- Proof Hamiltonian evolution generates a CT

We consider an infinitesimal evolution:

$$\underbrace{\xi_i(t)}_{\xi_i} \rightarrow \underbrace{\xi_i(t+dt)}_{\xi_i'} = \xi_i(t) + J_{ik} \frac{\partial H}{\partial \xi_k}\Big|_{\vec{\xi}(t)} dt + O(dt^2)$$

We have that $M_{ij} = \dfrac{\partial \xi_i'}{\partial \xi_j} = \delta_{ij} + J_{ir} \dfrac{\partial^2 H}{\partial \xi_j \partial \xi_r} dt + O(dt^2)$

Thus $M^t_{kl} = \delta_{kl} + J_{ls} \dfrac{\partial^2 H}{\partial \xi_k \partial \xi_s} dt$ and

$$\begin{aligned} M_{ij} J_{jk} M^t_{kl} &= \left(\delta_{ij} + J_{ir} \frac{\partial^2 H}{\partial \xi_j \partial \xi_r} dt\right) J_{jk} \left(\delta_{kl} + J_{ls} \frac{\partial^2 H}{\partial \xi_k \partial \xi_s} dt\right) \\ &= J_{il} + \Big(J_{ir} J_{jl} \frac{\partial^2 H}{\partial \xi_j \partial \xi_r} + \underbrace{J_{ik} J_{ls} \frac{\partial^2 H}{\partial \xi_k \partial \xi_s} dt}_{\text{take } k \to r,\ s \to j}\Big) + O(dt^2) \\ &= J_{il} + O(dt^2) \end{aligned}$$

Lecture 15 (November 23)

- **Generating functions for canonical transformations**

For a transformation to be canonical, we require

$$\delta \int_{t_a}^{t_b} dt \left[p_\sigma \dot{q}_\sigma - H(\vec{q}, \vec{p}, t)\right] = 0 = \delta \int_{t_a}^{t_b} dt \left[P_\sigma \dot{Q}_\sigma - \tilde{H}(\vec{Q}, \vec{P}, t)\right]$$

This is satisfied for all motions provided

$$p_\sigma \dot{q}_\sigma - H(\vec{q}, \vec{p}, t) = \lambda \left[P_\sigma \dot{Q}_\sigma - \tilde{H}(\vec{Q}, \vec{P}, t) + \frac{d}{dt} F(\vec{q}, \vec{Q}, t)\right]$$

where λ is a constant. We can always rescale coordinates and momenta to achieve $\lambda = 1$, which we henceforth assume.

Therefore,

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + P_\sigma \dot{Q}_\sigma - p_\sigma \dot{q}_\sigma + \overbrace{\frac{\partial F}{\partial Q_\sigma}\dot{Q}_\sigma + \frac{\partial F}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial F}{\partial t}}^{dF/dt}$$

To eliminate the terms proportional to $\dot{Q}_\sigma$ and $\dot{q}_\sigma$, demand

$$\frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial q_\sigma} = +p_\sigma$$

We then have

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F(\vec{q},\vec{Q},t)}{\partial t}$$

This is called a "type I canonical transformation".

By making Legendre transformations, we can extend this to a family of four types of CTs:

$$F(\vec{q},\vec{Q},t) = \begin{cases} F_1(\vec{q},\vec{Q},t) & \text{with} \quad p_\sigma = \dfrac{\partial F_1}{\partial q_\sigma} \, , \quad P_\sigma = -\dfrac{\partial F_1}{\partial Q_\sigma} \\ F_2(\vec{q},\vec{P},t) - P_\sigma Q_\sigma & \text{with} \quad p_\sigma = \dfrac{\partial F_2}{\partial q_\sigma} \, , \quad Q_\sigma = \dfrac{\partial F_2}{\partial P_\sigma} \\ F_3(\vec{p},\vec{Q},t) + p_\sigma q_\sigma & \text{with} \quad q_\sigma = -\dfrac{\partial F_3}{\partial p_\sigma} \, , \quad P_\sigma = -\dfrac{\partial F_3}{\partial q_\sigma} \\ F_4(\vec{p},\vec{P},t) + p_\sigma q_\sigma - P_\sigma Q_\sigma & \text{with} \quad q_\sigma = -\dfrac{\partial F_4}{\partial p_\sigma} \, , \quad Q_\sigma = \dfrac{\partial F_4}{\partial P_\sigma} \end{cases}$$

In each case, we have

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F_\gamma}{\partial t} \quad , \quad \gamma \in \{1,2,3,4\}$$

