Lecture 20 (Dec. 9) : MAPS $\left(\vec{\varphi}_{n+1} = \hat{T}\vec{\varphi}_n\right)$

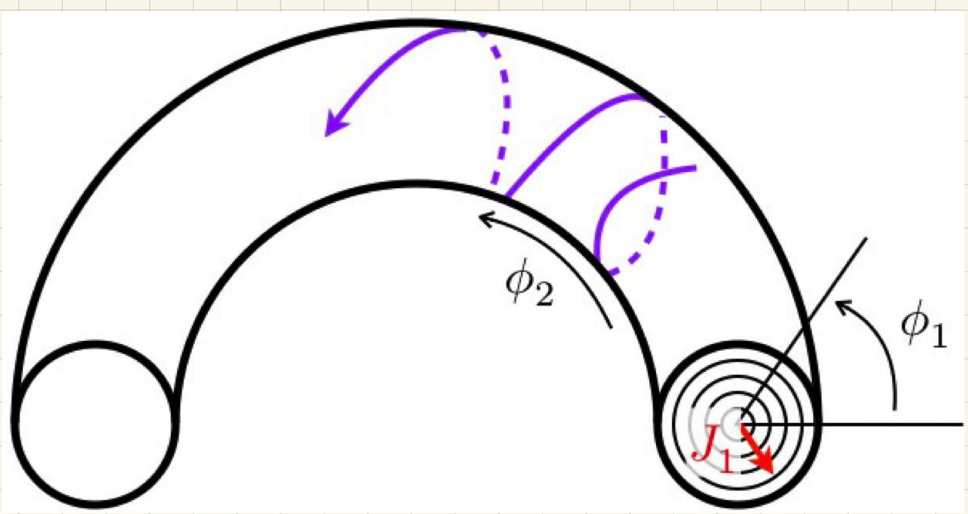
- **Motion on resonant tori**

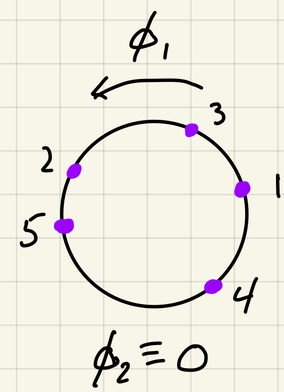
Consider the motion on a resonant torus in terms of the AAV:

$$\vec{\phi}(t) = \vec{\omega}(\vec{J})t + \vec{\phi}(0)$$

Resonance means that there exist some n-tuples $\vec{l} = \{l_1, \dots, l_n\}$ for which $\vec{l}\cdot\vec{\omega} = 0$. If the motion is periodic, so that $\omega_j = k_j \omega_0$ with $k_j \in \mathbb{Z}$ for each $j \in \{1, \dots, n\}$, then all of the frequencies are in resonance.

Let's consider the case $n=2$. Dynamics sketched below:





Since the energy E is fixed, we can regard $J_2 = J_2(J_1, E)$ and the motion as occurring in the 3-dim! space (ϕ_1, ϕ_2, J_1). Suppose we plot the consecutive intersections of the system's motion with the two-dim! subspace defined by fixing E and also ϕ_2 (say $\phi_2 \equiv 0$). Let's write $\phi \equiv \phi_1$ and $J \equiv J_1$, and define (ϕ_k, J_k) to be the values of (ϕ, J) at the k^{th} consecutive intersection of the system's motion with the subspace $(\phi_2 = 0, E \text{ fixed})$. The 2d space (ϕ_2, J_2) is called the **surface of section**. Since $\dot{\phi}_2 = \omega_2$, we have

or E

$$\phi_{k+1} - \phi_k = \omega_1 \cdot \frac{2\pi}{\omega_2} \equiv 2\pi\alpha$$

$\alpha(J) \equiv \dfrac{\omega_1(J)}{\omega_2(J)}$ (E suppressed)

and therefore

$$\phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1})$$

$$J_{k+1} = J_k$$

"twist map"

Note that we've written here $\alpha(J_{n+1})$ in the first equation. [Since $J_{k+1} = J_k$, it doesn't matter since J never changes for these dynamics. But writing the equations this way is more convenient.] Note that $(\phi_n, J_n) \to (\phi_{n+1}, J_{n+1})$ is canonical:

$$\{\phi_{k+1}, J_{k+1}\}_{(\phi_k, J_k)} = \det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)}$$

$$= \frac{\partial \phi_{k+1}}{\partial \phi_k}\frac{\partial J_{k+1}}{\partial J_k} - \frac{\partial \phi_{k+1}}{\partial J_k}\frac{\partial J_{k+1}}{\partial \phi_k} = 1 \cdot 1 - 0 \cdot 0 = 1$$

Formally, we may write this map as

$$\vec{\varphi}_{k+1} = \hat{T}\vec{\varphi}_k$$

where $\vec{\varphi}_k = (\phi_k, J_k)$ and $\hat{T}$ is the map. Note that if

$\alpha = \frac{r}{s} \in \mathbb{Q}$, then $\hat{T}^s$ acts as the identity, leaving every point in the (ϕ, J) plane fixed.

For systems with n degrees of freedom, and with the surface of section fixed by (ϕ_n, J_n) or (ϕ_n, E), define $\vec{\varphi} \equiv (\phi_1, \ldots, \phi_{n-1})$ and $\vec{J} \equiv (J_1, \ldots, J_{n-1})$. Then with $\vec{\alpha} \equiv \left(\frac{\omega_1}{\omega_n}, \ldots, \frac{\omega_{n-1}}{\omega_n}\right)$,

$$\vec{\varphi}_{k+1} = \vec{\varphi}_k + 2\pi \vec{\alpha}(\vec{J}_{k+1})$$
$$\vec{J}_{k+1} = \vec{J}_k$$

which is canonical. Note $\vec{\varphi}_k = (\varphi_{1,k}, \ldots, \varphi_{n-1,k})$ where $\varphi_{j,k}$ is the value of φ_j the k^{th} time the motion passes through the SOS. We call this map the **twist map**.

