Lecture 11 (Nov. 9)

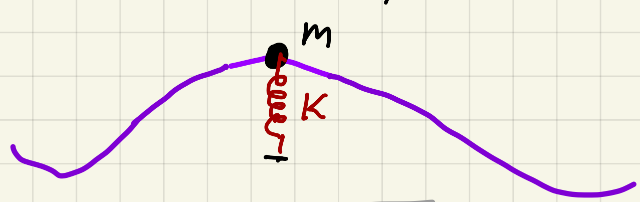
Start with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu(x)\dot{y}^2 - \frac{1}{2}\tau(x)y'^2 - \frac{1}{2}v(x)y^2$$

The last term corresponds to a harmonic potential attracting the string at each x value to $(x, y=0)$. In fact, if

$$\mu(x) = \mu_0 + m\delta(x) \quad , \quad v(x) = K\delta(x)$$

then we recover the problem of a string with an attached point mass that is connected to the point $(0,0)$ by a spring. The EL equations are found to be



$$\boxed{-\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] + v(x)\,y = -\mu(x)\frac{\partial^2 y}{\partial t^2}}$$

This equation is time-translation invariant because the coefficients are autonomous (i.e. $\tau(x)$, $v(x)$, and $\mu(x)$ do not depend on time t). This means that the partial differential operator (PDO)

$$\hat{Q} = \mu(x)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\tau(x)\frac{\partial}{\partial x} + v(x)$$

for which $\hat{Q}\, y(x,t) = 0$, commutes with the PDO $\partial/\partial t$: $[\hat{Q}, \partial/\partial t] = 0$. This means that the solutions to $\hat{Q}\, y(x,t) = 0$ may be written as

$$y(x,t) = \psi(x)\, e^{-i\omega t}$$

Furthermore, since $y^*(x,t)$ is a solution, then we may write

$$y(x,t) = \psi(x) \cos(\omega t + \phi)$$

We are left with the equation

$$\hat{K}\, \psi(x) = \mu(x)\, \omega^2 \psi(x)$$

where

$$\hat{K} = -\frac{d}{dx}\, \tau(x) \frac{d}{dx} + v(x)$$

is an ordinary differential operator (ODO). The equation

$$\hat{K}\, \psi(x) = -\frac{d}{dx}\left[\tau(x) \frac{d\psi(x)}{dx}\right] + v(x)\, \psi(x) = \mu(x) \omega^2 \psi(x)$$

is known as the **Sturm–Liouville equation.**

The simplest example is when $\tau(x) = \tau$ and $\mu(x) = \mu$ are constants, and $v(x) = 0$. Then $\hat{K} = -\tau \frac{d^2}{dx^2}$, and the solutions to the SL eqn are of the form

$$\psi(x) = A e^{ikx}$$

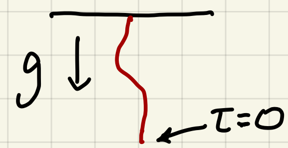
where $k^2 = \mu \omega^2/\tau = \omega^2/c^2$ with $c = (\tau/\mu)^{1/2}$ = wave speed.

I.e. $\psi(x) = A e^{\pm i\omega x/c}$, so $y(x) = f(ct-x) + g(ct+x)$.

- **Boundary conditions** – We consider four classes:

① Fixed endpoints : $\psi(x) = 0$ for $x = x_{L,R}$

② Natural : $\tau(x)\psi'(x) = 0$ for $x = x_{L,R}$



③ Periodic : $\psi(x+L) = \psi(x)$ where $L = x_R - x_L$
[Also require $\tau(x) = \tau(x+L)$.]

④ Mixed homogeneous: $\alpha\psi(x) + \beta\psi'(x) = 0$ for $x = x_{L,R}$
[Same α, β at both endpoints.]

