Lecture 12 (Nov. 11)

- Inhomogeneous Sturm – Liouville equation (§9.7):

$$\mu(x)\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\left[T(x)\frac{\partial y}{\partial x}\right] + v(x)\,y = \mu(x)\,\mathrm{Re}\left[f(x)e^{-i\omega t}\right]$$

Here the string is forced at frequency ω.
We write the solution as

$$y(x,t) = \mathrm{Re}\left[y(x)\,e^{-i\omega t}\right]$$

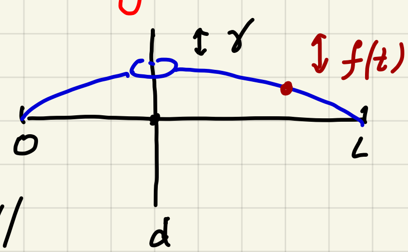
where

$$\left[\hat{K} - \omega^2\mu(x)\right]y(x) = \overbrace{\mu(x)f(x)}^{\text{could redefine as } \tilde{f}(x) \text{ but it is convenient if we include } \mu(x)}$$

$\longrightarrow y(x,t) = y_{hom}(x,t) + y_{inh}(x,t)$

with

$$\hat{K} = -\frac{d}{dx}T(x)\frac{d}{dx} + v(x)$$



the Sturm – Liouville operator. Recall

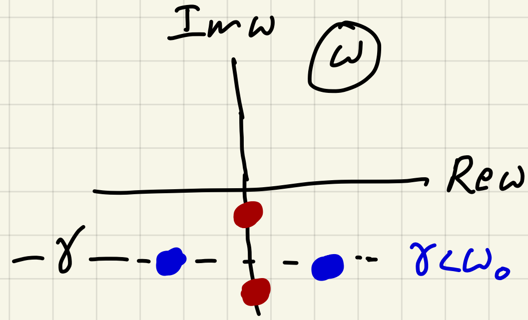
$$\hat{K}\psi_n(x) = \omega_n^2\mu(x)\psi_n(x)$$

$$\langle\psi_m|\psi_n\rangle = \int_{x_L}^{x_R} dx\,\mu(x)\,\psi_m^*(x)\psi_n(x) = \delta_{mn}$$

$$\mu(x)\sum_n \psi_n(x)\psi_n^*(x') = \delta(x-x')$$

Taking the inverse of $\hat{K} - \omega^2\mu(x)$, we have that the inhomogeneous solution is

Scratch



Unforced, damped SHO:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

Sol$^{\underline{n}}$: $x = A e^{-i\omega t} \Rightarrow -\omega^2 - 2i\gamma\omega + \omega_0^2 = 0$

$$\omega^2 + 2i\gamma\omega - \omega_0^2 = 0 \Rightarrow \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$

$e^{-i\omega_\pm t} \to 0$ as $t \to \infty$ due to $\gamma > 0$

$\gamma^2 < \omega_0^2 \Rightarrow$ underdamped , $\gamma^2 > \omega_0^2 \Rightarrow$ overdamped

Harmonic forcing: $f(t) = \int \frac{d\Omega}{2\pi} \hat{f}(\Omega) e^{-i\Omega t}$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \hat{f}(\Omega) e^{-i\Omega t}$$

$\hat{x}(\Omega) e^{-i\Omega t}$

Sol$^{\underline{n}}$: $x(t) = x_{hom}(t) + x_{inh}(t)$

$$A_+ e^{-i\omega_+ t} + A_- e^{-i\omega_- t} \to 0$$

$$(\omega_0^2 - 2i\gamma\Omega - \Omega^2)\,\hat{x}(\Omega) = \hat{f}(\Omega)$$

Single frequency: $x_{inh}(t) = A(\Omega)\cos[\Omega t + \delta(\Omega)]$

amplitude: $A(\Omega) = \left[(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2\right]^{-1/2}$

phase shift: $\delta(\Omega) = \tan^{-1}\left(\frac{2\gamma\Omega}{\Omega^2 - \omega_0^2}\right)$

$$y_{inh}(x) = \int_{x_L}^{x_R} dx'\, \mu(x')\, G_\omega(x,x')\, f(x')$$

where $G_\omega(x,x')$ is the **Green's function**, satisfying

$$\left[\hat{K} - \omega^2 \mu(x)\right] G_\omega(x,x') = \delta(x-x')$$

I.e. $G_\omega(x,x') = \left[\hat{K} - \omega^2 \mu\right]^{-1}_{x,x'}$. We may write