Perturbed twist map: Now consider a Hamiltonian $H(\vec{\phi}, \vec{J}) = H_0(\vec{J}) + \epsilon H_1(\vec{\phi}, \vec{J})$. Again we will take $n=2$. We expect the resulting map on the SOS to be given by

$$\hat{T}_\epsilon \vec{\varphi}_k = \varphi_{k+1} : \begin{cases} \phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1}) + \epsilon f(\phi_k, J_{k+1}) + \ldots \\ J_{k+1} = J_k + \epsilon g(\phi_k, J_{k+1}) + \ldots \end{cases}$$

Is this map canonical? Let's check that $\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = 1$:

$$d\phi_{k+1} = d\phi_k + 2\pi\alpha'(J_{k+1})\, dJ_{k+1} + \epsilon \frac{\partial f}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial f}{\partial J_{k+1}} dJ_{k+1}$$
$$dJ_{k+1} = dJ_k + \epsilon \frac{\partial g}{\partial \phi_k} d\phi_k + \epsilon \frac{\partial g}{\partial J_{k+1}} dJ_{k+1}$$

Now bring $d\phi_{k+1}$ and dJ_{k+1} to the LHS of each eqn and bring $d\phi_k$ and dJ_k to the RHS. We obtain

$$\underbrace{\begin{pmatrix} 1 & -2\pi\alpha'(J_{k+1}) - \epsilon\dfrac{\partial f}{\partial J_{k+1}} \\ 0 & 1-\epsilon\dfrac{\partial g}{\partial J_{k+1}} \end{pmatrix}}_{A_{k+1}} \begin{pmatrix} d\phi_{k+1} \\ dJ_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\epsilon\dfrac{\partial f}{\partial \phi_k} & 0 \\ \epsilon\dfrac{\partial g}{\partial \phi_k} & 1 \end{pmatrix}}_{B_k} \begin{pmatrix} d\phi_k \\ dJ_k \end{pmatrix}$$

Thus

$$\det \frac{\partial(\phi_{k+1}, J_{k+1})}{\partial(\phi_k, J_k)} = \frac{\det B_k}{\det A_{k+1}} = \frac{1+\epsilon\dfrac{\partial f}{\partial \phi_k}}{1-\epsilon\dfrac{\partial g}{\partial J_{k+1}}} \equiv 1$$

and we conclude the necessary condition is $\dfrac{\partial f}{\partial \phi_k} = \dfrac{\partial g}{\partial J_{k+1}}$.

This guarantees the map $\hat{T}_\epsilon$ is canonical.

If we restrict to $g = g(\phi)$, then we have $f = f(J)$.

We may then write $2\pi\alpha(J_{k+1}) + \epsilon f(J_{k+1}) \equiv 2\pi\alpha_\epsilon(J_{k+1})$. (We'll drop the ϵ subscript on α.) Thus, our perturbed twist map is given by

$$\phi_{k+1} = \phi_k + 2\pi\alpha(J_{k+1})$$

$$J_{k+1} = J_k + \epsilon g(\phi_k)$$

For $\alpha(J) = J$ and $g(\phi) = -\sin\phi$, we obtain the **standard map**

$$\phi_{k+1} = \phi_k + 2\pi J_{k+1} \quad , \quad J_{k+1} = J_k - \epsilon \sin\phi_k$$

- Maps from time-dependent Hamiltonians

- **Parametric oscillator**, e.g. pendulum with time-dependent length $\ell(t)$: $\ddot{x} + \omega_0^2(t)\,x = 0$ with $\omega_0(t) = \sqrt{g/\ell(t)}$. This describes pumping a swing by periodically extending and withdrawing one's legs. We have

$$\underbrace{\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix}}_{\dot{\vec{\varphi}}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}}_{A(t)} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\vec{\varphi}(t)} \qquad (v = \dot{x})$$

The formal sol$^{\underline{n}}$ to $\dot{\vec{\varphi}}(t) = A(t)\,\vec{\varphi}(t)$ is

$$\vec{\varphi}(t) = T \exp\left\{ \int_0^t dt'\, A(t') \right\} \vec{\varphi}(0)$$

where T is the **time ordering operator** which puts earlier times to the right. Thus

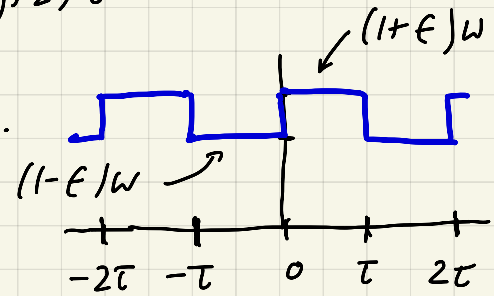
$$T \exp\left\{ \int_0^t dt'\, A(t') \right\} = \lim_{N\to\infty} \big(1 + A(t_{N-1})\,\delta\big) \cdots \big(1 + A(0)\,\delta\big)$$

where $t_j = j\delta$ with $\delta \equiv t/N$. Note if $A(t)$ is time independent then

$$T \exp\left\{ \int_0^t dt'\, A(t') \right\} = e^{At} = \lim_{N\to\infty} \left(1 + \frac{At}{N}\right)^N$$

There are no general methods for analytically evaluating time-ordered exponentials as we have here. But one tractable case is where the matrix $A(t)$ oscillates as a square wave:

$$\omega(t) = \begin{cases} (1+\epsilon)\,\omega_0 & \text{if } 2j\tau \leq t < (2j+1)\tau \\ (1-\epsilon)\,\omega_0 & \text{if } (2j+1)\tau \leq t < (2j+2)\tau \end{cases} \qquad (\text{for } j \in \mathbb{Z})$$



The period is 2τ. Define $\vec{\varphi}_n = \vec{\varphi}(t = 2n\tau)$. Then we have

$$\vec{\varphi}_{n+1} = e^{A_-\tau}\, e^{A_+\tau}\, \vec{\varphi}_n$$

NB: $e^{A_-\tau} e^{A_+\tau} \neq e^{(A_- + A_+)\tau}$

with

$$\vec{A}_\pm = \begin{pmatrix} 0 & 1 \\ -\omega_\pm^2 & 0 \end{pmatrix}, \qquad \omega_\pm \equiv (1 \pm \epsilon)\,\omega_0$$

Note that $A_\pm^2 = -\omega_\pm^2 \mathbb{1}$ and that

$$\begin{aligned} U_\pm \equiv e^{A_\pm \tau} &= \mathbb{1} + A_\pm \tau + \frac{1}{2!} A_\pm^2 \tau^2 + \frac{1}{3!} A_\pm^3 \tau^3 + \ldots \\ &= \left(1 - \frac{1}{2!}\omega_\pm^2 \tau^2 + \frac{1}{4!}\omega_\pm^4 \tau^4 + \ldots\right)\mathbb{1} \\ &\qquad + \left(\tau - \frac{1}{3!}\omega_\pm^2 \tau^3 + \frac{1}{5!}\omega_\pm^4 \tau^5 - \ldots\right) A_\pm \\ &= \cos(\omega_\pm \tau)\,\mathbb{1} + \omega_\pm^{-1} \sin(\omega_\pm \tau)\, A_\pm \\ &= \begin{pmatrix} \cos(\omega_\pm \tau) & \omega_\pm^{-1}\sin(\omega_\pm \tau) \\ -\omega_\pm \sin(\omega_\pm \tau) & \cos(\omega_\pm \tau) \end{pmatrix} \end{aligned}$$

Note also that $\det \mathcal{U}_\pm = 1$, since $\mathcal{U}_\pm$ is simply Hamiltonian evolution over half a period, and it must be canonical.

