for which we obtain

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \frac{m}{t}(\Lambda - q)$$

Inverting, we obtain the motion

$$q(t) = \Lambda - \frac{\Gamma t}{m} = q(0) + pt/m$$

We identify $\Lambda = q(0)$ as the initial value of q, and $\Gamma = -p$ as minus the (conserved) momentum.

The HJE may have many solutions, all yielding the same motion. For example,

$$S(q,\Lambda,t) = \sqrt{2m\Lambda}\, q - \Lambda t \qquad \frac{\partial S}{\partial q} = \sqrt{2m\Lambda}, \quad \frac{\partial S}{\partial t} = -\Lambda$$

This yields

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \sqrt{\frac{m}{2\Lambda}}\, q - t \;\Rightarrow\; q(t) = \sqrt{\frac{2\Lambda}{m}}\,(t+\Gamma)$$

Here $\Lambda = E$ is the energy and $q(0) = \sqrt{\frac{2\Lambda}{m}}\,\Gamma$.

- **Time-independent Hamiltonians** <u>Lecture 15 (Wed. Nov. 25)</u>

  When $\partial H/\partial t = 0$, we may reduce the order of the HJE by writing

$$S(\vec{q},\vec{\Lambda},t) = W(\vec{q},\vec{\Lambda}) + T(t,\vec{\Lambda})$$

The HJE then becomes

$$H\left(\vec{q}, \frac{\partial W}{\partial \vec{q}}\right) = -\frac{\partial T}{\partial t}$$

Since the LHS is independent of t and the RHS is independent of q, each side must be equal to the same constant, which we may take to be Λ_1. Therefore

$$S(\vec{q}, \Lambda, t) = W(\vec{q}, \Lambda) - \Lambda_1 t$$

We call $W(\vec{q}, \vec{\Lambda})$ **Hamilton's characteristic function.** The HJE now takes the form

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \Lambda_1$$

Note that adding an additional constant Λ_{n+1} to S simply shifts the time variable : $t \to t - \Lambda_{n+1}/\Lambda_1$.

One-dimensional motion

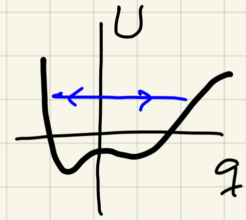
Consider the Hamiltonian $H(q,p) = \frac{p^2}{2m} + U(q)$. The HJE is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + U(q) = \Lambda \quad \leftarrow \text{clearly } \Lambda = E$$

with $\Lambda = \Lambda_1$. This may be recast as

$$\frac{\partial W}{\partial q} = \pm\sqrt{2m\left[\Lambda - U(q)\right]}$$

with a double-valued solution



$$W(q,\Lambda) = \pm\sqrt{2m}\int^{q} dq'\sqrt{\Lambda - U(q')}$$

The action (generating function) is $S(q,\Lambda,t) = W(q,\Lambda) - \Lambda t$.
The momentum is

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\left[\Lambda - U(q)\right]}$$

and

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \frac{\partial W}{\partial \Lambda} - t = \pm\sqrt{\frac{m}{2}}\int^{q(t)} dq' \frac{1}{\sqrt{\Lambda - U(q')}} - t$$

Thus the motion $q(t)$ is obtained by inverting

$$t + \Gamma = \pm\sqrt{\frac{m}{2}}\int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q)}} = I(q(t))$$

The lower limit on the integral is arbitrary and merely shifts t by a constant. Motion: $q(t) = I^{-1}(t+\Gamma)$

- **Separation of Variables**

If the characteristic function can be written as the sum

$$W(\vec{q},\vec{\Lambda}) = \sum_{\sigma=1}^{n} W_\sigma(q_\sigma,\vec{\Lambda})$$

the HJE is said to be completely separable. (A system may also be only partially separable.) In this case, each $W_\sigma(q_\sigma, \vec{\Lambda})$ is the solution of an equation of the form

$$H_\sigma\left(q_\sigma, \frac{\partial W_\sigma}{\partial q_\sigma}\right) = \Lambda_\sigma \,, \qquad p_\sigma = \frac{\partial W}{\partial q_\sigma} = \frac{\partial W_\sigma}{\partial q_\sigma}$$

NB: $H_\sigma(q_\sigma, p_\sigma)$ may depend on all the $\{\Lambda_1, \ldots, \Lambda_n\}$.