- **Eigenfunction properties** :

The SL equation is an eigenvalue equation:

$$-\frac{d}{dx}\left(\tau(x)\psi_n'(x)\right) + v(x)\psi_n(x) = \omega_n^2 \mu(x)\psi_n(x) \qquad (A)$$

for a given choice of BCs. Suppose we have a second sol$^{\underline{n}}$,

$$-\frac{d}{dx}\left(\tau(x)\psi_m'(x)\right) + v(x)\psi_m(x) = \omega_m^2 \mu(x)\psi_m(x) \qquad (B)$$

Multiply (B) by $\psi_n^*(x)$ and (A^*) by $\psi_m(x)$ and subtract :

$$\psi_n^* \frac{d}{dx}\left[\tau \psi_m'\right] - \psi_m \frac{d}{dx}\left[\tau \psi_n^{*\prime}\right] = (\omega_n^{*2} - \omega_m^2)\,\mu\, \psi_m \psi_n^*$$

$$= \frac{d}{dx}\left[\tau \psi_n^* \psi_m' - \tau \psi_m \psi_n^{\prime *}\right]$$

Now integrate from x_L to x_R :

$$(\omega_n^{*2} - \omega_m^2) \int_{x_L}^{x_R} dx\, \mu(x)\, \psi_n^*(x) \psi_m(x) = \tau(x) \left[\psi_n^*(x)\psi_m'(x) - \psi_m(x)\psi_n^{*\prime}(x)\right]_{x_L}^{x_R}$$

$$= 0$$

because the term in square brackets vanishes for any of the four boundary conditions. Thus,

$$(\omega_n^{*2} - \omega_m^2) \langle \psi_n | \psi_m \rangle = 0$$

where the inner product is

$$\langle \psi | \phi \rangle = \int_{x_L}^{x_R} dx\, \mu(x)\, \psi^*(x)\, \phi(x)$$

Since $\langle \psi_n | \psi_n \rangle > 0$, we have that $\omega_n^2 \in \mathbb{R}$. (Note this does not preclude $\omega_n^2 < 0$ in which case $\omega_n \in i\mathbb{R}$.) When $\omega_m^2 \neq \omega_n^2$, we have $\langle \psi_n | \psi_m \rangle = 0$. For degenerate eigenvalues, we may invoke the Gram-Schmidt method, which orthogonalizes the eigenfunctions within a degenerate subspace. Since the SLE is linear, we may then demand **orthonormality**:

$$\langle \psi_n | \psi_m \rangle = \delta_{mn}$$

Furthermore when the functions $\mu(x)$, $\tau(x)$, $v(x)$ are all real, and when, in the case of mixed homogeneous BCs, $\alpha/\beta \in \mathbb{R}$, we may choose $\psi_n(x) \in \mathbb{R} \;\forall\, n$.

Another aspect of the eigenspectrum, which is more difficult to prove (so we won't) is **completeness**:

$$\mu(x) \sum_{n=0}^{\infty} \psi_n^*(x)\, \psi_n(x') = \delta(x-x')$$

Note that we have labeled the eigenvalues and eigenfunctions with a discrete integer index $n \in \{0,1,\ldots,\infty\}$, and we may demand $\omega_0^2 \le \omega_1^2 \le \omega_2^2 \le \ldots$. Any square integrable, or $\mathcal{L}^2$, function $f(x)$, for which $\langle f|f\rangle < \infty$, can be expanded in the eigenfunctions, viz.

$$f(x) = \sum_{n=0}^{\infty} f_n\, \psi_n(x) \quad , \quad f_n = \langle \psi_n | f \rangle = \int_{x_L}^{x_R} dx\, \mu(x)\, \psi_n^*(x)\, f(x)$$

NB: What is true is that $\| f - \sum_{n=0}^{\infty} f_n \psi_n \| = 0$, where $\|h\| = \langle h|h\rangle$ is the norm of h. Note that this does not guarantee that $\sum_{n=0}^{\infty} f_n\, \psi_n(x)$ converges to $f(x)$ pointwise for all $x \in [x_L, x_R]$. Rather, the convergence holds "almost everywhere", which is to say for all $x \in [x_L, x_R]$ except on a set of measure zero.