Now we need

$$\mathcal{U} = \hat{T} \exp\left\{ \int_0^{2\tau} dt\, A(t) \right\} = \mathcal{U}_- \mathcal{U}_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(real, not symmetric)

$$a = \cos(\omega_- \tau)\cos(\omega_+ \tau) - \omega_-^{-1}\omega_+ \sin(\omega_-\tau)\sin(\omega_+\tau)$$
$$b = \omega_+^{-1}\cos(\omega_-\tau)\sin(\omega_+\tau) + \omega_-^{-1}\sin(\omega_-\tau)\cos(\omega_+\tau)$$
$$c = -\omega_+\cos(\omega_-\tau)\sin(\omega_+\tau) - \omega_-\sin(\omega_-\tau)\cos(\omega_+\tau)$$
$$d = \cos(\omega_-\tau)\cos(\omega_+\tau) - \omega_+^{-1}\omega_-\sin(\omega_-\tau)\sin(\omega_+\tau)$$

It follows from $\mathcal{U} = \mathcal{U}_-\mathcal{U}_+$ that $\mathcal{U}$ is also canonical (i.e. $\vec{\varphi}_{n+1} = \mathcal{U}\vec{\varphi}_n$ is a canonical transformation).

The eigenvalues $\lambda_\pm$ of $\mathcal{U}$ thus satisfy $\lambda_+\lambda_- = 1$. For a 2×2 matrix $\mathcal{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is

$$P(\lambda) = \det(\lambda \mathbb{1} - \mathcal{U}) = \lambda^2 - T\lambda + \Delta$$

where $T = \mathrm{tr}\,\mathcal{U} = a+d$ and $\Delta = \det \mathcal{U} = ad - bc$. The eigenvalues are then

$$\lambda_\pm = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4\Delta}$$

But in our case $\mathcal{U}$ is special, and $\det \mathcal{U} = 1$, so

$$\lambda_\pm = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2-4}$$

We therefore have:

$$|T| < 2 \ : \ \lambda_+ = \lambda_-^* = e^{i\delta} \quad \text{with } \delta = \cos^{-1}\left(\tfrac{1}{2}T\right)$$

$$|T| > 2 \ : \ \lambda_+ = \lambda_-^{-1} = e^{\mu}\,\mathrm{sgn}(T) \quad \text{with } \mu = \cosh^{-1}\left(\tfrac{1}{2}|T|\right)$$

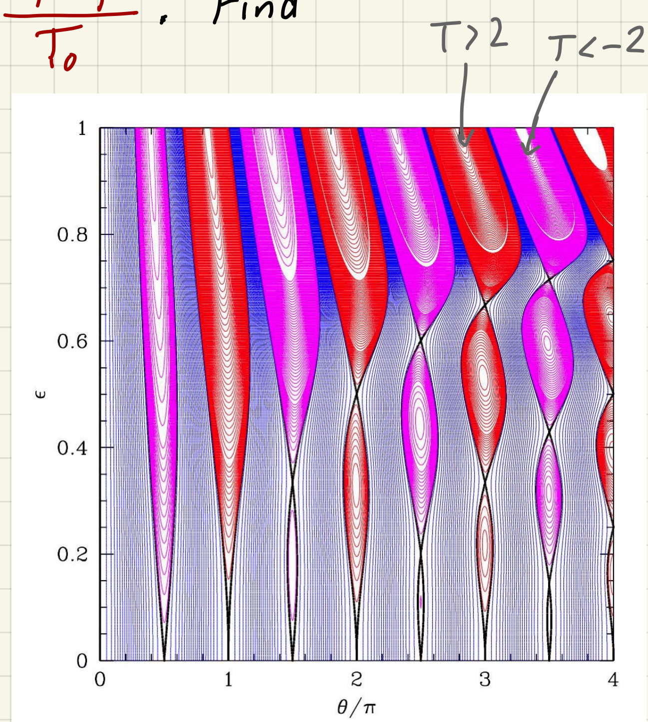
Note $\lambda_+\lambda_- = \det \mathcal{U} = 1$ always. Thus, for $|T|<2$, the motion is bounded, but for $|T|>2$ we have that $|\vec{\varphi}|$ increases exponentially with time, even though phase space volumes are preserved by the dynamics. I.e. we have exponential stretching along the eigenvector $\vec{\psi}_+$ and exponential squeezing along the eigenvector $\vec{\psi}_-$.

Let's set $\theta = \omega_0\tau = 2\pi\tau/T_0$ where T_0 is the natural oscillation period when $\epsilon = 0$. Since the period of the pumping is $T_{pump} = 2\tau$, we have $\frac{\theta}{\pi} = \frac{T_{pump}}{T_0}$. Find

$$\mathrm{Tr}\,\mathcal{U} = \frac{2\cos(2\theta) - 2\epsilon^2\cos(2\epsilon\theta)}{1-\epsilon^2}$$

$T=+2$: $\theta = n\pi + \delta$, $\epsilon = \pm\left|\frac{\delta}{n\pi}\right|^{1/2}$

$T=-2$: $\theta = (n+\frac{1}{2})\pi + \delta$, $\epsilon = \pm\,\delta$



The phase diagram in (θ,ϵ) space is shown at the right.

Kicked dynamics: Let $H(t) = T(p) + V(q)\,K(t)$, where

$$K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau)$$

"Dirac comb"

As $\tau \to 0$, $K(t) \to 1$ (constant).