Examples of CTs from generating functions

- Consider the type-II transformation generated by

$$F_2(\vec{q}, \vec{P}) = A_\sigma(\vec{q}) P_\sigma$$

where $A_\sigma(\vec{q})$ is an arbitrary function of $\{q_1, \cdots, q_n\}$. Then

$$Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} = A_\sigma(\vec{q}) \quad , \quad p_\sigma = \frac{\partial F_2}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} P_\alpha = \frac{\partial Q_\alpha}{\partial q_\sigma} P_\alpha$$

which is equivalent to: $Q_\sigma = A_\sigma(\vec{q})$, $P_\sigma = \dfrac{\partial q_\alpha}{\partial Q_\sigma} p_\alpha$

This is in fact the general point transformation discussed previously. For linear point transformations,

$$Q_\alpha = M_{\alpha\sigma} q_\sigma \quad , \quad P_\beta = p_{\sigma'} M^{-1}_{\sigma'\beta}$$

$$\{Q_\alpha, P_\beta\} = M_{\alpha\sigma} M^{-1}_{\sigma'\beta} \underbrace{\{q_\sigma, p_{\sigma'}\}}_{\delta_{\sigma\sigma'}} = \delta_{\alpha\beta}$$

Note that $F_2(\vec{q}, \vec{P}) = q_1 P_3 + q_3 P_1$ exchanges the labels 1 and 3: $Q_1 = \partial F_2/\partial P_1 = q_3$, $p_1 = \partial F_2/\partial q_1 = P_3$

$Q_3 = \partial F_2/\partial P_3 = q_1$, $p_3 = \partial F_2/\partial q_3 = P_1$

- Next, consider the type-I transformation generated by $F_1(\vec{q}, \vec{Q}) = A_\sigma(\vec{q}) Q_\sigma$. We then have

$$p_\sigma = \frac{\partial F_1}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma} Q_\alpha \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} = -A_\sigma(\vec{q})$$

Thus, $F_1(\vec{q},\vec{Q}) = q_\sigma Q_\sigma$, for which $A_\sigma(\vec{q}) = q_\sigma$, generates

$$p_\sigma = Q_\sigma \quad , \quad P_\sigma = -q_\sigma$$

$$\vec{\xi} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \rightarrow \begin{pmatrix} -\vec{P} \\ +\vec{Q} \end{pmatrix} = \vec{\Xi}$$

- A mixed generator:

$$F(\vec{q},\vec{Q}) = q_1 Q_1 + (q_3 - Q_2)P_2 + (q_2 - Q_3)P_3$$

which is type-I wrt index $\sigma = 1$ and type II wrt $\sigma = 2,3$. This generates

$$Q_1 = p_1 \ , \ Q_2 = q_3 \ , \ Q_3 = q_2 \ , \ P_1 = -q_1 \ , \ P_2 = p_3 \ , \ P_3 = p_2$$

(swaps p,q for label 1, swaps labels 2,3)

- $d=1$ simple harmonic oscillator: $H(q,p) = \dfrac{p^2}{2m} + \dfrac{1}{2}kq^2$

If we could find a CT for which

$$p = \sqrt{2mf(P)}\cos Q \quad , \quad q = \sqrt{\frac{2f(P)}{k}}\sin Q$$

then we'd have $\tilde{H}(Q,P) = f(P)$, which is cyclic in Q. The equations of motion are then $\dot{P} = -\partial\tilde{H}/\partial Q = 0$ and $\dot{Q} = \partial\tilde{H}/\partial P = f'(P)$. Taking the ratio gives

$$p = \sqrt{mk}\, q \,\mathrm{ctn}\, Q = \frac{\partial F}{\partial q}$$

This suggests a type-I transformation

$$F_1(q,Q) = \frac{1}{2}\sqrt{mk}\, q^2 \operatorname{ctn} Q$$

for which

$$p = \frac{\partial F_1}{\partial q} = \sqrt{mk}\, q \operatorname{ctn} Q$$

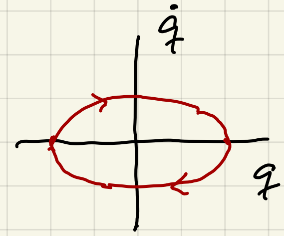
$$P = -\frac{\partial F_1}{\partial Q} = \frac{\sqrt{mk}\, q^2}{2\sin^2 Q}$$

