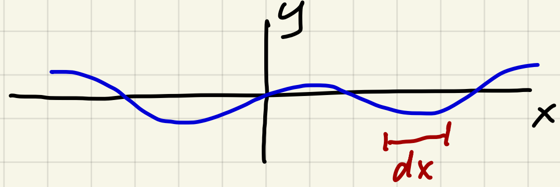
Lecture 9 (Nov. 2)



System: string of mass density $\mu(x)$ and tension $\tau(x)$. Instantaneous shape is $y(x,t)$.

Differential KE:

$$dT = \frac{1}{2}\mu(x)\left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx$$

Differential PE (relative to $y(x,t) = \text{const.}$):

$$dU = \tau(x)\, d\ell = \tau(x)\left\{\underbrace{\sqrt{dx^2+dy^2} - dx}_{d\ell}\right\}$$

Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - \tau(x)\left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right]$$

Assuming $\left|\frac{\partial y}{\partial x}\right| \ll 1$, $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{1}{2}\tau y_x^2 + \ldots$

Recall that for

$$S[y(x,t)] = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx\, \mathcal{L}(y, y_t, y_x; x, t)$$

that

$$\delta S = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right)\right]\delta y$$

$$+ \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial y_t}\delta y\right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y_x}\delta y\right]_{x_a}^{x_b}$$

First let's consider what is necessary in order that

The boundary terms both vanish. The first bounday term vanishes when $\delta y(x,t_a) = \delta y(x,t_b) = 0$. The second term vanishes when $\frac{\partial \mathcal{L}}{\partial y_x}\,\delta y$ vanishes at $x = x_{a,b}$ for all times t. For the case $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{1}{2}\tau y_x^2$, we have $\delta\mathcal{L}/\delta y_x = -\tau y_x$, thus, assuming $\tau(x_{a,b}) \neq 0$, the condition $y_x\,\delta y = 0$ at the end points means either (i) $y_x = 0$ or (ii) $\delta y = 0$ at each endpoint $x_{a,b}$. We then have the EL eqn,

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)}_{-\tau(x)\frac{\partial y}{\partial x} \text{ for our } \mathcal{L}} = 0$$

which for our case yields

$$\frac{\partial}{\partial x}\left[\tau(x)\,\frac{\partial y}{\partial x}\right] = \mu(x)\,\frac{\partial^2 y}{\partial t^2}$$

This equation, plus the spatial boundary conditions, governs the dynamics of the string. The simplest case is when $\mu(x) = \mu$ and $\tau(x) = \tau$ are both constants, whence we obtain the **Helmholtz equation**,

$$\frac{1}{c^2}\,y_{tt} = y_{xx} \Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) y(x,t) = 0$$

with $c = (\tau/\mu)^{1/2}$, which has units of velocity. This equation may be solved completely, and for arbitrary boundary conditions.

D'Alembert's solution

Define the variables $u \equiv x - ct$ and $v \equiv x + ct$. Then

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -c\frac{\partial}{\partial u} + c\frac{\partial}{\partial v}$$

Therefore

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 - \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2$$

wave operator

$$= 4\frac{\partial^2}{\partial u \partial v} = 4\frac{\partial}{\partial u}\frac{\partial}{\partial v}$$

Thus

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \Rightarrow y(u,v) = f(u) + g(v)$$

with $f(u)$ and $g(v)$ arbitrary functions as of yet. So:

$$y(x,t) = \underbrace{f(x-ct)}_{\text{right-mover}} + \underbrace{g(x+ct)}_{\text{left-mover}}$$

Now let's apply some initial conditions:

$$y(x,0) = f(x) + g(x)$$
$$c^{-1} y_t(x,0) = -f'(x) + g'(x)$$

Taking the spatial derivative of the first equation yields

$$y_x(x,0) = f'(x) + g'(x)$$

and thus we have

$$f'(\xi) = \frac{1}{2}\, y_x(\xi,0) - \frac{1}{2c}\, y_t(\xi,0)$$

$$g'(\xi) = \frac{1}{2}\, y_x(\xi,0) + \frac{1}{2c}\, y_t(\xi,0)$$

Now all we need to do is integrate $\int_0^{\xi} d\xi'$:

$$f(\xi) = \frac{1}{2}\, y(\xi,0) - \frac{1}{2c}\int_0^{\xi} d\xi'\, y_t(\xi',0) + C$$

$$g(\xi) = \frac{1}{2}\, y(\xi,0) + \frac{1}{2c}\int_0^{\xi} d\xi'\, y_t(\xi',0) - C$$

where $C = f(0) - \frac{1}{2}\, y(0,0) = \frac{1}{2}\, y(0,0) - g(0)$. Thus,

$$y(x,t) = \frac{1}{2}\left[y(x-ct,0) + y(x+ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\, y_t(\xi,0)$$

Thus we have a solution for all initial conditions.

