- **intrinsic degeneracy**

In this case, $H_0(J_1, J_2)$ is only a function of the action $\mathcal{J}_2 = (s/r) J_1 + J_2$. Then

$$K(\varphi_1, \vec{\mathcal{J}}) = \tilde{H}_0(\mathcal{J}_2) + \epsilon \hat{\tilde{V}}_{0,0}(\vec{\mathcal{J}}) + 2\epsilon \hat{\tilde{V}}_{r,s}(\vec{\mathcal{J}}) \cos\varphi_1$$

Since both $\Delta \mathcal{J}_1$ and $\Delta \varphi_1$ vary on the same $\mathcal{O}(\epsilon \hat{\tilde{V}}_{\cdot,\cdot})$, we can't expand in $\Delta \mathcal{J}_1$. However, in the vicinity of an EFP we may expand in both $\Delta \varphi_1$ and $\Delta \mathcal{J}_1$ to get

$$K(\Delta\varphi_1, \Delta\mathcal{J}_1) = \frac{1}{2} G (\Delta \mathcal{J}_1)^2 + \frac{1}{2} F (\Delta \varphi_1)^2$$

with

$$G(\mathcal{J}_2) = \left[\frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2} + \epsilon \frac{\partial^2 \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1^2} + 2\epsilon \frac{\partial^2 \hat{\tilde{V}}_{r_1 -s}}{\partial \mathcal{J}_1^2} \right]_{(\mathcal{J}_1^{(0)}, \mathcal{J}_2)}$$

$$F(\mathcal{J}_2) = -2\epsilon \hat{\tilde{V}}_{r_1 -s}(\mathcal{J}_1^{(0)}, \mathcal{J}_2)$$

This expansion is general, but for intrinsic case $\frac{\partial^2 \tilde{H}_0}{\partial \mathcal{J}_1^2} = 0$. Thus both F and G are $\mathcal{O}(\epsilon \hat{\tilde{V}}_{\cdot,\cdot})$ and $\nu_1 = \sqrt{FG} = \mathcal{O}(\epsilon)$ and the ratio of semimajor to semiminor axis lengths is

$$\frac{(\Delta \mathcal{J}_1)_{max}}{(\Delta \varphi_1)_{max}} = \sqrt{\frac{F}{G}} = \mathcal{O}(1)$$

(2) Secondary resonances

Details to be found in §15.9.3. Here just a sketch:

- CT $(\Delta\varphi_1, \Delta J_1) \to (I_1, \chi_1)$, given by

$$\Delta\varphi_1 = \left(2\sqrt{G/F}\, I_1\right)^{1/2} \sin\chi_1 \qquad \Delta J_1 = \left(2\sqrt{F/G}\, I_1\right)^{1/2} \cos\chi_1$$

- Define $\chi_2 \equiv \varphi_2$ and $I_2 \equiv J_2$. Then

$$K_0(\vec{\varphi}, \vec{J}) \to \tilde{K}_0(\vec{I}) = \tilde{H}_0(J_1^{(0)}, I_2) + \nu_1(I_2)\, I_1 - \tfrac{1}{16} G(I_2)\, I_1^2 + \ldots$$

- To this we add back the terms with $s\ell_1 + r\ell_2 \neq 0$ which we previously dropped:

$$\tilde{K}_1(\vec{\chi}, \vec{I}) = \sum_{\vec{\ell}} \sum_n W_{\vec{\ell},n}(\vec{I})\, e^{in\chi_1} e^{i(s\ell_1 + r\ell_2)\chi_2/r}$$

where

$$W_{\vec{\ell},n}(\vec{I}) = \hat{\tilde{V}}_{\vec{\ell}}(J_1^{(0)}, I_2)\, J_n\!\left(\frac{\ell_1}{r} \sqrt[4]{\frac{G}{F}} \sqrt{2I_1}\right)$$

Bessel function

- We now have $\tilde{K}(\vec{\chi}, \vec{I}) = \tilde{K}_0(\vec{I}) + \epsilon' \tilde{K}_1(\vec{\chi}, \vec{I})$

Note that ϵ also appears within $\tilde{K}_0$, and $\epsilon' = \epsilon$.

- A secondary resonance occurs if $r'\nu_1 = s'\nu_2$, where

$$\nu_{1,2}(\vec{I}) = \frac{\partial \tilde{K}_0(\vec{I})}{\partial I_{1,2}}$$

- Do as we did before: CT $(\vec{\chi}, \vec{I}) \to (\vec{\psi}, \vec{M})$ via

$$F_2'(\vec{\chi}, \vec{M}) = (r'\chi_1 - s'\chi_2)M_1 + \chi_2 M_2$$

Then

$$n\chi_1 + \left(\frac{s}{r}l_1 + l_2\right)\chi_2 = \frac{n}{r'}\psi_1 + \left(\frac{ns'}{r'} + \frac{s}{r}l_1 + l_2\right)\psi_2$$

and averaging over ψ_2 yields $nrs' + sr'l_1 + rr'l_2 = 0$, which entails

$$n = jr' \quad , \quad l_1 = kr \quad , \quad l_2 = -js' - ks$$

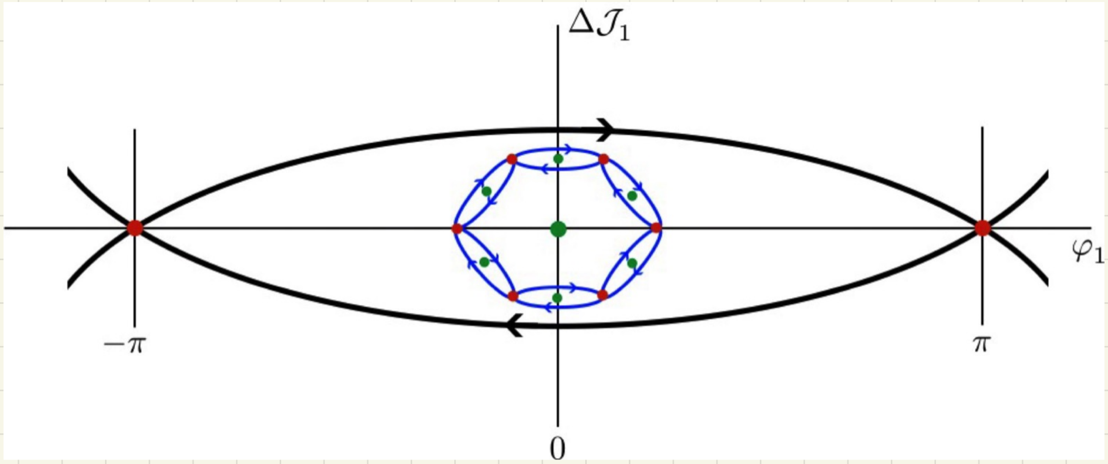
with $j, k \in \mathbb{Z}$.

- Averaging results in

see eqn. 15.304

$$\langle \tilde{K} \rangle_{\psi_2} = \tilde{\tilde{K}}_0(\vec{M}) + \epsilon' \sum_j \Gamma_{jr', -js'}(\vec{M})\, e^{-ij\psi_1}$$

- $M_2 = (s'/r')I_1 + I_2$ is the adiabatic invariant for the new oscillation



Motion in the vicinity of a secondary resonance with $r' = 6$ and $s' = 1$. EFPs in green, HFPs in red. Separatrices in black and blue. Note self-similarity.