Thus,

$$q = \frac{(2P)^{1/2}}{(mk)^{1/4}} \sin Q \Rightarrow f(P) = \sqrt{\frac{k}{m}}\, P \equiv \omega P$$

where $\omega = (k/m)^{1/2}$ is the oscillation frequency. We also have $\tilde{H}(Q,P) = \omega P = E$, the conserved energy, i.e. $P = \frac{E}{\omega}$. The equations of motion are $\dot{P} = 0$ and $\dot{Q} = f'(P) = \omega$, so the motion is $Q(t) = \omega t + \phi_0$, $P(t) = P = E/\omega \Rightarrow$

$$q(t) = \sqrt{\frac{2f(P)}{k}} \sin Q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \phi_0)$$



- **Hamilton-Jacobi theory**

General form of CT:

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial F(\vec{q},\vec{Q},t)}{\partial t}$$

with $$\frac{\partial F}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial p_\sigma} = \frac{\partial F}{\partial P_\sigma} = 0$$

Let's be audacious and demand $\tilde{H}(\vec{Q},\vec{P},t)=0$!

This entails

$$\frac{\partial S}{\partial q_\sigma}=p_\sigma \quad , \quad \frac{\partial S}{\partial t}=-H$$

$$\frac{\partial F}{\partial t}=-H \quad , \quad \frac{\partial F}{\partial q_\sigma}=p_\sigma$$

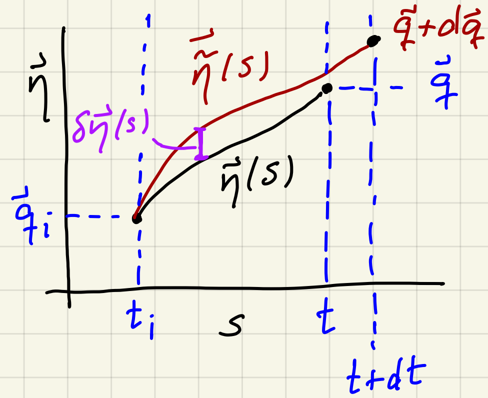
The remaining functional dependence of F may either be on $\vec{Q}$ (type I) or on $\vec{P}$ (type II). It turns out that the function we seek is none other than the action, S, expressed as a function of its endpoint values.

- **Action as a function of coordinates and time**

Consider a path $\vec{\eta}(s)$ interpolating between $(\vec{q}_i, t_i)$ and $(\vec{q}, t)$ which satisfies

$$\frac{\partial L}{\partial \eta_\sigma}-\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right)=0$$

Now consider a new path $\tilde{\vec{\eta}}(s)$ starting at $(\vec{q}_i, t_i)$ but ending at $(\vec{q}+d\vec{q}, t+dt)$, which also satisfies the equations of motion. We wish to compute the differential



$$dS = S[\tilde{\vec{\eta}}(s)] - S[\vec{\eta}(s)]$$

$$=\int_{t_i}^{t+dt} ds\, L(\tilde{\vec{\eta}},\dot{\tilde{\vec{\eta}}},s) - \int_{t_i}^{t} ds\, L(\vec{\eta},\dot{\vec{\eta}},s)$$

$$= L(\tilde{\vec{\eta}}(t),\dot{\tilde{\vec{\eta}}}(t),t)\,dt + \int_{t_i}^{t} ds \left\{ \frac{\partial L}{\partial \eta_\sigma}\left[\tilde{\eta}_\sigma-\eta_\sigma\right] + \frac{\partial L}{\partial \dot{\eta}_\sigma}\left[\dot{\tilde{\eta}}_\sigma-\dot{\eta}_\sigma\right]\right\}$$

$$= L(\vec{\tilde{\eta}}(t), \dot{\vec{\tilde{\eta}}}(t), t)\, dt + \left.\frac{\partial L}{\partial \dot{\eta}_\sigma}\right|_t [\tilde{\eta}_\sigma(t) - \eta_\sigma(t)]$$

$$+ \int_{t_i}^{t} ds \underbrace{\left\{ \frac{\partial L}{\partial \eta_\sigma} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right) \right\}}_{=0} [\tilde{\eta}_\sigma(s) - \eta_\sigma(s)]$$

$$= L(\vec{\eta}(t), \dot{\vec{\eta}}(t), t)\, dt + \pi_\sigma(t)\, \delta\eta_\sigma(t) + \mathcal{O}(\delta\vec{q}\, dt)$$

where $\pi_\sigma \equiv \partial L / \partial \dot{\eta}_\sigma$ and $\delta\eta_\sigma(s) \equiv \tilde{\eta}_\sigma(s) - \eta_\sigma(s)$.