As an example, consider

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + \overbrace{A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\theta}}^{U(r,\theta,\phi)}$$

This is a real mess to tackle using the Lagrangian formalism. We seek a characteristic function of the form

$$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

The HJE then takes the form

$$\frac{1}{2m}\Big(\underbrace{\frac{\partial W_r}{\partial r}}_{p_r}\Big)^2 + \frac{1}{2mr^2}\Big(\underbrace{\frac{\partial W_\theta}{\partial \theta}}_{p_\theta}\Big)^2 + \frac{1}{2mr^2\sin^2\theta}\Big(\overbrace{\frac{\partial W_\phi}{\partial \phi}}^{p_\phi}\Big)^2 + A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\theta} = \Lambda_1 = E$$

Multiply through by $r^2\sin^2\theta$ to obtain

$$\underbrace{\frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi)}_{\text{depends only on } \phi} = \underbrace{-\sin^2\theta\left\{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta)\right\} - r^2\sin^2\theta\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\}}_{\text{depends only on } r,\,\theta}$$

Thus we must have

$$(\phi) \qquad \frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) = \Lambda_2 = \text{Constant}$$

Now replace the LHS of the penultimate equation by Λ_2 and divide by $\sin^2\theta$ to get

$$\underbrace{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\theta}}_{\text{depends only on } \theta} = \underbrace{-r^2\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\}}_{\text{depends only on } r}$$

Same story. We set

$$(\theta) \qquad \frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\theta} = \Lambda_3 = \text{constant}$$

We are now left with

$$(r) \qquad \frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) + \frac{\Lambda_3}{r^2} = \Lambda_1$$

Thus,

$$S(\vec{q},\vec{\Lambda},t) = \sqrt{2m}\int^r dr' \sqrt{\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2}} + \sqrt{2m}\int^\theta d\theta' \sqrt{\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'}} + \sqrt{2m}\int^\phi d\phi' \sqrt{\Lambda_2 - C(\phi')} - \Lambda_1 t$$

Now differentiate with respect to $\Lambda_{1,2,3}$ to obtain

$$(1) \quad \Gamma_1 = \frac{\partial S}{\partial \Lambda_1} = \sqrt{\frac{m}{2}} \int^{r(t)} dr' \left[\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2} \right]^{-1/2} - t$$

$$(2) \quad \Gamma_2 = \frac{\partial S}{\partial \Lambda_2} = -\sqrt{\frac{m}{2}} \int^{\theta(t)} \frac{d\theta'}{\sin^2\theta'} \left[\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'} \right]^{-1/2} + \sqrt{\frac{m}{2}} \int^{\phi(t)} d\phi' \left[\Lambda_2 - C(\phi') \right]^{-1/2}$$

$$(3) \quad \Gamma_3 = \frac{\partial S}{\partial \Lambda_3} = -\sqrt{\frac{m}{2}} \int^{r(t)} \frac{dr'}{(r')^2} \left[\Lambda_1 - A(r') - \frac{\Lambda_3}{(r')^2} \right]^{-1/2} + \sqrt{\frac{m}{2}} \int^{\theta(t)} d\theta' \left[\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\theta'} \right]^{-1/2}$$

Order of solution:

1. Invert (1) to obtain $r(t)$.

2. Insert this result for $r(t)$ into (3), then invert to obtain $\theta(t)$.

3. Insert $\theta(t)$ into (2) and invert to obtain $\phi(t)$.

NB: Varying the lower limits on the integrals in (1, 2, 3) just redefines the constants $\Gamma_{1,2,3}$.

