Lecture 10 (Nov. 4)

Recall we were discussing the dynamics of a string (mass density μ, tension τ) with an attached point mass m at $x=0$. We wrote

$$y(x,t) = \underbrace{f(ct-x)}_{\text{incident}} + \underbrace{g(ct+x)}_{\text{reflected}} \qquad (x<0)$$
$$= \underbrace{h(ct-x)}_{\text{transmitted}} \qquad (x>0)$$

At $x=0$, we have $F=ma$ for the mass point, i.e.

$$m\ddot{y}(0,t) = \tau y'(0^+,t) - \tau y'(0^-,t)$$

as well as continuity $y(0^-,t) = y(0^+,t)$. Expressed in terms of the functions f, g, and h, we have

$$f''(\xi) + g''(\xi) = -\frac{2\tau}{mc^2}\, g'(\xi)$$
$$f(\xi) + g(\xi) = h(\xi)$$

which we solved by going to Fourier space:

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \quad , \quad \hat{f}(k) = \int_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi}$$

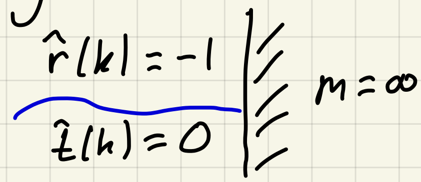
etc. Note $\hat{f}(-k) = \hat{f}(k)^*$ since $f(\xi) \in \mathbb{R}$. We found

$$\hat{g}(k) = \hat{r}(k)\hat{f}(k) \quad , \quad \hat{h}(k) = \hat{t}(k)\hat{f}(k)$$

where, with $Q = \frac{2\tau}{mc^2} = \frac{2\mu}{m}$, $\qquad [Q] = L^{-1}$

$$\hat{r}(k) = -\frac{k}{k - iQ} \quad , \quad \hat{t}(k) = -\frac{iQ}{k - iQ}$$

are, respectively, the **reflection** and **transmission amplitudes**. Note that $\hat{t}(k) = 1 + \hat{r}(k)$, which follows directly from the continuity relation $h = f + g$. Another result is that



$$|\hat{r}(k)|^2 + |\hat{t}(k)|^2 = 1$$

We call $R(k) \equiv |\hat{r}(k)|^2$ and $T(k) \equiv |\hat{t}(k)|^2$ the reflection and transmission **coefficients**. These are the modulus squared, respectively, of the reflection and transmission **amplitudes**. By the way, note that $\hat{r}(-k) = \hat{r}(k)^*$ and $\hat{t}(-k) = \hat{t}(k)^*$.

Energy

The energy in the string is

$$E_{string}(t) = \int_{-\infty}^{\infty} dx \left\{ \tfrac{1}{2}\mu \dot{y}^2 + \tfrac{1}{2}\tau y'^2 \right\}$$
$$= \tau \int_{ct}^{\infty} d\xi \, [f'(\xi)]^2 + \tau \int_{-\infty}^{ct} d\xi \left([g'(\xi)]^2 + [h'(\xi)]^2 \right)$$

The total energy of the system is $E = E_{string} + E_{mass}$, with

$$E_{mass}(t) = \tfrac{1}{2} mc^2 \left[h'(ct) \right]^2$$

Scratch

$$E_{string}(t) = \int_{-\infty}^{0} dx \left\{ \frac{1}{2}\mu \left[c f'(ct-x) + c g'(ct+x) \right]^2 + \frac{1}{2}\tau \left[-f'(ct-x) + g'(ct+x) \right]^2 \right\} + \int_{0}^{\infty} dx\ \frac{1}{2}(\mu c^2 + \tau)\left[h'(ct) \right]^2$$

But $\mu c^2 = \tau$! Thus

$$E_{string}(t) = \tau \int_{-\infty}^{0} dx \left\{ \left[f'(ct-x) \right]^2 + \left[g'(ct+x) \right]^2 \right\} + \tau \int_{0}^{\infty} dx \left[h'(ct-x)^2 \right]$$

$$= \tau \int_{ct}^{\infty} d\xi \left[f'(\xi) \right]^2 + \tau \int_{-\infty}^{ct} d\xi \left(\left[g'(\xi) \right]^2 + \left[h'(\xi) \right]^2 \right)$$