Equations of motion:

$$\dot{q} = T'(p) \quad , \quad \dot{p} = -V'(q)\,K(t)$$

Define $q_n \equiv q(t = n\tau^+)$ and $p_n = p(t = n\tau^+)$ and integrate from $t = n\tau^+$ to $t = (n+1)\tau^+$:

$$q_{n+1} = q_n + \tau\, T'(p_n)$$
$$p_{n+1} = p_n - \tau\, V'(q_{n+1})$$

This is our map $\vec{\varphi}_{n+1} = \hat{T}\,\vec{\varphi}_n$. Note that it is q_{n+1} which appears as the argument of V' in the second equation. This is crucial in order that $\hat{T}$ be canonical:

$$dq_{n+1} = dq_n + \tau\, T''(p_n)\, dp_n$$
$$dp_{n+1} = dp_n - \tau\, V''(q_{n+1})\, dq_{n+1}$$

$$\begin{pmatrix} 1 & 0 \\ \tau V''(q_{n+1}) & 1 \end{pmatrix} \begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

$$\begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \tau T''(p_n) \\ -\tau V''(q_{n+1}) & 1 - \tau^2 T''(p_n) V''(q_{n+1}) \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix}$$

and thus

$$\det \frac{\partial(q_n, p_n)}{\partial(q_{n+1}, p_{n+1})} = 1$$

The **standard map** is obtained from

$$H(t) = \frac{L^2}{2I} - V\cos\phi\, K(t)$$

resulting in

$$\begin{aligned} \phi_{n+1} &= \phi_n + \frac{\tau}{I} L_n \\ L_{n+1} &= L_n - \tau V \sin\phi_{n+1} \end{aligned}$$

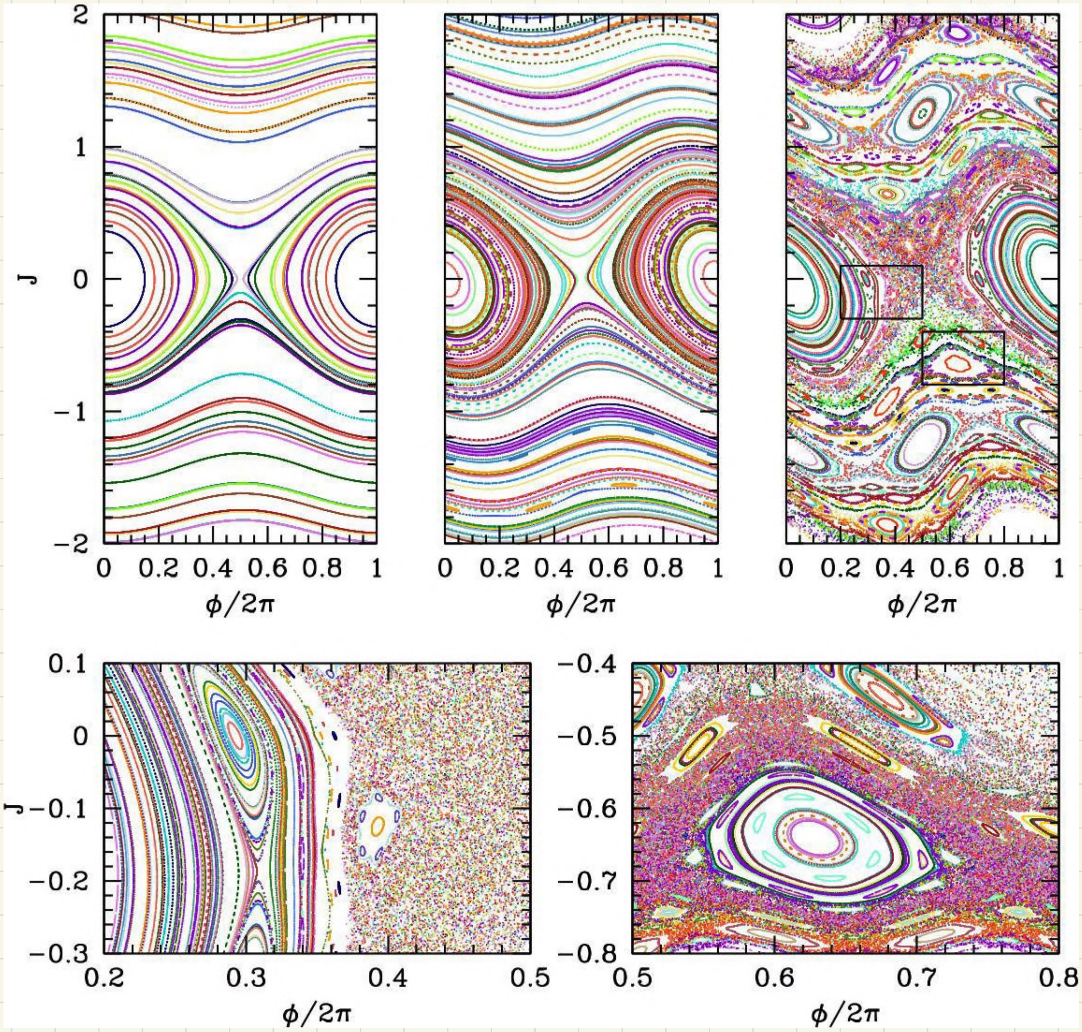
Defining $J_n \equiv L_n / \sqrt{2\pi I V}$ and $\epsilon \equiv \tau\sqrt{V/2\pi I}$ we arrive at

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\epsilon J_n \\ J_{n+1} &= J_n - \epsilon \sin\phi_{n+1} \end{aligned}$$

The phase space (ϕ, J) is thus a cylinder. As $\epsilon \to 0$,

$$\left.\begin{aligned} \frac{\phi_{n+1}-\phi_n}{\epsilon} &\to \frac{d\phi}{ds} = 2\pi J \\ \frac{J_{n+1}-J_n}{\epsilon} &\to \frac{dJ}{ds} = -\sin\phi \end{aligned}\right\} \Rightarrow \begin{array}{l} E = \pi J^2 - \cos\phi \\ \text{is preserved} \end{array}$$

This is because $\epsilon \to 0$ means $\tau \to 0$ hence $K(t) \to 1$, which is the simple pendulum. There is a separatrix at $E = 1$, along which $J(\phi) = \pm \frac{2}{\pi} \left|\cos(\phi/2)\right|$.