- Variational method

Define the functional $\omega^2[\psi(x)] \equiv \dfrac{N[\psi(x)]}{D[\psi(x)]}$ with

$$N[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx \left\{ \tau(x)\,\psi'(x)^2 + v(x)\,\psi(x)^2 \right\}$$

$$D[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\, \psi(x)^2$$

Then the variation of $\omega^2[\psi]$ is

$$\delta\omega^2 = \frac{\delta N}{D} - \frac{N\delta D}{D^2}$$

Thus, if we demand $\delta\omega^2 \equiv 0$, we have

$$\delta N = \frac{N}{D}\,\delta D = \omega^2\,\delta D$$

and since

$$\frac{\delta N}{\delta\psi(x)} = -\frac{d}{dx}\left[\tau(x)\psi'(x)\right] + v(x)\psi(x)$$

$$\frac{\delta D}{\delta\psi(x)} = \mu(x)\,\psi(x)$$

we see that $\delta\omega^2 \equiv 0$ yields the SLE,

$$\frac{\delta N}{\delta\psi(x)} = -\frac{d}{dx}\left[\tau(x)\psi'(x)\right] + v(x)\psi(x) = \omega^2\mu(x)\,\psi(x) = \omega^2\frac{\delta D}{\delta\psi(x)}$$

Note also that the variation of δN contains

Scratch

$$N[\psi(x)] \equiv \frac{1}{2}\int_{x_L}^{x_R} dx \left\{\tau(x)\,\psi'(x)^2 + v(x)\,\psi(x)^2\right\} \equiv \int_{x_L}^{x_R} dx\, L_N(\psi,\psi',x)$$

$$D[\psi(x)] = \frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\,\psi(x)^2 \equiv \int_{x_L}^{x_R} dx\, L_D(\psi,\psi',x)$$

$$L_N(\psi,\psi',x) = \frac{1}{2}\tau(x)\psi'^2 + \frac{1}{2}v(x)\psi^2$$

$$L_D(\psi,\psi',x) = \frac{1}{2}\mu(x)\,\psi^2$$

$$\frac{\delta N}{\delta\psi(x)} = \frac{\partial L_N}{\partial\psi} - \frac{d}{dx}\frac{\partial L_N}{\partial\psi'} = v(x)\psi - \frac{d}{dx}\left[\tau(x)\,\psi'\right]$$

$$\frac{\delta D}{\delta\psi(x)} = \frac{\partial L_D}{\partial\psi} - \cancel{\frac{d}{dx}\frac{\partial L_D}{\partial\psi'}} = \mu(x)\,\psi$$

Fourier analysis : $\psi_n(x) \rightarrow \psi_k(x) = e^{ikx}$

$$f(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\hat{f}(k)\,e^{ikx}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\,e^{-ikx} = \langle\psi_k|f\rangle$$

$$\langle k|k'\rangle = \int_{-\infty}^{\infty} dx\, e^{i(k'-k)x} = 2\pi\,\delta(k-k')$$ replaces $\delta_{kk'}$

completeness : $\delta(x-x') = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ik(x-x')}$

a boundary term $\tau(x)\psi'(x)\delta\psi(x)\Big]_{x_L}^{x_R}$, which vanishes for any of our first three classes of boundary conditions, i.e. fixed endpoints ($\delta\psi(x_{L,R})=0$), natural ($\tau(x_{L,R})\psi'(x_{L,R})=0$), or periodic ($f(x)=f(x+L)$ for $f(x)=\psi(x)$ and $f(x)=\tau(x)$). In order to accommodate the fourth class of BC, i.e. mixed homogeneous, with $\alpha\psi(x)+\beta\psi'(x)=0$ for $x=x_{L,R}$, if we redefine $\omega^2=\tilde{N}/D$, where

$$\tilde{N}[\psi(x)] = N[\psi(x)] + \frac{\alpha}{2\beta}\left\{\tau(x_R)\psi(x_R)^2 - \tau(x_L)\psi(x_L)^2\right\}$$