$x < 0$: $\quad \xi = ct - x \in [ct, \infty]$

$\xi = ct + x \in [-\infty, ct]$

$x > 0$: $\quad \xi = ct - x \in [-\infty, ct]$

$$\int_{-\infty}^{\infty} d\xi \left[f'(\xi) \right]^2 = \int_{-\infty}^{\infty} d\xi \left[\frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} \right] \left[\frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \hat{f}^*(k') e^{-ik'\xi} \right]$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} (ik)(-ik') \hat{f}(k) \hat{f}^*(k') \underbrace{\int_{-\infty}^{\infty} d\xi\, e^{i(k-k')\xi}}_{2\pi\,\delta(k-k')}$$

Let's evaluate the total energy in the limits $t \to \pm\infty$. For $|t| \to \infty$, $E_{mass} \to 0$ because we assume the mass starts from rest, and by late times it has shaken off all the energy it acquired into vibrations of the string. So we have

$$E_{string}(-\infty) = \tau \int_{-\infty}^{\infty} d\xi \, [f'(\xi)]^2 = \tau \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 |\hat{f}(k)|^2$$

$$E_{string}(+\infty) = \tau \int_{-\infty}^{\infty} d\xi \left([g'(\xi)]^2 + [h'(\xi)]^2 \right) = \tau \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 \left(|\hat{g}(k)|^2 + |\hat{h}(k)|^2 \right)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 \underbrace{\left(|\hat{r}(k)|^2 + |\hat{t}(k)|^2 \right)}_{=1} |\hat{f}(k)|^2 = E_{string}(-\infty)$$

In fact, we can show with a bit more work that $E(t) = E_{string}(-\infty)$ for all times $t \in \mathbb{R}$, including the contribution from $E_{mass}(t)$. I.e. total energy is conserved.

- Back to real space!

We have

$$h(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) \hat{f}(k) e^{ik\xi} = \int_{-\infty}^{\infty} d\xi' \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) e^{ik(\xi-\xi')} \right] f(\xi')$$

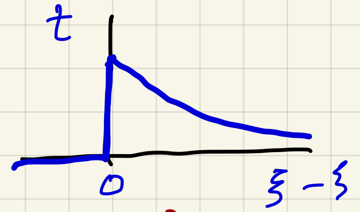
$$= \int_{-\infty}^{\infty} d\xi' \, t(\xi-\xi') f(\xi')$$

where $\quad t(\xi-\xi') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) e^{ik(\xi-\xi')}$

is the transmission kernel in real space. For our case,

$$\hat{t}(k) = \frac{-iQ}{k-iQ} \Rightarrow t(\xi-\xi') = Q e^{-Q(\xi-\xi')}\,\Theta(\xi-\xi')$$

Note that for a δ-function pulse $f(\xi) = C\delta(\xi)$ we have that

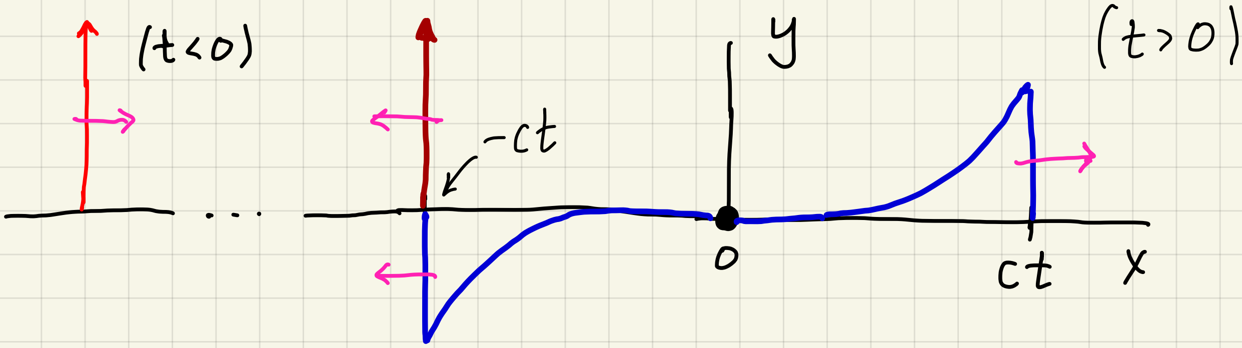


$$f(\xi) = C\,\delta(\xi) \Rightarrow \begin{aligned} h(\xi) &= C\,t(\xi) \\ g(\xi) &= C\{\delta(\xi) - t(\xi)\} \end{aligned}$$