Top: $\epsilon = 0.01$ (left), $\epsilon = 0.2$ (center), $\epsilon = 0.4$ (right)
Bottom: details from $\epsilon = 0.4$ (upper right)

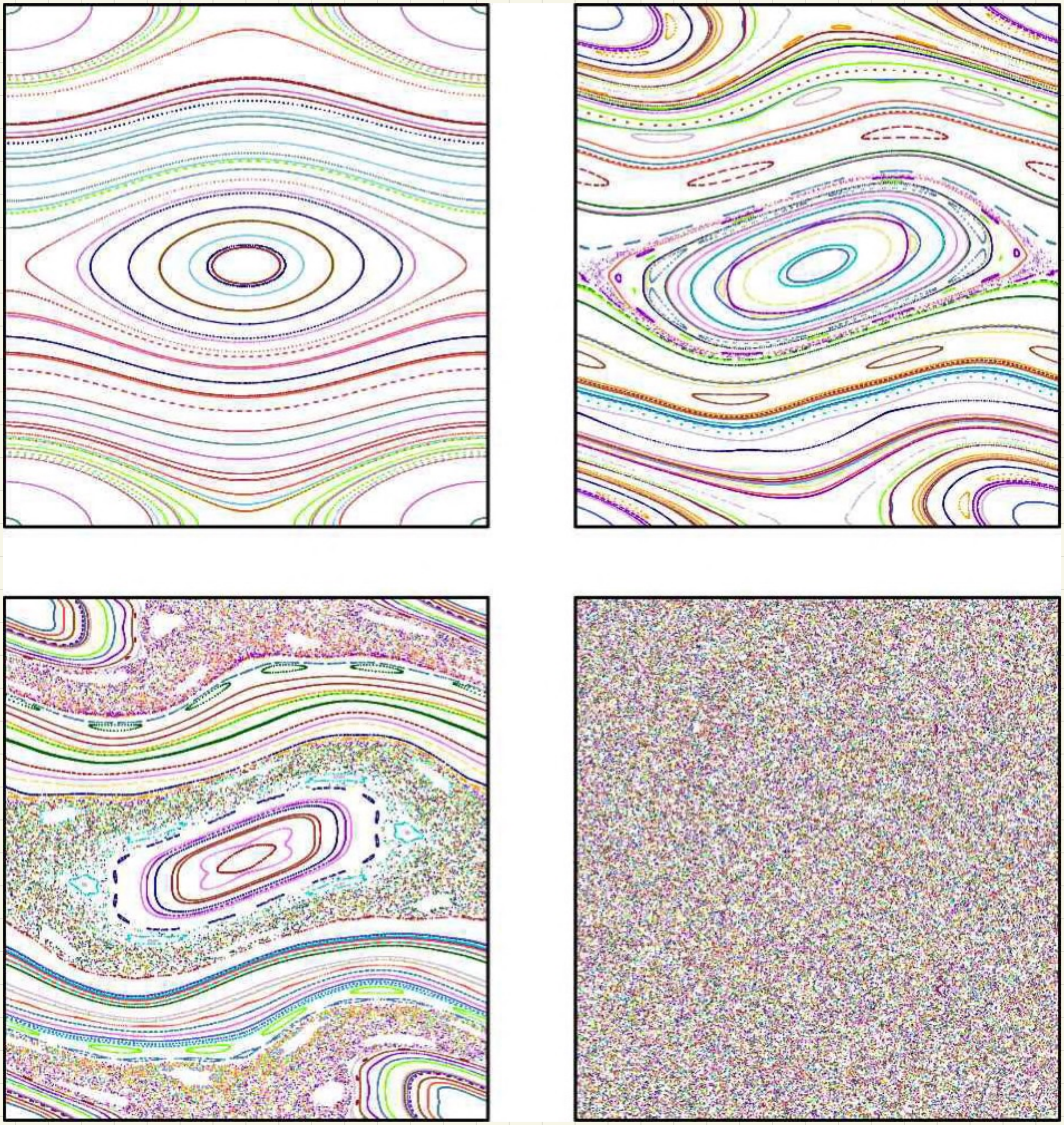
Another example is the **kicked Harper map**, when

$$H(t) = -V_1 \cos\left(\frac{2\pi p}{P}\right) - V_2 \cos\left(\frac{2\pi q}{Q}\right) K(t)$$

This generates the map

$$x_{n+1} = x_n + \alpha \epsilon \sin(2\pi y_n) \qquad x \equiv q/Q \quad \alpha = \sqrt{V_1/V_2}$$
$$y_{n+1} = y_n - \alpha^{-1} \epsilon \sin(2\pi x_{n+1}) \qquad y \equiv p/P \quad \epsilon = \frac{2\pi\tau\sqrt{V_1 V_2}}{PQ}$$

on the torus $T^2 = [0,1] \times [0,1]$ with $x = 0,1$ identified and $y = 0,1$ identified.



Kicked Harper map with $\alpha = 2$ and $\epsilon = 0.01$ (UL), $\epsilon = 0.125$ (UR), $\epsilon = 0.2$ (LL), and $\epsilon = 5.0$ (LR).

Note PSF says $K(t) = \tau \sum_{-\infty}^{\infty} \delta(t - n\tau) = \sum_{-\infty}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)$

and a kicked Hamiltonian may be written

$$H(J, \phi, t) = \underbrace{H_0(J) + V(\phi)}_{\text{integrable}} + \underbrace{2V(\phi) \sum_{m=1}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right)}_{\text{resonances}}$$

Poincaré-Birkhoff Theorem

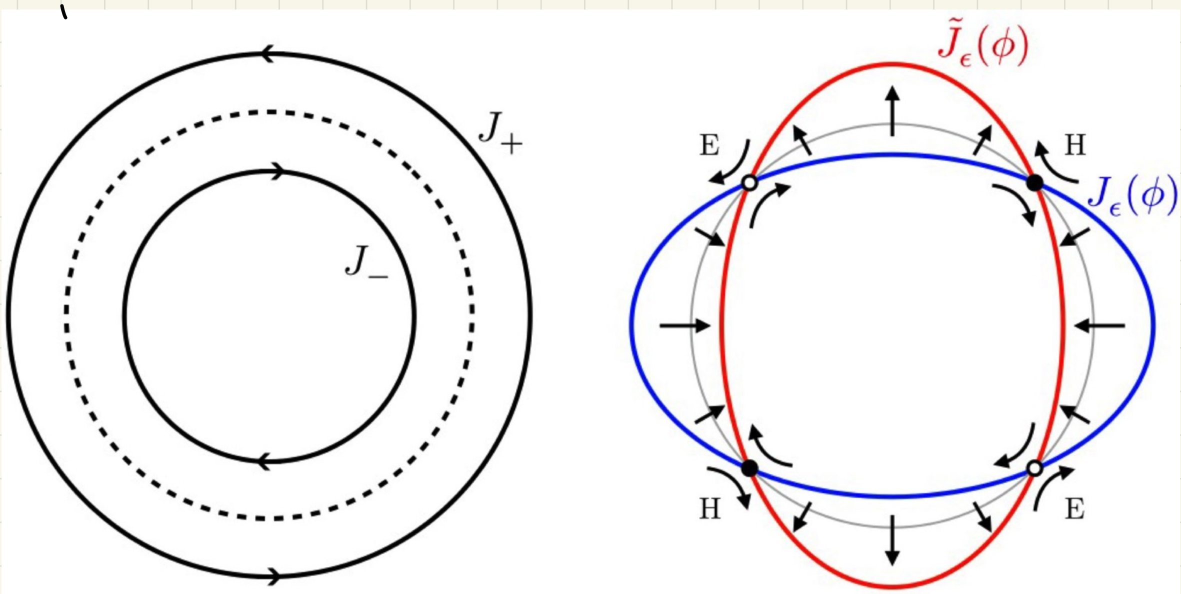
Back to our perturbed twist map, $\hat{T}_\epsilon$:

$$\phi_{n+1} = \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon f(\phi_n, J_{n+1})$$
$$J_{n+1} = J_n + \epsilon g(\phi_n, J_{n+1})$$