Hamiltonian density

We define the momentum density as $\wp = \partial\mathcal{L}/\partial y_t$.
The Hamiltonian density is then $\mathcal{H} = \wp\, y_t - \mathcal{L}$.
Typically $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \mathcal{U}(y, y_x)$, hence $\wp = \mu y_t$ and

$$\mathcal{H} = \frac{\wp^2}{2\mu} + \mathcal{U}(y, y_x)$$

Expressed in terms of y_t rather than $\wp$, we have

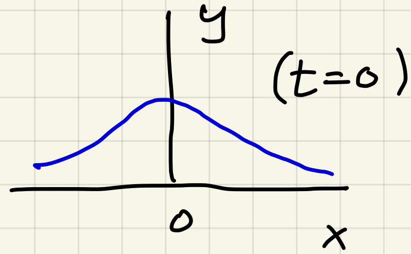
Scratch

$$y(x,t) = \frac{1}{2}\left[y(x-ct,0) + y(x+ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi \, y_t(\xi,0)$$

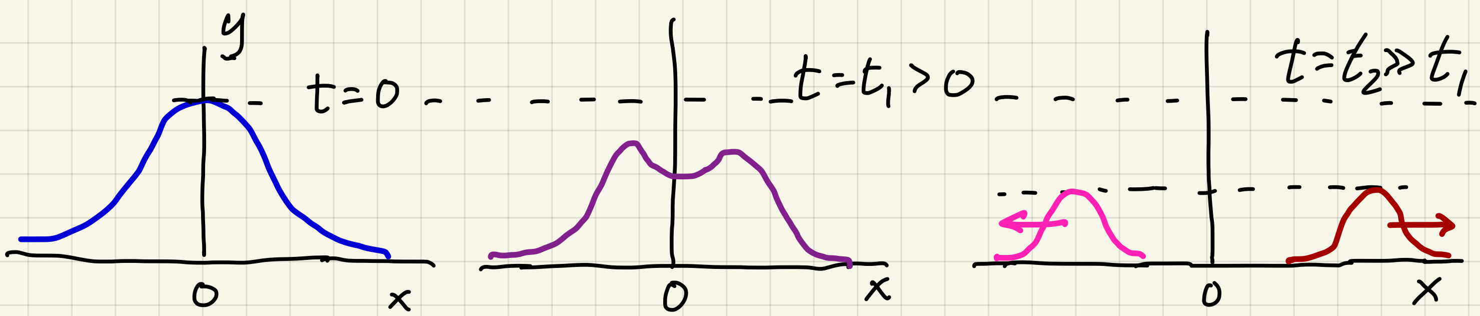
Suppose $y(x,0) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$, $y_t(x,0) = 0$.

Then:

$$y(x,t) = \frac{\gamma/2\pi}{(x-ct)^2+\gamma^2} + \frac{\gamma/2\pi}{(x+ct)^2+\gamma^2}$$



Evolution:



the energy density,

$$\mathcal{E}(x,t) = \tfrac{1}{2}\mu y_t^2 + \mathcal{U}(y, y_x; x)$$

The equations of motion are

$$-\frac{\partial \mathcal{U}}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) = 0$$

Now note that

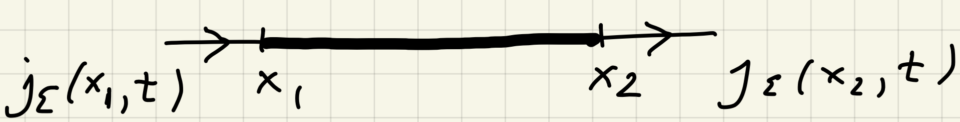
$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= \mu y_t y_{tt} + \frac{\partial \mathcal{U}}{\partial y} y_t + \frac{\partial \mathcal{U}}{\partial y_x} y_{xt} \\
&= \mu y_t y_{tt} - \mu y_t y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) y_t + \frac{\partial \mathcal{U}}{\partial y_x} y_{xt} \\
&= \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{U}}{\partial y_x} y_t\right] \equiv -\frac{\partial j_\varepsilon}{\partial x} \quad ; \quad j_\varepsilon \equiv -\frac{\partial \mathcal{U}}{\partial y_x} y_t
\end{aligned}$$

where j_ε is the energy current along the string. For the case $\mathcal{U} = \frac{1}{2}\tau y_x^2$, we have $j_\varepsilon = -\tau y_x y_t$. Note that