So for our example,

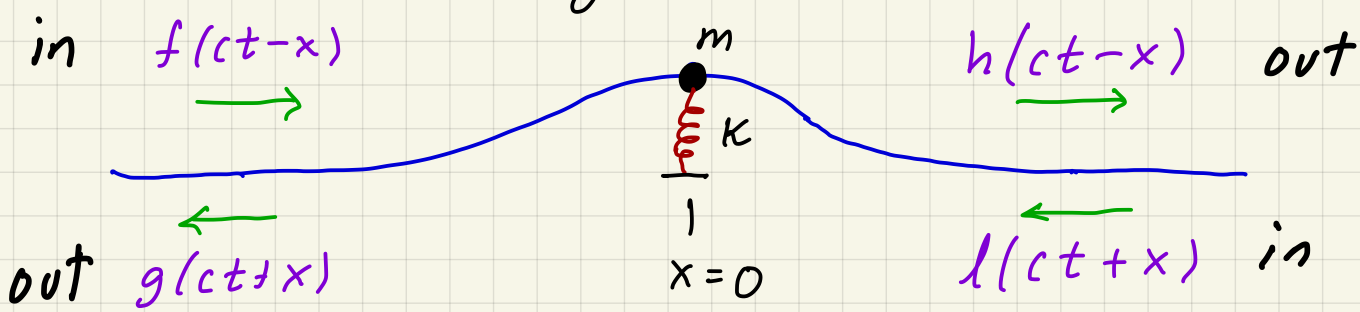
$$h(ct-x) = CQ\,e^{-Q(ct-x)}\,\Theta(ct-x)$$

so the late time shape of $y(x,t)$ looks like this



- S-matrix

Consider a more general state of affairs:



Continuity at $x=0$ says $f(\xi)+g(\xi)=h(\xi)+\ell(\xi)$.
Newton's law $F=ma$ for the mass point is now

$$m\ddot{y}(0,t) = \tau\left[y'(0^+,t)-y'(0^-,t)\right]-K\,y(0,t)$$

which says

$$mc^2\left[f''(\xi)+g''(\xi)\right] = \tau\left[\ell'(\xi)-h'(\xi)-g'(\xi)+f'(\xi)\right] - K\left[f(\xi)+g(\xi)\right]$$

Now take the FT:

$$\hat{f}(k)+\hat{g}(k) = \hat{h}(k)+\hat{\ell}(k)$$

$$-mc^2k^2\left[\hat{f}(k)+\hat{g}(k)\right] = i\tau k\left[\hat{\ell}(k)-\hat{h}(k)-\hat{g}(k)+\hat{f}(k)\right] - K\left[\hat{f}(k)+\hat{g}(k)\right]$$

Divide now by $\frac{1}{2}mc^2$, with

$$Q \equiv \frac{2\tau}{mc^2} \quad , \quad P^2 \equiv \frac{K}{mc^2} \qquad \text{units: } [Q]=[P]=L^{-1}$$

to obtain (suppressing k in $\hat{f}(k)$ etc.)

$$-k^2\left[\hat{f}+\hat{g}+\hat{h}+\hat{\ell}\right] = iQk\left[\hat{\ell}-\hat{h}-\hat{g}+\hat{f}\right] - P^2\left[\hat{f}+\hat{g}+\hat{h}+\hat{\ell}\right]$$

The **S-matrix** relates outgoing states ($\hat{h}$ and $\hat{g}$) to the incoming ones ($\hat{f}$ and $\hat{\ell}$). We have

$$\text{(i)} \quad \hat{f}-\hat{\ell} = \hat{h}-\hat{g}$$

and

$$\text{(ii)}\quad \overbrace{(k^2+iQk-P^2)}^{\Lambda(k)}(\hat{f}+\hat{\ell}) = -\overbrace{(k^2-iQk-P^2)}^{\Lambda^*(k)}(\hat{h}+\hat{g})$$

In matrix form,

$$\begin{pmatrix} 1 & -1 \\ \Lambda & \Lambda \end{pmatrix}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -\Lambda^* & -\Lambda^* \end{pmatrix}\begin{pmatrix} \hat{h} \\ \hat{g} \end{pmatrix}$$

where $\Lambda(k) \equiv k^2+iQk-P^2$. Thus

$$\begin{pmatrix} \hat{h} \\ \hat{g} \end{pmatrix} = -\frac{1}{2\Lambda^*}\begin{pmatrix} -\Lambda^* & 1 \\ \Lambda^* & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \Lambda & \Lambda \end{pmatrix}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix}$$

$$= \underbrace{-\frac{1}{2\Lambda^*}\begin{pmatrix} \Lambda-\Lambda^* & \Lambda+\Lambda^* \\ \Lambda+\Lambda^* & \Lambda-\Lambda^* \end{pmatrix}}_{S(k)\ =\ \text{“scattering matrix”}}\begin{pmatrix} \hat{f} \\ \hat{\ell} \end{pmatrix}$$