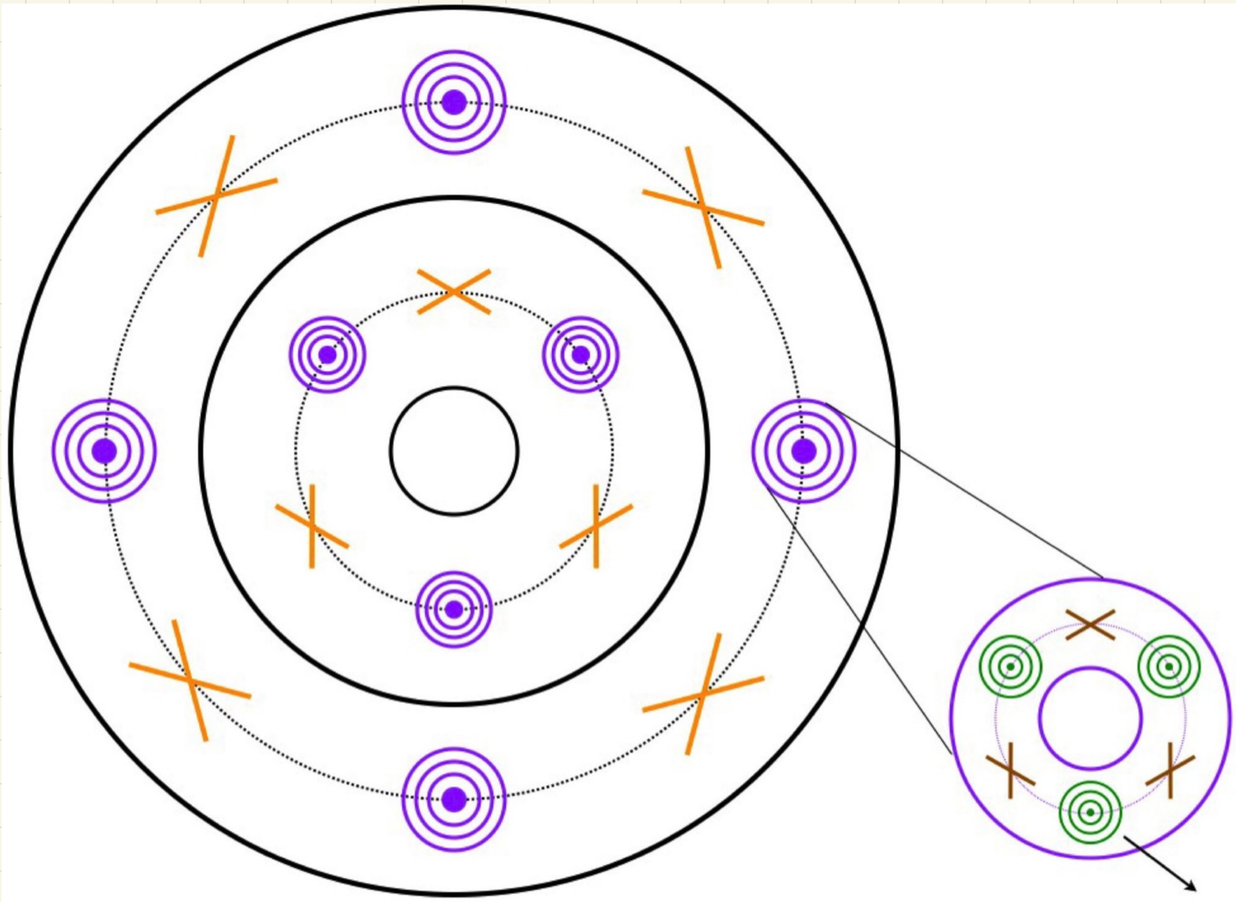
with

$$\frac{\partial f}{\partial \phi_n} + \frac{\partial g}{\partial J_{n+1}} = 0 \quad \Rightarrow \quad \hat{T}_\epsilon \text{ canonical}$$

For $\epsilon = 0$, the map $\hat{T}_0$ leaves J invariant, and thus maps circles to circles. If $\alpha(J) \notin \mathbb{Q}$, the images of the iterated map $\hat{T}_0$ become dense on the circle. Suppose $\alpha(J) = \frac{r}{s} \in \mathbb{Q}$, and wolog assume $\alpha'(J) > 0$, so that on circles $J_\pm = J \pm \Delta J$ we have $\alpha(J_+) > r/s$ and $\alpha(J_-) < r/s$. Under $\hat{T}_0^{\,s}$, all points on the circle $C = C(J)$ are fixed. The circle $C_+ = C(J_+)$ rotates slightly counterclockwise while $C_- = C(J_-)$ rotates slightly clockwise. Now consider the action of $\hat{T}_\epsilon^{\,s}$, assuming that $\epsilon \ll \Delta J / J$. Acting on C_+, the result is still a net counterclockwise shift plus a small radial component of $\mathcal{O}(\epsilon)$. Similarly, C_- continues to rotate clockwise plus an $\mathcal{O}(\epsilon)$ radial component. By the Intermediate Value Theorem, for each value of ϕ there is some point $J = J_\epsilon(\phi)$ where the angular shift vanishes. Thus, along the curve $J_\epsilon(\phi)$ the action of $\hat{T}_\epsilon^s$ is **purely radial**. Next consider the curve $\tilde{J}_\epsilon(\phi) = \hat{T}_\epsilon^s J_\epsilon(\phi)$. Since $\hat{T}_\epsilon^s$ is volume-preserving, these curves must intersect at an even number of points.