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j_\varepsilon}{\partial x} = 0 \quad ; \quad [\mathcal{E}] = E L^{-1}, \quad [j_\varepsilon] = E T^{-1}$$

which is the **continuity equation** for energy. Thus,

$$\frac{d}{dt}\int_{x_1}^{x_2} dx\, \mathcal{E}(x,t) = -\int_{x_1}^{x_2} dx\, \frac{\partial j_\varepsilon(x,t)}{\partial x} = \underbrace{j_\varepsilon(x_1,t)}_{\text{rate in}} - \underbrace{j_\varepsilon(x_2,t)}_{\text{rate out}}$$



For $\mathcal{U} = \frac{1}{2}\tau y_x^2$ with $\mu(x) = \mu$ and $\tau(x) = \tau$ constant, writing $y(x,t) = f(x-ct) + g(x+ct)$ we find

$$\mathcal{E}(x,t) = \tau\left[f'(x-ct)\right]^2 + \tau\left[g'(x+ct)\right]^2$$

$$j_\mathcal{E}(x,t) = c\tau\left[f'(x-ct)\right]^2 - c\tau\left[g'(x+ct)\right]^2$$

which are each sums over right-moving and left-moving contributions.

Example : Klein-Gordon system $\mathcal{U}(y, y_x) = \frac{1}{2}\tau y_x^2 + \frac{1}{2}\beta y^2$
Then $\mathcal{E} = \frac{1}{2}\mu y_t^2 + \frac{1}{2}\tau y_x^2 + \frac{1}{2}\beta y^2$. Eqns of motion:

$$\mathcal{L} = \frac{1}{2}\mu y_t^2 - \mathcal{U}(y, y_x) \Rightarrow$$

$$-\frac{\partial \mathcal{U}}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{U}}{\partial y_x}\right) = 0$$

$$-\beta y - \mu y_{tt} + \tau y_{xx} = 0$$

Thus we have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) y = m^2 y \quad ; \quad m \equiv \sqrt{\frac{\beta}{\mu}}$$

This is not the Helmholtz eqn (it is the KG eqn).
D'Alembert's solution does not pertain. Still,

$$j_\mathcal{E} = -\frac{\partial \mathcal{U}}{\partial y_x} y_t = -\tau y_x y_t$$

Momentum flux density and stress-energy tensor:

$$\mathcal{E} = \tfrac{1}{2}\mu y_t^2 + \tfrac{1}{2}\tau y_x^2 \Rightarrow \frac{\partial \mathcal{E}}{\partial x} = \frac{\partial}{\partial t}(\mu y_t y_x)$$

Thus, with

momentum current

momentum flux density

$$J_\pi \equiv \mathcal{E} \quad , \quad \pi \equiv -\mu y_t y_x = \frac{J_\varepsilon}{c^2}$$

we may write

$$\underbrace{\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)}_{\partial/\partial x^\mu} \overbrace{\begin{pmatrix} c\mathcal{E} & -c\pi \\ J_\varepsilon & -J_\pi \end{pmatrix}}^{T^\mu{}_\nu} = 0$$

or $\partial_\mu T^\mu{}_\nu = 0$, where $T^\mu{}_\nu$ is the **stress-energy tensor.** Note that while π and $g = \mu y_t$ have the same dimensions, π is the momentum density <u>along</u> the string while g is the momentum density <u>transverse</u> to the string. General result:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu y)}\partial_\nu y - \delta^\mu_\nu \mathcal{L}$$

This satisfies $\partial_\mu T^\mu{}_\nu = 0$ for all ν.