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\, \psi_n^*(x')}{\omega_n^2 - \omega^2} \quad , \quad [G_\omega] = \frac{T^2}{M}$$

You can read about how to obtain $G_\omega(x,x')$ without having to do the infinite sum over all the eigenfunctions in §9.7.1. For now, I just quote the result for the case where $\mu(x) = \mu$, $\tau(x) = \tau$, $v(x) = 0$, and $[x_L, x_R] = [0, L]$. Then

$$G_\omega(x,x') = \frac{\sin(\omega x_< / c)\, \sin(\omega(L - x_>)/c)}{(\omega \tau / c) \sin(\omega L / c)}$$

where $x_< = \min(x,x')$ and $x_> = \max(x,x')$, $c = \sqrt{\frac{\tau}{\mu}}$

Example: Let $f(x) = f_0\, \delta(x - x_0)$. Then

$$y_{inh}(x) = \mu f_0\, G_\omega(x, x_0)$$

Note that there are no constants of integration.

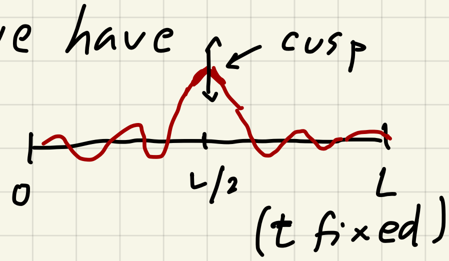
The full sol$^{\underline{n}}$ is then

homogeneous sol$^{\underline{n}}$ (e.g. Bernoulli)

inhomogeneous sol$^{\underline{n}}$

$$y(x,t) = y_{hom}(x,t) + y_{inh}(x,t)$$

The initial conditions enter in $y_{hom}(x,t)$ as we have learned from the Bernoulli solution. If there is some small damping, then at long times we have



$$y(x,t \gg \gamma^{-1}) = y_{inh}(x,t)$$
$$= \mu f_0 \, G_\omega(x,x_0)\cos(\omega t)$$

where γ is the damping rate (i.e. rate of energy loss for unforced system). If $x_0 = \frac{1}{2}L$, then

$$G_\omega(x,\tfrac{1}{2}L) = \frac{c}{2\omega\tau\cos(\omega L/2c)} \times \begin{cases} \sin(\omega x/c) & \text{if } x < L/2 \\ \sin(\omega(L-x)/c) & \text{if } x > L/2 \end{cases}$$

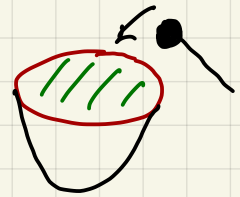
Note that $y_{inh}(x,t)$ is continuous at $x = \frac{1}{2}L$ but its spatial derivative $y'_{inh}(x,t)$ is discontinuous at $x = \frac{1}{2}L$.

- **Continua in higher dimensions** : $h(\vec{x},t)$ displacement

Generalization of wave operator :

e.g. drumhead :

$$\hat{K} = -\frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\vec{x})\,\frac{\partial}{\partial x^\beta} + v(\vec{x})$$



kettle drum

This arises from

$$\mathcal{L} = \frac{1}{2}\mu(\vec{x})\left(\frac{\partial h}{\partial t}\right)^2 - \frac{1}{2}\tau_{\alpha\beta}(\vec{x})\frac{\partial h}{\partial x^\alpha}\frac{\partial h}{\partial x^\beta} - \frac{1}{2}v(\vec{x})h^2$$

The wave equation is

$$\hat{K}\,h(\vec{x},t) = -\mu(\vec{x})\frac{\partial^2}{\partial t^2}h(\vec{x},t)$$

Since $[\hat{K},\partial_t] = 0$, solutions may be written as

$$h(\vec{x},t) = \mathrm{Re}\left[h(\vec{x})\,e^{-i\omega t}\right]$$

where

$$\left[\hat{K} - \omega^2\mu(\vec{x})\right]h(\vec{x}) = 0$$

This is again an eigenvalue equation, with solutions

$$\psi_n(\vec{x}) \implies \hat{K}\psi_n(\vec{x}) = \omega_n^2\,\mu(\vec{x})\,\psi_n(\vec{x})$$