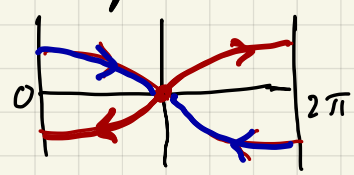


The situation is depicted in the above figure. The intersections of $J_\epsilon(\phi)$ and $\tilde{J}_\epsilon(\phi)$ are thus **fixed points** of the map $\hat{T}_\epsilon^s$. We furthermore see that the intersection $J_\epsilon(\phi) \cap \tilde{J}_\epsilon(\phi)$ consists of an alternating sequence of elliptic and hyperbolic fixed points. This is the content of the PBT: a small perturbation of a resonant torus with $\alpha(J) = r/s$ results in an equal number of elliptic and hyperbolic fixed points for $\hat{T}_\epsilon^s$. Since T_ϵ has period s acting on these fixed points, the number of EFPs and HFPs must be equal and a multiple of s. **In the vicinity of each EFP, this structure repeats** (see the figure below).



Self-similar structures in the iterated twist map.

Stable and unstable manifolds

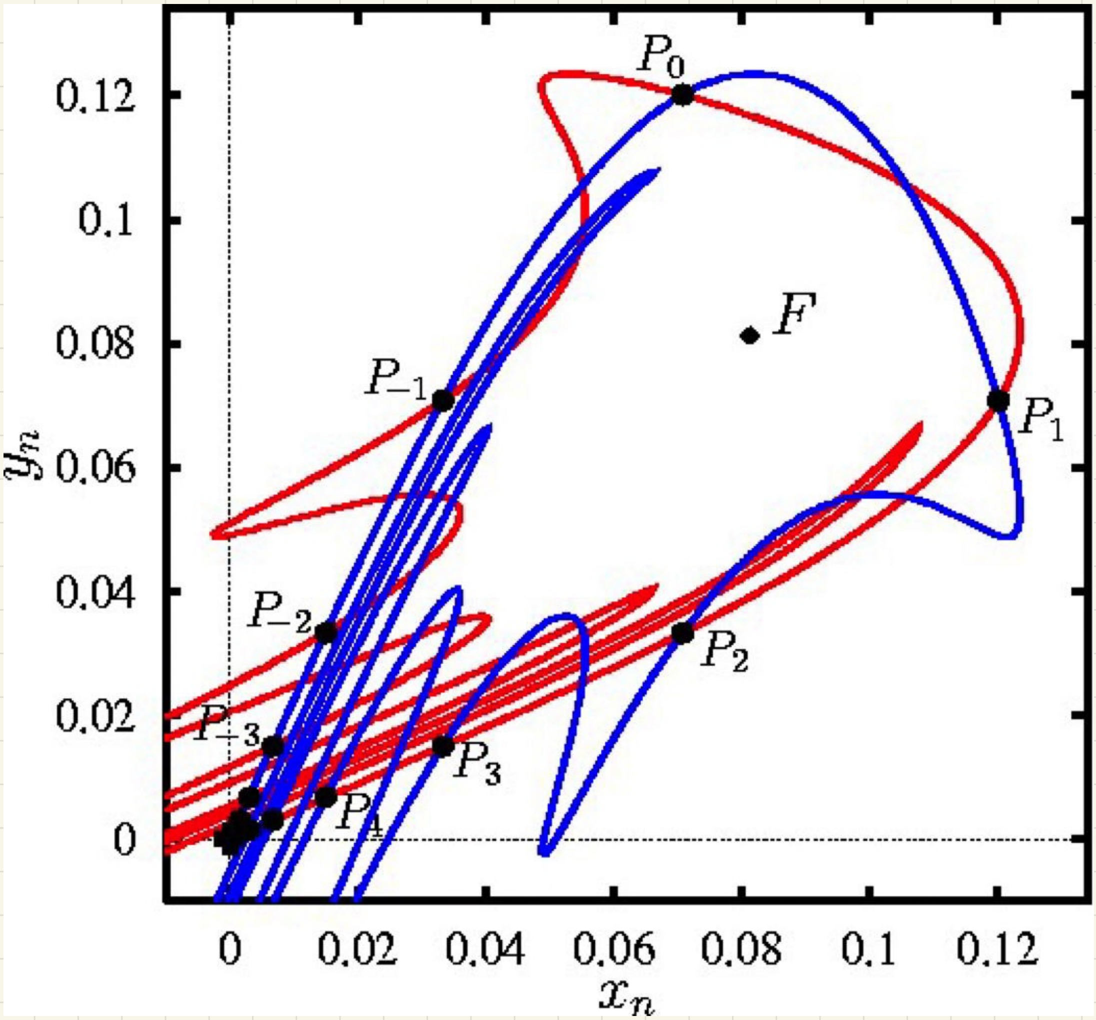


Emanating from each HFP are stable and unstable manifolds:

$$\vec{\varphi} \in \Sigma^{s}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_{\epsilon}^{ns}\vec{\varphi} = \vec{\varphi}^{*} \quad \text{(flows \textbf{to} } \vec{\varphi}^{*}\text{)}$$

$$\vec{\varphi} \in \Sigma^{u}(\vec{\varphi}^{*}) \Rightarrow \lim_{n\to\infty} \hat{T}_{\epsilon}^{-ns}\vec{\varphi} = \vec{\varphi}^{*} \quad \text{(flows \textbf{from} } \vec{\varphi}^{*}\text{)}$$

Note $\Sigma^{s}(\vec{\varphi}_i^{*}) \cap \Sigma^{s}(\vec{\varphi}_j^{*}) = \phi$ and $\Sigma^{u}(\vec{\varphi}_i^{*}) \cap \Sigma^{u}(\vec{\varphi}_j^{*}) = \phi$ for $i \neq j$ (no s/s or u/u intersections). However, $\Sigma^{s}(\vec{\varphi}_i^{*})$ and $\Sigma^{u}(\vec{\varphi}_j^{*})$ **can** intersect. For $i=j$, this is called a **homoclinic point**. (On its way from $\vec{\varphi}_j^{*}$ to $\vec{\varphi}_i^{*}$.) For $i \neq j$, this is a **heteroclinic point**.