Electromagnetism: $\mathcal{E} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \Rightarrow$

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\right)$$

$$= \frac{1}{4\pi}\vec{E}\cdot(c\vec{\nabla}\times\vec{\nabla} - 4\pi\vec{J}) + \frac{1}{4\pi}\vec{B}\cdot(-c\vec{\nabla}\times\vec{E})$$

$$= -\vec{E}\cdot\vec{J} - \vec{\nabla}\cdot\vec{S}$$

where $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$ = Poynting vector. The stress-energy tensor is

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \mathcal{E} & -c^{-1}S_x & -c^{-1}S_y & -c^{-1}S_z \\ c^{-1}S_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ c^{-1}S_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ c^{-1}S_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

with

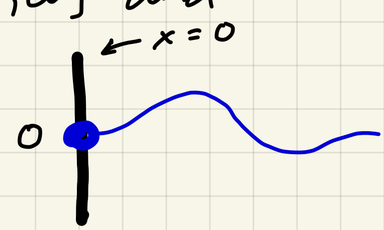
$$\sigma_{ij} = \frac{1}{4\pi}\left\{-E_iE_j - B_iB_j + \frac{1}{2}(\vec{E}^2+\vec{B}^2)\delta_{ij}\right\}$$

which is the **Maxwell stress tensor**. Now

$$\partial_\mu T^{\mu}{}_{\nu} = 0 \quad ; \quad \partial_\mu = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)$$

- Reflection at an interface

Consider a semi-infinite string with $x \in [0,\infty]$ and with $y(0,t) = 0 \;\forall t$. We write



$$y(x,t) = f(x-ct) + g(x+ct)$$

and impose the boundary condition at $x=0$:

$$f(-ct) + g(ct) = 0 \;\Rightarrow\; f(\xi) = -g(-\xi) \;\forall \xi$$

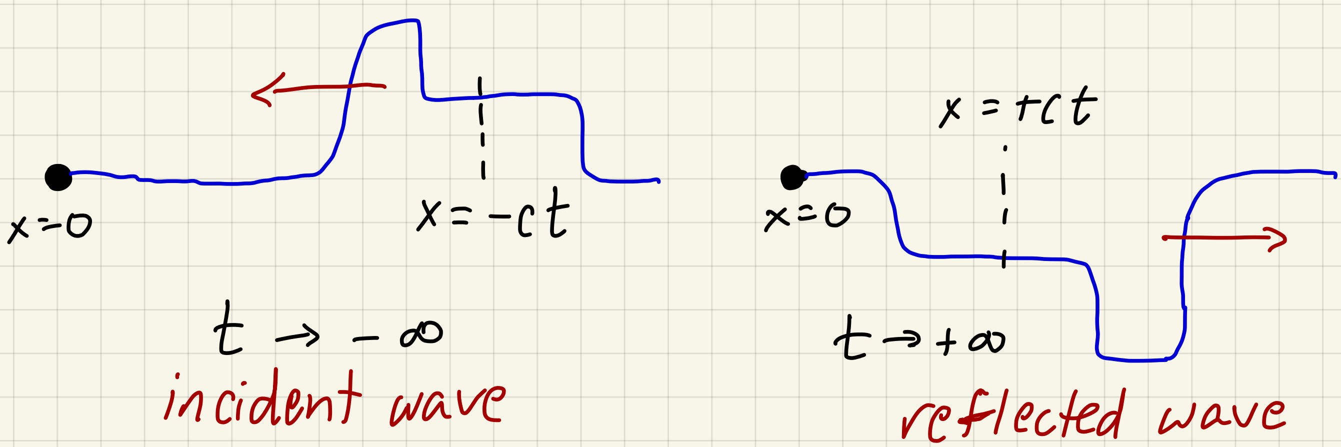
$$\xi \equiv -ct$$

Therefore, we have

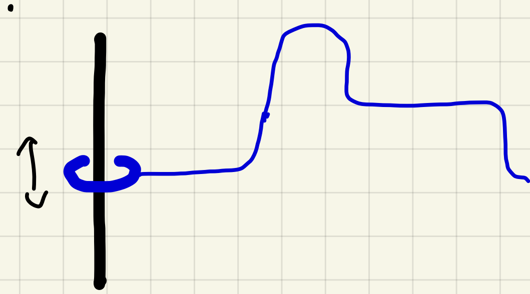
$$y(x,t) = g(ct+x) - \underbrace{g(ct-x)}_{f(x-ct)}$$

This is the general solution. Now suppose $g(\xi)$ resembles a pulse localized around $\xi \approx 0$.
In the distant past, $t \to -\infty \Rightarrow ct - x \to -\infty$
Hence no contribution from right mover.