- ## Action - Angle Variables

In a system which is "completely integrable", the HJE may be solved by separation of variables. Each momentum p_σ is then a function of its conjugate coordinate q_σ plus constants : $p_\sigma = \dfrac{\partial W_\sigma}{\partial q_\sigma} = p_\sigma(q_\sigma, \vec{\Lambda})$. This satisfies $H_\sigma(q_\sigma, p_\sigma) = \Lambda_\sigma$. The level sets of each $H_\sigma(q_\sigma, p_\sigma)$ are curves $C_\sigma(\vec{\Lambda})$, which describe projections of the full motion onto the (q_σ, p_σ) plane. We will assume in general that the motion is <u>bounded</u>, which means only two types of projected motion are possible :

librations : periodic oscillations about an equilibrium

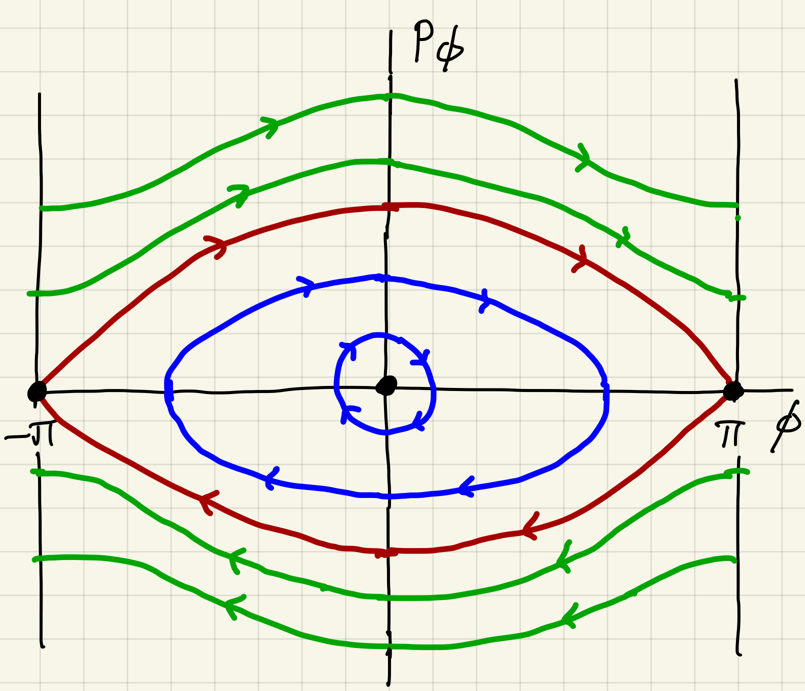
rotations : in which an angular coordinate advances by 2π in each cycle

Example : simple pendulum $H(\phi, p_\phi) = \dfrac{p_\phi^2}{2I} + \frac{1}{2} I\omega^2 (1-\cos\phi)$

rotations : $E > I\omega^2$

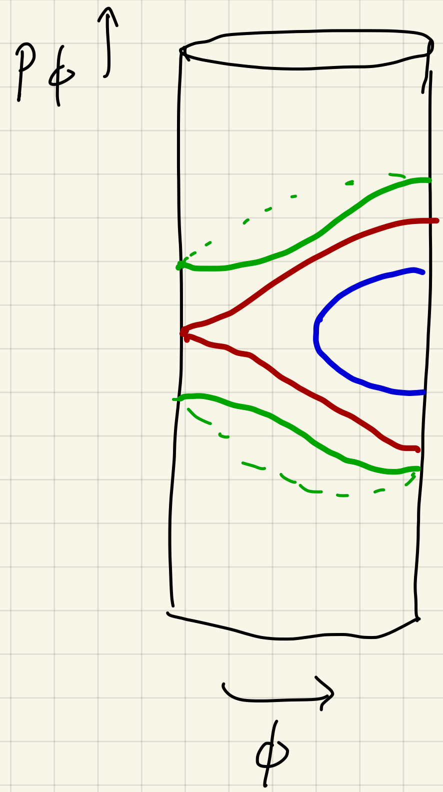
librations : $0 < E < I\omega^2$

separatrix : $E = I\omega^2$

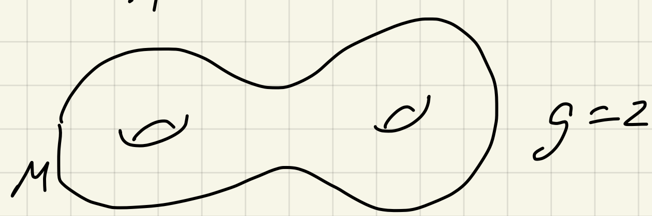


Generically, each $C_\sigma(\vec{\Lambda})$ is either a libration or a rotation.

