coordinatize each such torus. Invariance of the tori means that

$$\dot{J}_\sigma = -\frac{\partial H}{\partial \phi_\sigma} = 0 \Rightarrow H = H(\vec{J})$$

Each coordinate ϕ_σ describes the projected motion around C_σ, and is normalized so that

$$\oint_{C_\sigma} d\phi_\sigma = 2\pi \quad \text{(once around } C_\sigma\text{)}$$

The dynamics of the angle variables are given by

$$\dot{\phi}_\sigma = \frac{\partial H}{\partial J_\sigma} = \nu_\sigma(\vec{J})$$

Thus $\phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\vec{J})\,t$. The n frequencies $\{\nu_\sigma(\vec{J})\}$ describe the rates at which the circles C_σ (topologically!) are traversed.

Lecture 17 (Nov. 30)

- **Canonical transformation to action-angle variables**

These AAVs sound great! Very intuitive! But how do we find them? Since the $\{J_\sigma\}$ determine the $\{C_\sigma\}$ and since each q_σ determines a point (two points, in the case of librations) on C_σ, this suggests a type-II

CT with generator $F_2(\vec{q}, \vec{J})$:

$$p_\sigma = \frac{\partial F_2}{\partial q_\sigma} \quad , \quad \phi_\sigma = \frac{\partial F_2}{\partial J_\sigma}$$

Now

$$2\pi = \oint_{C_\sigma} d\phi_\sigma = \oint_{C_\sigma} d\left(\frac{\partial F_2}{\partial J_\sigma}\right) = \oint_{C_\sigma} dq_\sigma \frac{\partial^2 F_2}{\partial J_\sigma \partial q_\sigma} = \frac{\partial}{\partial J_\sigma} \oint dq_\sigma \, p_\sigma$$

we are led to define

$$J_\sigma = \frac{1}{2\pi} \oint_{C_\sigma} dq_\sigma \, p_\sigma$$

Procedure:

(1) Separate and solve the HJE for $W(\vec{q}, \vec{\Lambda}) = \sum_\sigma W_\sigma(q_\sigma, \vec{\Lambda})$.

(2) Find the orbits $C_\sigma(\vec{\Lambda})$, i.e. the level sets satisfying the conditions $H_\sigma(q_\sigma, p_\sigma; \vec{\Lambda}) = \Lambda_\sigma$.

(3) Invert the relation $J_\sigma(\vec{\Lambda}) = \frac{1}{2\pi} \oint_{C_\sigma} dq_\sigma \, p_\sigma$ to obtain $\vec{J}(\vec{\Lambda}) \xrightarrow{\text{(invert)}} \vec{\Lambda}(\vec{J})$

(4) The type-II generator to AAVs is

$$F_2(\vec{q}, \vec{J}) = \sum_\sigma W_\sigma(q_\sigma, \vec{\Lambda}(\vec{J}))$$

Let's now work through some examples.

Harmonic oscillator

Our Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$, so the HJE equation is

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2} m\omega_0^2 q^2 = \Lambda$$

We have

$$p = \frac{\partial W}{\partial q} = \pm \sqrt{2m\Lambda - m^2\omega_0^2 q^2}$$

Simplify by defining

$$q = \sqrt{\frac{2\Lambda}{m\omega_0^2}} \sin\theta \Rightarrow p = \sqrt{2m\Lambda} \cos\theta$$

and so

$$J = \frac{1}{2\pi} \oint dq\, p = \frac{1}{2\pi} \cdot \frac{2\Lambda}{\omega_0} \int_0^{2\pi} d\theta \cos^2\theta = \frac{\Lambda}{\omega_0}$$

We still must solve the HJE:

$$\frac{dW}{d\theta} = \frac{dW}{dq} \cdot \frac{\partial q}{\partial \theta} = \sqrt{2m\Lambda} \cos\theta \cdot \sqrt{\frac{2\Lambda}{m\omega_0^2}} \cos\theta = 2J\cos^2\theta$$