The eigenfunctions and eigenvalues satisfy

$$\langle\psi_m|\psi_n\rangle = \int_\Omega d^dx\,\mu(\vec{x})\,\psi_m^*(\vec{x})\,\psi_n(\vec{x}) = \delta_{mn}$$

$$\mu(\vec{x})\sum_n \psi_n(\vec{x})\,\psi_n^*(\vec{x}\,') = \delta(\vec{x}-\vec{x}\,')$$

where the medium is confined to a region $\Omega \subset \mathbb{R}^d$. We must also apply boundary conditions of the form

(i) $h(\vec{x})\big|_{\partial\Omega} = 0$, where $\partial\Omega$ = boundary of Ω

(ii) $\tau(\vec{x})\,\hat{n}\cdot\vec{\nabla}h\big|_{\partial\Omega} = 0$, where $\hat{n}$ is normal to $\partial\Omega$

(iii) PBCs , e.g. in a box of dim$^{\underline{ns}}$ $L_1 \times L_2 \times \cdots \times L_d$

(iv) $\left[\alpha\,\psi(\vec{x}) + \beta\,\hat{n}\cdot\vec{\nabla}\psi(\vec{x})\right]_{\partial\Omega} = 0$

The Green's function is

$$G_\omega(\vec{x},\vec{x}\,') = \sum_n \frac{\psi_n(\vec{x})\,\psi_n^*(\vec{x}\,')}{\omega_n^2 - \omega^2}$$

with

$$\left[\hat{K} - \omega^2\mu(\vec{x})\right] G_\omega(\vec{x},\vec{x}\,') = \delta(\vec{x}-\vec{x}\,')$$

The variational approach generalizes as well, with

$$\omega^2[\psi(\vec{x})] \equiv \frac{N[\psi(\vec{x})]}{D[\psi(\vec{x})]}$$

and

$$N[\psi(\vec{x})] = \int_\Omega d^d x\; \psi^*(\vec{x}) \Big\{ \overbrace{-\frac{\partial}{\partial x^\alpha}\tau_{\alpha\beta}(\vec{x})\frac{\partial}{\partial x^\beta} + v(\vec{x})}^{\hat{K}} \Big\}\, \psi(\vec{x})$$

$$D[\psi(\vec{x})] = \int_\Omega d^d x\; \mu(\vec{x})\,\psi^2(\vec{x})$$

Demanding $\delta\omega^2 = 0$ yields the wave equation

$$\hat{K}\psi(\vec{x}) = \omega^2\mu(\vec{x})\,\psi(\vec{x})$$

- **Membranes** : $z = h(x,y)$

The equation of a surface is $F(x,y,z) = z - h(x,y) = 0$.
Let the differential surface area be dS. The projection onto the (x,y) plane is then

$$dA = dx\,dy = \hat{n}\cdot\hat{z}\,dS = n^z\,dS$$

The unit normal is

$$\hat{n} = \frac{\vec{\nabla}F}{|\vec{\nabla}F|} = \frac{\hat{z} - \vec{\nabla}h}{\sqrt{1+(\vec{\nabla}h)^2}} \qquad (\text{note } \hat{z}\cdot\vec{\nabla}h = 0)$$

Thus,

$$dS = \frac{dx\,dy}{\hat{n}\cdot\hat{z}} = \sqrt{1+(\vec{\nabla}h)^2}\,dx\,dy$$

before: $ds = \sqrt{1+h'^2}\,dx$

We consider a model where

$$U[h(x,y,t)] = \int dS\,\sigma = U_0 + \frac{1}{2}\int d^2x\,\sigma(\vec{x})\,(\vec{\nabla}h)^2 + \ldots$$

with σ the surface tension. Other energy functions are possible. The kinetic energy is

$$T[h(x,y,t)] = \frac{1}{2}\int d^2x\,\mu(\vec{x})\left(\frac{\partial h}{\partial t}\right)^2$$

Thus

$$S = \int dt \int d^2x\,\mathcal{L}(h, \partial_t h, \vec{\nabla}h, t, \vec{x})$$

$$\mathcal{L} = \frac{1}{2}\mu(\vec{x})(\partial_t h)^2 - \frac{1}{2}\sigma(\vec{x})(\vec{\nabla}h)^2$$