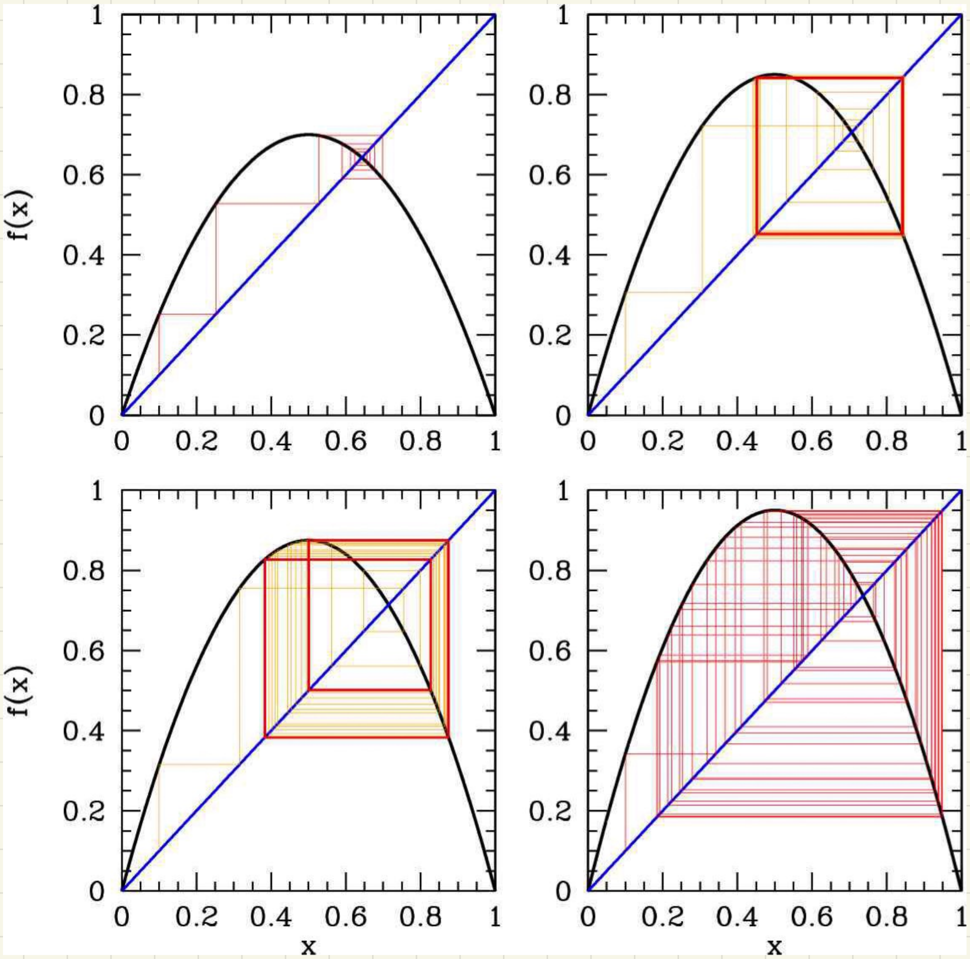
Homoclinic tangle for $x_{n+1} = y_n$ and $y_{n+1} = (a + b y_n^2) y_n - x_n$ with $a = 2.693$, $b = -104.888$. Blue curve is the stable manifold. Red curve is the unstable manifold. HFP at $(0,0)$. The fact that neither red nor blue curve can self intersect requires them to become increasingly tortured.

But since $\hat{T}_\epsilon^s$ is continuous and invertible, its action on a homoclinic (heteroclinic) point will produce a <u>new</u> homoclinic (heteroclinic) point, ad infinitum! For homoclinic intersections, the result is known as a **homoclinic tangle**.

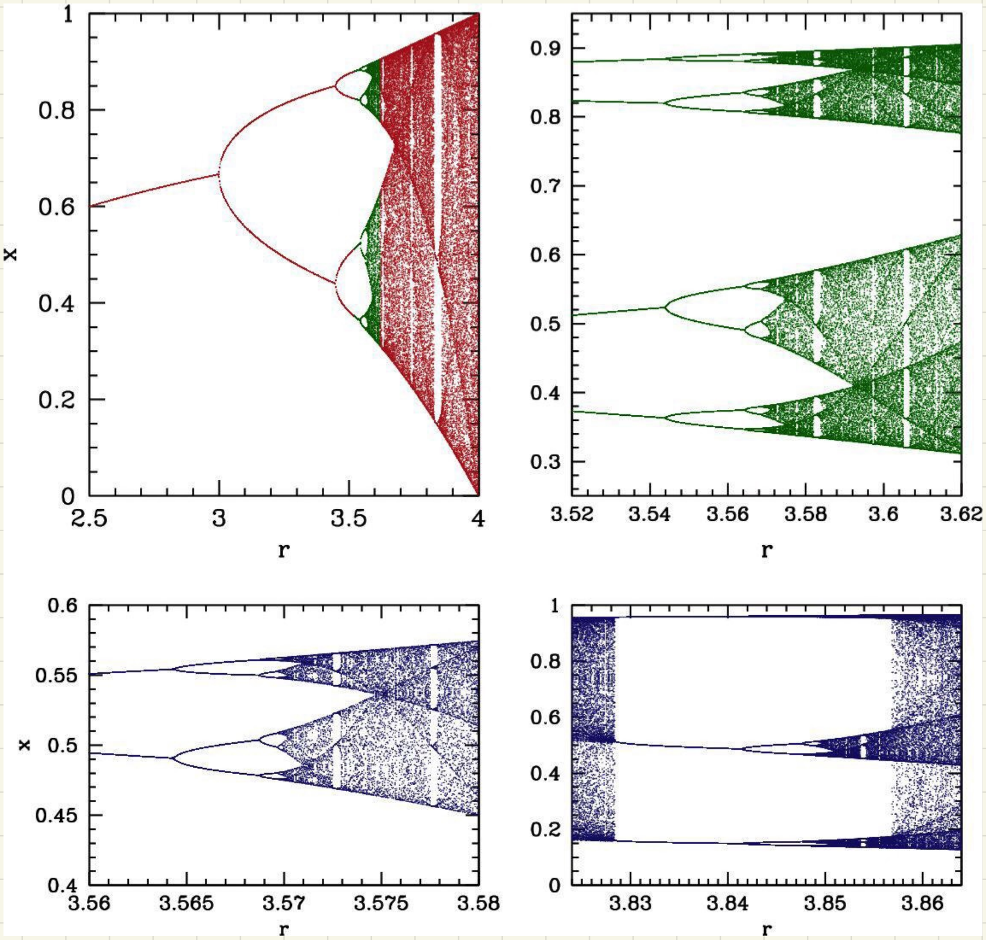
- **Maps in $d=1$:** $x_{n+1} = f(x_n)$; fixed point $x^* = f(x^*)$

  If $x = x^* + u$, then $u_{n+1} = f'(x^*)\, u_n + \mathcal{O}(u^2)$

  FP is **stable** if $|f'(x^*)| < 1$, **unstable** if $|f'(x^*)| > 1$.



Cobweb diagram for $f(x) = r\,x(1-x)$



Fixed points and cycles for $f(x) = r\,x(1-x)$