In fact, for all for classes of BC we can take

$$\omega^2[\psi(x)] = \frac{N[\psi(x)]}{D[\psi(x)]} = \frac{\frac{1}{2}\int_{x_L}^{x_R} dx\, \psi(x)\Big[\overbrace{-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)}^{\hat{K}}\Big]\psi(x)}{\frac{1}{2}\int_{x_L}^{x_R} dx\, \mu(x)\psi^2(x)}$$

Thus, expanding $\psi(x)=\sum_{n=0}^{\infty} C_n\psi_n(x)$, we have

$$\omega^2[\psi(x)] = \omega^2(C_0,\dots,C_\infty) = \frac{\frac{1}{2}\sum_{n=0}^{\infty}\omega_n^2 C_n^2}{\frac{1}{2}\sum_{m=0}^{\infty} C_m^2}$$

Then $$\frac{\partial\omega^2}{\partial C_j} = \frac{(\omega_j^2-\omega^2)C_j}{\frac{1}{2}\sum_m C_m^2} = 0 \quad \text{for all } j\in\{0,1,\dots,\infty\}$$

any $\psi(x) \rightarrow \omega^2[\psi] \geq \omega_0^2$

Solutions :

$$C_j^{(k)} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases} \qquad \text{with } \omega^2=\omega_k^2$$

(k^{th} sol^n)

Example : string with mass point in center

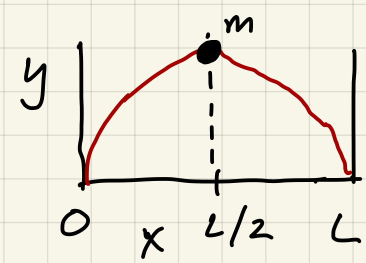
$$\mu(x) = \mu + m\,\delta(x - \tfrac{1}{2}L) \;;\quad \tau(x) = \tau \;;\quad v(x) = 0$$

Here $x_L = 0$ and $x_R = L$. Then

$$\omega^2[\psi] = \frac{\frac{1}{2}\tau \int_0^L dx\, \psi'^2(x)}{\frac{1}{2}\mu \int_0^L dx\, \psi^2(x) + \frac{1}{2} m\, \psi^2(\tfrac{1}{2}L)}$$

Now consider a **trial function**

$$\psi(x) = \begin{cases} A x^\alpha & \text{for } x \in [0, \frac{L}{2}] \\ A(L-x)^\alpha & \text{for } x \in [\frac{L}{2}, L] \end{cases}$$



Here we have a single variational parameter, α.

- $\displaystyle\int_0^L dx\, \psi'^2(x) = 2A^2 \int_0^{L/2} dx\, \alpha^2 x^{2\alpha-2} = A^2 \cdot \frac{2\alpha^2}{2\alpha-1}\left(\frac{L}{2}\right)^{2\alpha-1}$
- $\displaystyle\int_0^L dx\, \psi^2(x) = 2A^2 \int_0^{L/2} dx\, x^{2\alpha} = A^2 \cdot \frac{2}{2\alpha+1}\left(\frac{L}{2}\right)^{2\alpha+1}$
- $\displaystyle\psi^2(\tfrac{1}{2}L) = A^2\left(\frac{L}{2}\right)^{2\alpha}$

$c = (\tau/\mu)^{1/2}$

$$\omega^2[\psi] = \frac{\tau\left(\frac{\alpha^2}{2\alpha-1}\right)\left(\frac{L}{2}\right)^{2\alpha-1}}{\mu\left(\frac{1}{2\alpha+1}\right)\left(\frac{L}{2}\right)^{2\alpha+1} + \frac{1}{2}m\left(\frac{L}{2}\right)^{2\alpha}} = \left(\frac{c}{L}\right)^2 \frac{4\alpha^2(2\alpha+1)}{(2\alpha-1)\left[1+(2\alpha+1)\frac{m}{M}\right]}$$