The equations of motion are then

$$\frac{\partial\mathcal{L}}{\partial h} - \underbrace{\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\partial_t h}} - \underbrace{\vec{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\vec{\nabla}h}} = 0$$

$$\shortparallel \qquad\qquad \shortparallel \qquad\qquad\qquad \shortparallel$$

$$0 - \left(\mu(\vec{x})\frac{\partial^2 h}{\partial t^2}\right) - \left(-\vec{\nabla}\cdot\left[\sigma(\vec{x})\vec{\nabla}h\right]\right) = 0$$

Thus

$$\vec{\nabla}\cdot\left[\sigma(\vec{x})\vec{\nabla}h(\vec{x},t)\right] = \frac{\partial^2 h(\vec{x},t)}{\partial t^2}$$

which is a generalization of the Helmholtz equation. When μ and σ are constants, we get Helmholtz:

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h(\vec{x},t) = 0$$

Note $[\mu] = ML^{-2}$ and $[\sigma] = EL^{-2} = MT^{-2}$, thus with $c \equiv (\sigma/\mu)^{1/2}$ we have $[c] = LT^{-1}$ as before.

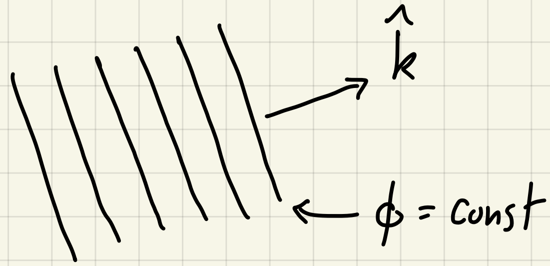
d'Alembert solution:

$$h(\vec{x},t) = f(\hat{k}\cdot\vec{x} - ct)$$

where $\hat{k}$ is a fixed direction in space. These are plane waves (really "line waves"). The locus of points of constant $h(\vec{x},t)$ satisfies

$$\phi(\vec{x},t) = \hat{k}\cdot\vec{x} - ct = \text{constant}$$

and setting $d\phi = 0$ then yields $\hat{k} \cdot \frac{d\vec{x}}{dt} = c$, i.e. the velocity along $\hat{k}$ is c. The component of $\vec{x}$ lying perpendicular to $\hat{k}$ is arbitrary, so constant $\phi(\vec{x},t)$ corresponds to lines orthogonal to $\hat{k}$.



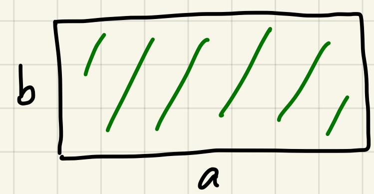
Due to linearity of the wave eqn, we can superpose plane wave solutions to arrive at the general solution,

$$h(\vec{x},t) = \int \frac{d^2k}{(2\pi)^2} \Big[\underbrace{A(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - ckt)}}_{+\hat{k}\text{ mover}} + \underbrace{B(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} + ckt)}}_{-\hat{k}\text{ mover}} \Big]$$

$k = |\vec{k}|$

- Rectangles : $\Omega = [0,a] \times [0,b]$

Separation of variables solves PDE:



$$h(x,y,t) = X(x)\, Y(y)\, T(t)$$

Helmholtz eqn $\frac{1}{h}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) h = 0$ yields

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\text{depends only on } x} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\text{depends only on } y} = \frac{1}{c^2} \cdot \underbrace{\frac{1}{T}\frac{\partial^2 T}{\partial t^2}}_{\text{depends only on } t}$$

So we conclude

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2}$$

Thus, $\omega = c|\vec{k}|$. Most general $\text{sol}^{\underline{n}}$:

$$\begin{aligned} X(x) &= A \sin(k_x x + \alpha) \\ Y(y) &= B \sin(k_y y + \beta) \quad , \quad h(x,y,t) = X(x)Y(y)T(t) \\ T(t) &= C \sin(\omega t + \gamma) \end{aligned}$$

but imposing boundary conditions $h(\vec{x},t)\big|_{\partial\Omega} = 0$ then requires

$$\alpha = \beta = 0 \quad , \quad \sin(k_x a) = \sin(k_y b) = 0 \quad \Rightarrow \quad \begin{cases} k_x = m\pi/a \\ k_y = n\pi/b \end{cases}$$

The most general $\text{sol}^{\underline{n}}$ consistent with the BCs is then

$$h(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega_{mn} t + \gamma_{mn})$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2}$$

and the constants $\{A_{mn}, \gamma_{mn}\}$ are determined by the initial conditions.