Integrate to get

$$W(\theta, J) = J\theta + \frac{1}{2} J \sin 2\theta + \text{const.}$$

$$\theta = \cos^{-1}\left[q / \sqrt{2m\Lambda(J)}\right] \longrightarrow W(q, J)$$

Then

$$\phi = \left.\frac{\partial W}{\partial J}\right|_q = \theta + \frac{1}{2}\sin 2\theta + J(1+\cos 2\theta)\left.\frac{\partial \theta}{\partial J}\right|_q$$

Now $q = \sqrt{2J/m\omega_0}\,\sin\theta$ so

$$dq = \frac{\sin\theta}{\sqrt{2m\omega_0 J}}\,dJ + \sqrt{\frac{2J}{m\omega_0}}\cos\theta\, d\theta \Rightarrow \left.\frac{\partial\theta}{\partial J}\right|_q = -\frac{1}{2J}\tan\theta$$

Plugging into our expression for ϕ, we obtain $\phi = \theta$. (Not much of a surprise.) Thus, the full CT is

$$q = \left(\frac{2J}{m\omega_0^2}\right)\sin\phi \quad , \quad p = \sqrt{2m\omega_0 J}\,\cos\phi$$

and the Hamiltonian is $H(\phi, J) = \omega_0 J$. The equations of motion are

call it $H \equiv \tilde{H}$

$$\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0 \quad , \quad \dot{J} = -\frac{\partial H}{\partial \phi} = 0$$

with solution

$$\phi(t) = \phi(0) + \omega_0 t$$
$$J(t) = J(0)$$

and of course $\nu(J) = \omega_0$ (independent of J).

- Please read § 15.5.5 (AAV for particle in a box)

- Integrability and motion on invariant tori

Recall that a **completely integrable** system may be solved by separation of variables, and that

$$H(\vec{q},\vec{p}) \to \tilde{H}(\vec{\phi},\vec{J}) = \tilde{H}(\vec{J})$$

$$\dot{J}_\sigma = -\frac{\partial \tilde{H}}{\partial \phi_\sigma} = 0 \Rightarrow J_\sigma(t) = J_\sigma(0)$$

$$\dot{\phi}_\sigma = +\frac{\partial \tilde{H}}{\partial J_\sigma} = \nu_\sigma(\vec{J}) \Rightarrow \phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\vec{J})\, t$$

Thus, the angle variables wind around the invariant torus at constant rates $\nu_\sigma(\vec{J})$. While each $\phi_\sigma(t)$ winds around its own circle, the motion of the system as a whole will not be periodic unless the frequencies $\nu_\sigma(\vec{J})$ are commensurate, which means that there exists a time T (i.e. the period) such that $\nu_\sigma T = 2\pi k_\sigma$ with $k_\sigma \in \mathbb{Z}$ $\forall\, \sigma \in \{1,\ldots,n\}$. Thus

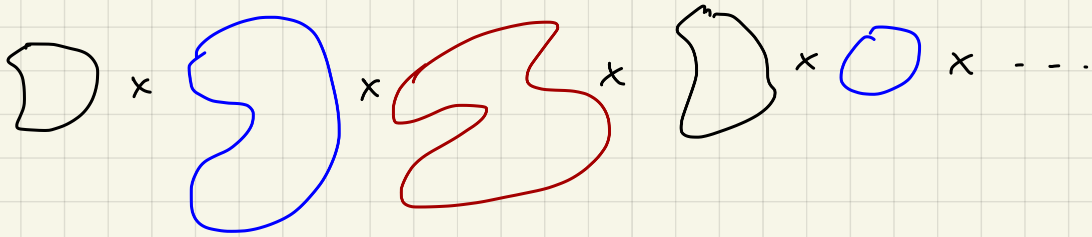
$$\frac{\nu_\alpha}{\nu_\beta} = \frac{k_\alpha}{k_\beta} \in \mathbb{Q} \quad \forall\ \alpha,\beta \in \{1,\ldots,n\}$$

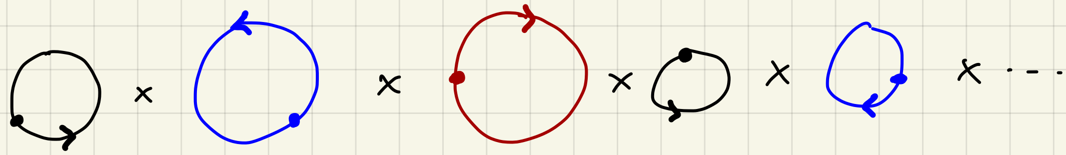
T is the smallest such period if $\{k_1,\ldots,k_n\}$ have no common factors. On a given torus, either all orbits are periodic or none is periodic.