$M = \mu L$

Best variational estimate ⇒ set $\frac{d\omega^2(\alpha)}{d\alpha} = 0$:

$$\frac{d\omega^2}{d\alpha} = 0 \Rightarrow 4\alpha^2 - 2\alpha - 1 + (\alpha - 1)(2\alpha+1)^2 \frac{m}{M} = 0$$

This is a cubic equation. For $m/M \to 0$, we have $4\alpha^2 - 2\alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{4}(1+\sqrt{5}) \simeq 0.809$. Find then $\omega^2 \approx 11.09 \frac{c^2}{L^2} \Rightarrow \omega \simeq 3.330 \frac{c}{L}$. The exact result we know is $\psi_0(x) = (2/L)^{1/2} \sin(\pi x/L)$ with $\omega_0 = \pi c/L$, and our variational frequency is about 6.00% higher.
For $m/M \to \infty$, the string's inertia is negligible. Then $\psi(x)$ describes an isoceles triangle, and

$$m\ddot{y} = -2\tau \cdot \left(\frac{y}{\frac{1}{2}L}\right) \Rightarrow \omega_0 = 2\sqrt{\frac{\tau}{mL}} = \frac{2}{L}\sqrt{\frac{\tau}{\mu} \cdot \frac{\mu L}{m}} = \frac{2c}{L}\sqrt{\frac{M}{m}}$$

The variational sol$^{\underline{n}}$ yields $\alpha = 1$ and $\omega^2 = \omega_0^2$ exactly.
Note $\alpha = 1$ corresponds to a triangular shape

Our example involved just one variational parameter.
We could have more, e.g.

$$\psi(x) = A x^{\alpha} + B x^{\beta} \qquad (0 \le x \le \tfrac{L}{2})$$
$$\psi(L-x) = \psi(x)$$

Variation parameters: 3 $(\alpha, \beta, B/A)$

Or : $A \equiv C\cos\gamma$, $B \equiv C \sin\gamma \Rightarrow (\alpha, \beta, \gamma)$

Another basis : $\psi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$

- $\int_0^L dx\, \psi_m(x)\, \psi_n(x) = \delta_{mn}$
- $\int_0^L dx\, \psi_m'(x)\, \psi_n'(x) = -\int_0^L dx\, \psi_m(x)\, \psi_n''(x) = \left(\frac{n\pi}{L}\right)^2 \delta_{mn}$

$\psi_n'' = -\left(\frac{n\pi}{L}\right)^2 \psi_n$

So take $\psi(x) = \sum_{n=1}^{\infty} C_n\, \psi_n(x)$

variational parameters $\{C_1, \dots, C_\infty\}$

$$w^2[\psi] = \frac{\frac{1}{2}\tau \int_0^L dx\, \psi'^2(x)}{\frac{1}{2}\mu \int_0^L dx\, \psi^2(x) + \frac{1}{2} m\, \psi^2(\tfrac{1}{2}L)}$$

$$= \frac{\frac{1}{2}\tau \sum_n \left(\frac{n\pi}{L}\right)^2 C_n^2}{\frac{1}{2}\mu \sum_j C_j^2 + \frac{1}{L} m \left(\sum_j C_j \underbrace{\sin\left(\frac{j\pi}{2}\right)}_{(-1)^k \delta_{j,2k-1}}\right)^2}$$

$$\underbrace{\left(\sum_j C_j \sin\left(\frac{j\pi}{2}\right)\right)^2}_{\left[\sum_{k=1}^{\infty} (-1)^k C_k\right]^2}$$

$C_1, \dots, C_\ell$ finite subset

$$w^2(\underbrace{C_1, \dots, C_\infty}) = \frac{\sum_{n=1}^{\infty} n^2 C_n^2}{\sum_{j=1}^{\infty} C_j^2 + \frac{2m}{M}\left[\sum_{k=1}^{\infty} (-1)^k C_k\right]^2} \cdot \left(\frac{\pi c}{L}\right)^2$$