Hence

$$S(k) = \begin{pmatrix} \hat{t}(k) & \hat{r}'(k) \\ \hat{r}(k) & \hat{t}'(k) \end{pmatrix}$$

with

$$\hat{r}(k) = \hat{r}'(k) = -\frac{k^2-P^2}{k^2-iQk-P^2} \xrightarrow[P\to 0]{} \frac{-k}{k-iQ}$$

$$\hat{t}(k) = \hat{t}'(k) = -\frac{iQk}{k^2-iQk-P^2} \xrightarrow[P\to 0]{} \frac{-iQ}{k-iQ}$$

Here $\hat{r}=\hat{r}'$ and $\hat{t}=\hat{t}'$ due to time-reversal symmetry.

Note: (i) $\hat{t}(k) = 1 + \hat{r}(k)$

(ii) $|\hat{r}(k)|^2 + |\hat{t}(k)|^2 = 1$

The first of these again comes from continuity of $y(x,t)$ at $x=0$, which says

$$f(\xi) + g(\xi) = h(\xi) + \ell(\xi) \Rightarrow \hat{f}(k) + \hat{g}(k) = \hat{h}(k) + \hat{\ell}(k)$$

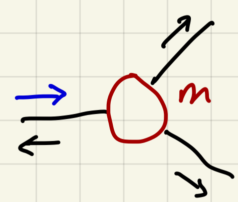
But since $\hat{h} = \hat{t}\hat{f} + \hat{r}'\hat{\ell}$ and $\hat{g} = \hat{r}\hat{f} + \hat{t}'\hat{\ell}$ we have

$$(1 + \hat{r} - \hat{t})\hat{f} = (1 - \hat{r}' - \hat{t}')\hat{\ell}$$

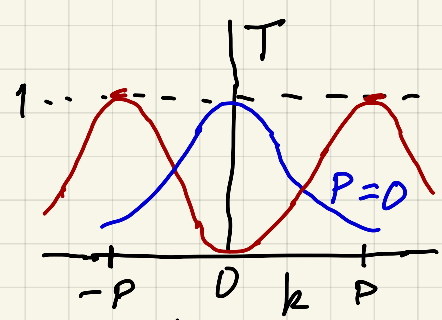
Since the inputs $\hat{f}$ and $\hat{\ell}$ are arbitrary, we must have

$$\hat{t}(k) = 1 + \hat{r}(k) \quad , \quad \hat{t}'(k) = 1 + \hat{r}'(k)$$

for all values of k. The reflection and transmission coefficients are



$$R+T=1 \left\{ \begin{aligned} R(k) &= |\hat{r}(k)|^2 = \frac{(k^2-P^2)^2}{(k^2-P^2)^2 + Q^2k^2} \\ T(k) &= |\hat{t}(k)|^2 = \frac{Q^2k^2}{(k^2-P^2)^2 + Q^2k^2} \end{aligned} \right.$$



Note that setting $P \to 0$ recovers our previous results.

Also note that maximizing $T(k)$ with respect to k yields $k^2 = P^2$, and that $T(k = \pm P) = 1$.

- Finite strings : Bernoulli's method

Let $x_L = 0$ and $x_R = L$, with $y(0,t) = y(L,t) = 0$ (fixed ends). Again we write

$$y(x,t) = f(x-ct) + g(x+ct)$$

Invoking the BC at $x=0$ yields $f(\xi) = -g(-\xi)$, hence we have

$$y(x,t) = g(ct+x) - g(ct-x)$$

We next demand $y(L,t) = 0$, which yields

$$g(ct+L) = g(ct-L) \Rightarrow g(\xi+2L) = g(\xi)$$

which says that $g(\xi)$ is periodic with period $2L$. Any such periodic function may be expressed as a Fourier series, viz.

$$g(\xi) = \sum_{n=1}^{\infty} \left\{ \tilde{A}_n \cos\left(\frac{n\pi\xi}{L}\right) + \tilde{B}_n \sin\left(\frac{n\pi\xi}{L}\right) \right\}$$

The full, time-dependent solution is then given by

$$A_n \equiv \sqrt{2\mu L}\,\tilde{B}_n \qquad B_n \equiv -\sqrt{2\mu L}\,\tilde{A}_n$$