In terms of the original $\{q_1,\ldots,q_n\}$ coordinates,

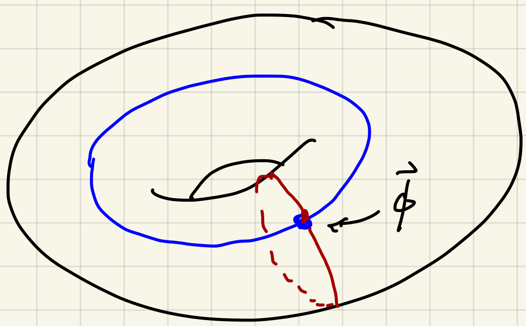
Scratch

n-torus : $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}}$

$T^n =$ 

$=$ 

$\vec{\phi}(t) = \{ \phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t), \dots \}$



there are two possibilities:

(i) libration: $q_\sigma(t) = \sum_{\vec{l}\in\mathbb{Z}^n} A^{(\sigma)}_{l_1\cdots l_n} e^{il_1\phi_1(t)} \cdots e^{il_n\phi_n(t)}$

(ii) rotation: $q_\sigma(t) = q^0_\sigma \phi_\sigma(t) + \sum_{\vec{l}\in\mathbb{Z}^n} B^{(\sigma)}_{l_1\cdots l_n} e^{il_1\phi_1(t)} \cdots e^{il_n\phi_n(t)}$

where a complete rotation results in $\Delta q_\sigma = 2\pi q^0_\sigma$.

- **Liouville-Arnol'd Theorem**

This is another statement of what it means for a Hamiltonian system to be integrable. Suppose a Hamiltonian $H(\vec{q},\vec{p})$ has n **first integrals** $I_k(\vec{q},\vec{p})$, where $k\in\{1,\dots,n\}$. This means

Poisson bracket

$$\frac{dI_k}{dt} = \sum_{\sigma=1}^{n}\left(\frac{\partial I_k}{\partial q_\sigma}\frac{dq_\sigma}{dt} + \frac{\partial I_k}{\partial p_\sigma}\frac{dp_\sigma}{dt}\right) = \{I_k, H\} = 0$$

If the $\{I_k\}$ are independent functions, meaning that $\{\vec{\nabla} I_k\}$ form a set of n linearly independent vectors at almost every point in phase space M, and if all the first integrals commute with respect to the Poisson bracket, i.e. $\{I_k, I_l\} = 0$ for all k,l ($\Leftrightarrow$ I_k and I_l **in involution**), then:

(i) The space $\underset{\dim(M_I)=n}{M_I} \equiv \{(\vec{q},\vec{p})\in \underset{\dim(M)=2n}{M} \mid I_k(\vec{q},\vec{p}) = C_k \ \forall\, k\in\{1,\dots,n\}\}$ is diffeomorphic to an n-torus $T^n = S^1\times S^1\times\cdots\times S^1$, on which one can introduce action-angle variables on a set of overlapping patches whose union contains M_I, where the angle variables are coordinates on M_I and the action variables are the first integrals.

(ii) The transformed Hamiltonian is $\tilde{H} = \tilde{H}(\vec{I})$, hence

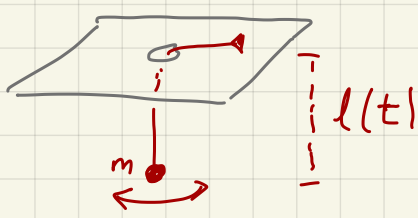
$$\dot{I}_k = -\frac{\partial \tilde{H}}{\partial \phi_k} = 0$$

$$\dot{\phi}_k = +\frac{\partial \tilde{H}}{\partial I_k} = \nu_k(\vec{I}) \Rightarrow \phi_k(t) = \phi_k(0) + \nu_k(\vec{I})\, t$$

Note this does not require $\tilde{H} = \sum_k \tilde{H}_k(I_k)$.