Scratch



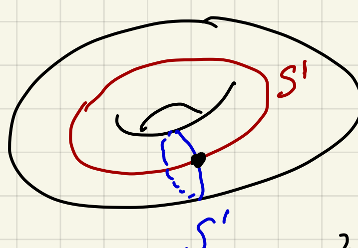
$$\underset{M}{\int} dK = -4\pi$$



$$\mathbb{R}/\mathbb{Z} \cong S^1$$

$$S^1 \times \mathbb{R} = \text{cylinder}$$

$$S^1 \times S^1 = T^2$$



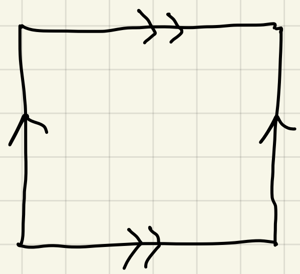
$g = 1$

$$\underset{T^2}{\int} dK = 0$$

$$\underset{M}{\int} d\Sigma \, K = 2\pi(2-2g)$$

$g = \#$ holes

$\chi \equiv 2 - 2g$



$K = 0$

$$\underset{S^2}{\int} d\Sigma \frac{1}{R^2} = \underset{S^2}{\int} d\hat{n} = 4\pi$$

$$aba^{-1}b^{-1} \neq 1$$

Topologically, both librations and rotations are homotopic to (= "can be continuously distorted to") a circle, S^1. Note though that they cannot be continuously distorted into each other, since librations can continuously be deformed to the point of static equilibrium, while rotations cannot. For a system with n freedoms, the motion is thus confined to n-tori:

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$$

$C_1(\vec{\lambda})$ $C_n(\vec{\lambda})$

These are called **invariant tori**, because for a given set of initial conditions, the motion is confined to one such n-torus. **Invariant tori never intersect!** Note that phase space is of dimension 2n, while the invariant tori, which fill phase space, are of dimension n. (Think about the phase space for the simple pendulum, which is topologically a cylinder, covered by librations and rotations which themselves are topologically circles.)

Action-angle variables $(\vec{\phi}, \vec{J})$ are a set of coordinates $(\vec{\phi})$ and momenta $(\vec{J})$ which cover phase space with invariant n-tori. The n **actions** $\{J_1, \dots, J_n\}$ specify a particular n-torus, and the n **angles** $\{\phi_1, \dots, \phi_n\}$

coordinatize each such torus. Invariance of the tori means that

$$\dot{J}_\sigma = -\frac{\partial H}{\partial \phi_\sigma} = 0 \Rightarrow H = H(\vec{J})$$

Each coordinate ϕ_σ describes the projected motion around C_σ, and is normalized so that

$$\oint_{C_\sigma} d\phi_\sigma = 2\pi \quad \text{(once around } C_\sigma\text{)}$$

The dynamics of the angle variables are given by

$$\dot{\phi}_\sigma = \frac{\partial H}{\partial J_\sigma} = \nu_\sigma(\vec{J})$$

Thus $\phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\vec{J})\, t$. The n frequencies $\{\nu_\sigma(\vec{J})\}$ describe the rates at which the circles C_σ are traversed.

(topologically!)

Lecture 17 (Nov. 30)

- **Canonical transformation to action-angle variables**

These AAVs sound great! Very intuitive! But how do we find them? Since the $\{J_\sigma\}$ determine the $\{C_\sigma\}$ and since each q_σ determines a point (two points, in the case of librations) on C_σ, this suggests a type-II