• Circles: $\Omega = \{ (x,y) \mid x^2+y^2 \leq a^2 \}$

It is convenient to work in 2d polar coordinates (r,φ).
The Helmholtz equation takes the form

$$\vec{\nabla}^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \varphi^2} = \frac{1}{c^2}\frac{\partial^2 h}{\partial t^2}$$

Separation of variables:

$$h(r,\varphi,t) = R(r)\,\Phi(\varphi)\,T(t)$$

Again we have

$$\frac{1}{R}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\cdot\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}$$

with

$$\Phi(\varphi) = \cos(m\varphi + \beta)$$
$$T(t) = \cos(\omega t + \gamma)$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{m^2}{r^2} - \frac{\omega^2}{c^2}\right)R = 0$$

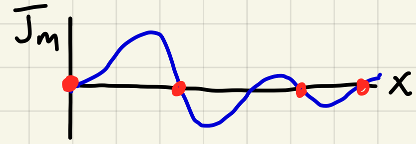
Since $h(r,\varphi+2\pi,t) = h(r,\varphi,t)$, we must have $m \in \mathbb{Z}$.
This is Bessel's equation, with solutions

$$R(r) = A\,J_m\left(\frac{\omega r}{c}\right) + B\,N_m\left(\frac{\omega r}{c}\right)$$

with $J_m(z)$ and $N_m(z)$ the Bessel and Neumann functions of order m, respectively. Since $N_m(z)$ diverges as $z \to 0$ for all m, we must have $B = 0$. (For an annulus, we may have $B \neq 0$.) The boundary condition at $r = a$ yields

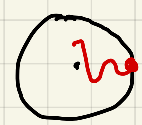
$$J_m\left(\frac{\omega a}{c}\right) = 0 \Rightarrow \omega = \omega_{m\ell} = x_{m\ell} \cdot \frac{c}{a}$$

where $J_m(x_{m\ell}) = 0$, i.e. $x_{m\ell}$ is the ℓ^{th} zero $(\ell = 1, 2, \ldots, \infty)$ of $J_m(x)$. Thus,



$$h(r, \varphi, t) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{\infty} A_{m\ell} J_m(x_{m\ell} r/a) \cos(m\varphi + \beta_{m\ell}) \cos(\omega_{m\ell} t + \gamma_{m\ell})$$

The constants $A_{m\ell}$, $\beta_{m\ell}$, and $\gamma_{m\ell}$ are set by the initial conditions. Note $h(r=a, \varphi, t) = 0$ for all φ and for all t.



- Read §9.3.6 (sound in fluids) and §9.4 (dispersion)

- Classical Field Theory

  Independent variables : $\{x^1, \ldots, x^n\} \in \Omega \subset \mathbb{R}^n$ or $\{x^0, x^1, \ldots, x^d\}$, $n = d+1$

  Real fields : $\{\phi_1, \ldots, \phi_K\}$

  Lagrangian density : $\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu \phi_a, x^\mu)$

  Action : $S = \int d^n x \, \mathcal{L}$

  Let's compute the variation of S :

$$\delta S = \int_\Omega d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \delta\phi_a}{\partial x^\mu} \right\}$$

$$= \int_\Omega d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a$$

differential surface area

$$+ \oint_{\partial\Omega} d\Sigma \, n^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a$$

The surface term vanishes if we demand

$$\delta\phi_a(\vec{x})\Big|_{\partial\Omega} = 0 \qquad \text{or} \qquad n^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\Big|_{\partial\Omega} = 0$$

Then we have

$$\frac{\delta S}{\delta \phi_a(\vec{x})} = \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right]_{\vec{x}}$$

evaluate at $\vec{x}$

Thus $\delta S = 0$ entails the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

When $\mathcal{L}$ is independent of the independent variables x^μ, the stress-energy tensor is conserved:

$$\partial_\mu T^\mu{}_\nu = 0 \qquad \text{with} \qquad T^\mu{}_\nu = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}$$

This is analogous to $\frac{dH}{dt} = 0$ in particle mechanics.