How about the left-mover? Set $ct + x \approx 0 \Rightarrow x \approx -ct \in [0, \infty]$. I.e. incoming left-mover at $x \approx -ct$. For $t \to +\infty$, $ct + x \to +\infty \Rightarrow$ left-mover is gone. $ct - x \approx 0 \Rightarrow x \approx ct \in [0, \infty]$ I.e. outgoing right mover at $x \approx ct$. Sketch:



Suppose instead $y_x(0,t) = 0 \ \forall t$.



From $\delta S = \cdots - \frac{\partial \mathcal{L}}{\partial y_x} \delta y \Big|_0$

must vanish free

$$\partial \mathcal{L} / \partial y_x = -\tau y_x \Rightarrow y_x(0,t) = 0 \quad \forall t$$

Shape of string:

$$y(x,t) = f(x-ct) + g(x+ct)$$
$$y_x(x,0) = f'(-ct) + g'(ct)$$

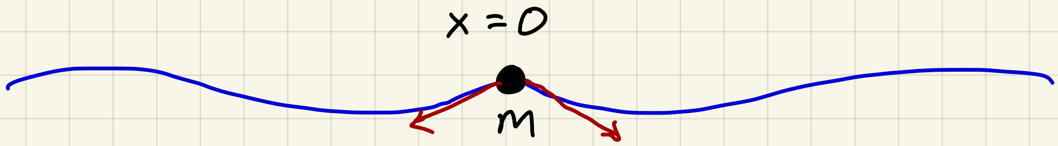
Thus $f'(\xi) = -g'(-\xi)$. Integrate to get

$$f(\xi) = g(-\xi)$$

So the shape is

$$y(x,t) = g(ct+x) + g(ct-x)$$
$$y_x(x,t) = g'(ct+x) - g'(ct-x)$$
$$= 0 \text{ when } x=0$$

- Mass point on a string:



$x<0$: $y(x,t) = f(ct-x) + g(ct+x)$

$x>0$: $y(x,t) = h(ct-x)$

Interpretation: f = incident wave

g = reflected wave

h = transmitted wave

Newton's law for mass at $x=0$:

$$m\ddot{y}(0,t) = \tau y'(0^+,t) - \tau y'(0^-,t)$$

Discontinuous $y'(0,t) = y_x(0,t) \Rightarrow$ acceleration of m.

Furthermore:

$$y'(0^-,t) = -f'(ct) + g'(ct)$$
$$y'(0^+,t) = h'(ct)$$

Continuity $\Rightarrow y(0^-,t) = y(0^+,t) \Rightarrow$

$$h(ct) = f(ct) + g(ct)$$

Let $\xi = ct \Rightarrow$

$$h(\xi) = f(\xi) + g(\xi)$$

$$f''(\xi) + g''(\xi) = -\frac{2\tau}{mc^2} g'(\xi)$$

From these, get $g(\xi)$ and $h(\xi)$ in terms of $f(\xi)$.

Fourier transforms:

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} \quad , \quad \hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ik\xi}$$

Derivatives wrt ξ replaced by $ik \times \hat{f}(k)$ etc.

Then we have

$$(-k^2 + iQk)\hat{g}(k) = k^2 \hat{f}(k)$$

$$\hat{h}(k) = \hat{f}(k) + \hat{g}(k)$$

with $Q \equiv 2\tau/mc^2 = 2\mu/m$; $[Q] = L^{-1}$.

Solution:

$$\hat{g}(k) = \hat{r}(k)\,\hat{f}(k) \quad , \quad \hat{h}(k) = \hat{t}(k)\,\hat{f}(k)$$

with

$$\hat{r}(k) = -\frac{k}{k-iQ} \quad , \quad \hat{t}(k) = -\frac{iQ}{k-iQ}$$

Note $t = 1+r$ since $h = f+g$.

Shape of transmitted wave:

$$h(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{t}(k)\, \hat{f}(k)$$

$$= \int_{-\infty}^{\infty} d\xi'\; t(\xi-\xi')\, f(\xi')$$

$$t(\xi-\xi') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')}$$

and for

$$\hat{t}(k) = -\frac{iQ}{k-iQ}$$

find

$$t(\xi-\xi') = Q\, e^{-Q(\xi-\xi')}\, \Theta(\xi-\xi')$$