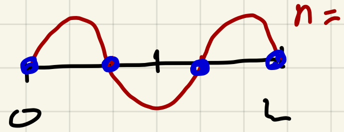
$$y(x,t) = g(ct+x) - g(ct-x)$$

$$= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \Bigg\{ \underbrace{A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)}_{\equiv\, C_n \cos\left(\frac{n\pi c t}{L} + \phi_n\right)} \Bigg\}$$

We define

$$k_n \equiv \frac{n\pi}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad \psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$$

for $n \in \{1, 2, \ldots, \infty\}$. Thus, $\psi_n(x) = (2/\mu L)^{1/2} \sin(k_n x)$ has $(n+1)$ nodes, located at $x_{j,n} = jL/n$, for $j \in \{0, \ldots, n\}$. We further define the inner product,



$$\langle \phi \,|\, \chi \rangle \equiv \mu \int_0^L dx\, \phi(x)\, \chi(x)$$

where ϕ and χ are real functions of $x \in [0, L]$ that satisfy $\phi(0) = \phi(L) = \chi(0) = \chi(L) = 0$. Our basis functions $\psi_n(x)$ are orthonormal with respect to this IP:

$$\langle \psi_m | \psi_n \rangle = \frac{2}{L} \int_0^L dx\, \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \delta_{mn}$$

Furthermore, this basis is <u>complete</u>, i.e.

$$\mu \sum_{n=1}^{\infty} \psi_n(x)\, \psi_n(x') = \delta(x - x')$$

We may express the constants $\{A_n, B_n\}$ in terms of our initial conditions, viz.

$$y(x,0) = \sum_n A_n \psi_n(x) \quad , \quad \dot{y}(x,0) = \sum_{n=1}^{\infty} \omega_n B_n \psi_n(x)$$

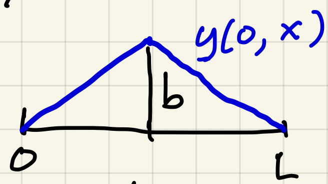
Multiplying by $\mu \psi_m(x)$ and integrating over $[0,L]$,

$$A_m = \mu \int_0^L dx\, y(x,0)\, \psi_m(x) \quad , \quad B_m = \mu \omega_n^{-1} \int_0^L dx\, \dot{y}(x,0)\, \psi_m(x)$$

Example : $y(x,0) = \begin{cases} 2bx/L & \text{if } x \in [0, \frac{1}{2}L] \\ 2b(L-x)/L & \text{if } x \in [\frac{1}{2}L, L] \end{cases}$

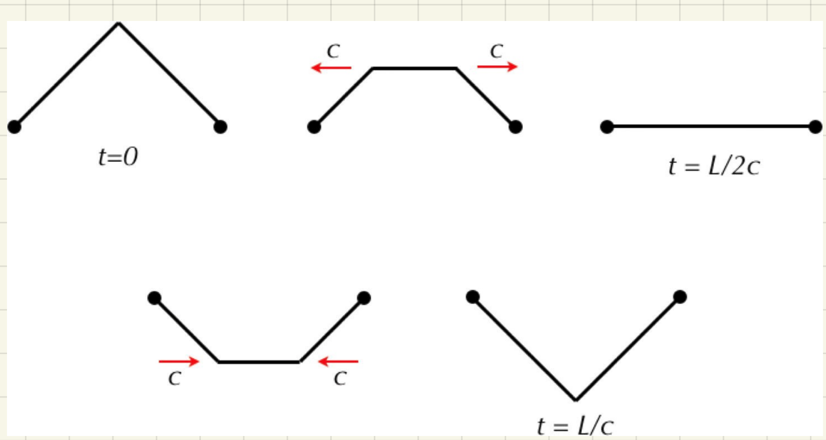
and $\dot{y}(x,0) = 0$ (release string from rest). Find

$$A_n = (2\mu L)^{1/2} \frac{4b}{\pi^2 n^2} \sin\left(\tfrac{1}{2} n\pi\right)$$



i.e. $A_{2k} = 0$ and $A_{2k+1} = (2\mu L)^{1/2} \cdot \dfrac{4b}{\pi^2} \cdot \dfrac{(-1)^k}{(2k+1)^2}$.

Also $B_n = 0 \;\forall n$. Note that $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is odd under reflection about the midpoint $x = \frac{L}{2}$, whereas our initial condition $y(x,0) = y(L-x,0)$ was even. Here's a set of images of the evolution:



This is the d'Alembert solution, extending $g(x)$ to the entire real line, with $g(x) = g(x+2L) = -g(-x)$.