Note that

$$\begin{aligned}
dq_\sigma &= \tilde{\eta}_\sigma(t+dt) - \eta_\sigma(t) \\
&= \tilde{\eta}_\sigma(t+dt) - \tilde{\eta}_\sigma(t) + \overbrace{\tilde{\eta}_\sigma(t) - \eta_\sigma(t)}^{\delta\eta_\sigma(t)} \\
&= \dot{\tilde{\eta}}_\sigma(t)\, dt + \delta\eta_\sigma(t) \\
&= \dot{\eta}_\sigma(t)\, dt + \delta\eta_\sigma(t) + \overbrace{[\dot{\tilde{\eta}}_\sigma(t) - \dot{\eta}_\sigma(t)]}^{\delta\dot{\eta}_\sigma(t)}\, dt
\end{aligned}$$

and therefore

$$\delta\eta_\sigma(t) = dq_\sigma - \dot{\eta}_\sigma(t)\, dt - \cancel{\delta\dot{\eta}_\sigma(t)}\, dt$$

Thus, we have

$$dS = \pi_\sigma(t)\, dq_\sigma + \left[L(\vec{\eta}(t), \dot{\vec{\eta}}(t), t) - \pi_\sigma(t)\, \dot{\eta}_\sigma(t) \right] dt$$

$$= p_\sigma\, dq_\sigma - H\, dt$$

We then conclude

$$\frac{\partial S}{\partial q_\sigma} = p_\sigma \ , \quad \frac{\partial S}{\partial t} = -H \ , \quad \frac{dS}{dt} = L$$

What about the lower limit at t_i? Clearly there are $(n+1)$ constants associated with this limit, viz.

$$\{q_1(t_i), \ldots, q_n(t_i); t_i\}$$

We'll call these constants $\{\Lambda_1, \ldots, \Lambda_{n+1}\}$ and write

$$S = S(q_1, \ldots, q_n; \Lambda_1, \ldots, \Lambda_n; t) + \Lambda_{n+1}$$

We may regard each Λ_σ as either Q_σ or P_σ, i.e. that S is in general a mixed type I - type II generator. That is to say, for $\sigma \in \{1, \ldots, n\}$,

$$\Gamma_\sigma \equiv \frac{\partial S}{\partial \Lambda_\sigma} = \begin{cases} -P_\sigma & \text{if } \Lambda_\sigma = Q_\sigma \\ +Q_\sigma & \text{if } \Lambda_\sigma = P_\sigma \end{cases}$$

The last constant Λ_{n+1} will be associated with time translation.

- **Hamilton-Jacobi equation**

  Since $S(\vec{q}, \vec{\Lambda}, t)$ generates a CT for which $\tilde{H}(\vec{Q}, \vec{P}, t) = 0$, we must have $\partial F/\partial t = -H \Rightarrow$

$$H\Big(q_1, \ldots, q_n, \overbrace{\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}}^{p_\sigma}, t\Big) + \frac{\partial S}{\partial t} = 0$$

which is known as the Hamilton-Jacobi equation (HJE). The HJE is a PDE in $(n+1)$ variables $\{q_1, \ldots, q_n, t\}$.

Since $\tilde{H}(\vec{Q},\vec{P},t) = 0$, the equations of motion are utterly trivial:

$$Q_\sigma(t) = \text{const.} \quad , \quad P_\sigma(t) = \text{const.} \qquad \forall \sigma \quad !$$

How can this yield any nontrivial dynamics? Well what we really want is the motion $\{q_\sigma(t)\}$, and to obtain this we must **invert** the relation

$$\Gamma_\sigma = \frac{\partial S(\vec{q},\vec{\Lambda},t)}{\partial \Lambda_\sigma}$$

in order to arrive at $q_\sigma(\vec{Q},\vec{P},t)$. This is possible only if

$$\det\left(\frac{\partial^2 S}{\partial q_\alpha \partial \Lambda_\beta}\right) \neq 0$$

known as the **Hessian condition.**

Example

Consider $H = \frac{p^2}{2m}$, i.e. a free particle in $d=1$ dimension. The HJE is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0$$

One solution is

$$S(q,\Lambda,t) = \frac{m(q-\Lambda)^2}{2t}$$

$$\frac{\partial S}{\partial q} = \frac{m(q-\Lambda)}{t}$$

$$\frac{\partial S}{\partial t} = -\frac{m(q-\Lambda)^2}{2t^2}$$