Maxwell theory

The Lagrangian density, with sources, is

$$\mathcal{L}(A^\nu, \partial_\mu A^\nu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ with $x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$
and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; ; \quad A_\nu = g_{\nu\lambda} A^\lambda \quad , \; g = \mathrm{diag}(+,-,-,-)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = g^{\mu\alpha} g^{\nu\beta} A_{\alpha\beta} \; ; \quad g_{\mu\nu} = g^{\mu\nu}$$

The EL equations are

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

Conserved currents in field theory

In particle mechanics, a one-parameter family of transformations $\tilde{q}_\sigma(q, \zeta)$ which leaves $L(q, \dot{q}, t)$ invariant results in a conserved "charge"

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \; ; \quad \tilde{q}_\sigma(q, \zeta=0) = q_\sigma$$

with $d\Lambda/dt = 0$. We generalize to field theory by taking $q_\sigma(t) \to \phi_a(x^\mu)$. Then

$$\frac{d}{d\zeta}\bigg|_{\zeta=0} \mathcal{L}(\tilde{\phi}_a, \partial_\mu \tilde{\phi}_a, x^\mu) = \frac{\partial \mathcal{L}}{\partial \phi_a} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial}{\partial x^\mu} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0}$$

$$= \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)\bigg|_{\zeta=0} ,$$

where we have invoked the EL eqns,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)$$

Thus we have

$$\partial_\mu J^\mu = 0 \qquad \text{with} \qquad J^\mu = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0}$$

Let us write $x^\mu = \{x^0, x^1, \ldots, x^d\}$ with $n = d+1$. Then with $x^0 \equiv ct$ and $Q_\Omega \equiv c^{-1}\int d^d x\, J^0$, we have

$$\frac{dQ_\Omega}{dt} = \int_\Omega d^d x\, \partial_0 J^0 = -\int_\Omega d^3 x\, \vec{\nabla}\cdot\vec{J} = -\oint_{\partial\Omega} d\Sigma\, \hat{n}\cdot\vec{J} = 0$$

provided $\hat{n}\cdot\vec{J}\big|_{\partial\Omega} = 0$. Thus, the rate of change of Q_Ω is minus the integrated flux exiting the region Ω.

Example:

$$\mathcal{L}(\psi, \psi^*, \partial_\mu \psi, \partial_\mu \psi^*) = \tfrac{1}{2}K(\partial_\mu \psi^*)(\partial^\mu \psi) - U(\psi^*\psi)$$

The Lagrangian density is invariant under

$$\psi \to \tilde{\psi} = e^{i\zeta}\psi \quad , \quad \psi^* \to \tilde{\psi}^* e^{-i\zeta}\psi$$

We regard ψ and ψ^* as independent fields. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = ie^{i\zeta}\psi \quad , \quad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -ie^{-i\zeta}\tilde{\psi}^*$$

and thus

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}(-i\psi^*)$$

$$= \frac{K}{2i}\left(\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*\right) = K\,\mathrm{Im}\left(\psi^*\partial^\mu\psi\right)$$

Note that $U(\tilde{\psi}^*\tilde{\psi}) = U(\psi^*\psi)$ is independent of ζ.

- Gross – Pitaevskii model

This is a model of nonrelativistic interacting bosons, with

$$\mathcal{L} = i\hbar\,\psi^*\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi - g\left(\psi^*\psi - n_0\right)^2$$

Details in §9.5.3 of the notes. The EL equations are

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + 2g\left(|\psi|^2 - n_0\right)\psi$$

and its complex conjugate. This is called the nonlinear Schrödinger equation (NLSE). The one-parameter invariance of $\mathcal{L}$ is again

$$\psi(\vec{x},t) \to \tilde{\psi}(\vec{x},t) \equiv e^{-i\zeta}\,\psi(\vec{x},t)$$

$$\psi^*(\vec{x},t) \to \tilde{\psi}^*(\vec{x},t) \equiv e^{+i\zeta}\,\psi^*(\vec{x},t)$$

The conserved current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0}$$

with components

$$J^0 = \hbar |\psi|^2 \equiv \hbar \rho$$

$$\vec{J} = \frac{\hbar^2}{2im}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right) \equiv \hbar \vec{j}$$

Thus,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \text{(continuity eqn.)}$$

In this example, $x^\mu = x_\mu$ and there is no difference between raised and lowered indices.