- **Adiabatic invariants**

Adiabatic processes in thermodynamics are ones in which no heat is exchanged between a system and its environment. In mechanics, adiabatic perturbations are slow, smooth changes to a Hamiltonian system's parameters. A typical example: slowly changing the length $\ell(t)$ of a pendulum. General setting: $H = H(\vec{q}, \vec{p}; \lambda(t))$. All explicit time dependence in H is through $\lambda(t)$. If ω_0 is a characteristic frequency of the motion when λ is constant, then



$$\epsilon \equiv \omega_0^{-1} \left| \frac{d \ln \lambda}{dt} \right|$$

provides a dimensionless measure of the rate of change of $\lambda(t)$. We require $\epsilon \ll 1$ for adiabaticity. Under such conditions, the action variables are preserved to exponential accuracy. (We will see just what this means.) For the SHO, the energy, action, and oscillation frequency are related according to $J = E/\nu$. During an adiabatic process, $E(t)$ and $\nu(t)$ may vary appreciably, but $J(t)$ remains very nearly constant. Thus, if θ_0 is the oscillation amplitude, then assuming small oscillations,

$$E = \frac{1}{2} mg\ell\,\theta_0^2 = \nu J = \sqrt{\frac{g}{\ell}}\,J$$

$$\Rightarrow \theta_0(\ell) \approx \frac{2J}{m\sqrt{g}\,\ell^{3/2}}$$

Adiabatic invariance then says $\theta_0(\ell) \propto \ell^{-3/2}$.

Consider now an $n=1$ system, and suppose that for fixed λ the type-II generator to action-angle variables is $S(q, J; \lambda)$. Now let $\lambda = \lambda(t)$, in which case

$$\tilde{H}(\phi, J, t) = H(J; \lambda) + \frac{\partial S}{\partial \lambda}\frac{d\lambda}{dt}$$

ϕ - dependence through $S(q(\phi, J; \lambda), J; \lambda)$

where

$$H(J; \lambda) = H\big(q(\phi, J; \lambda), p(\phi, J; \lambda); \lambda\big)$$

Note that $H(J; \lambda)$ is independent of ϕ, because for fixed λ the function $S(q, J; \lambda)$ generates the AAV.

Hamilton's equations are now

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial J} = \nu(J;\lambda) + \frac{\partial^2 S}{\partial\lambda\,\partial J}\frac{d\lambda}{dt}$$

$$\dot{J} = -\frac{\partial \tilde{H}}{\partial \phi} = -\frac{\partial^2 S}{\partial\lambda\,\partial\phi}\frac{d\lambda}{dt}$$

where $\nu(J;\lambda) \equiv \partial H(J;\lambda)/\partial J$ and where

$$S(\phi, J;\lambda) = S(q(\phi,J;\lambda), J;\lambda) \equiv \sum_{m=-\infty}^{\infty} S_m(J;\lambda)\, e^{im\phi}$$

Fourier analyzing the equation for $\dot{J}$, we have

$$\dot{J} = -i\dot{\lambda}\sum_{m=-\infty}^{\infty} m\,\frac{\partial S_m}{\partial\lambda}\, e^{im\phi}$$

Now,

$$\Delta J = J(\infty) - J(-\infty) = \int_{-\infty}^{\infty} dt\, \dot{J}$$

$$= -i\sum_{m=-\infty}^{\infty} m \int_{-\infty}^{\infty} dt\, \frac{\partial S_m(J;\lambda)}{\partial\lambda}\frac{d\lambda}{dt}\, e^{im\phi}$$

(m = 0 term is cancelled)

Now $\phi(t) = \nu t + \phi(0)$ to good accuracy, since $\dot{\lambda}$ is small. So we must evaluate expressions such as

$$m \neq 0: \quad \mathcal{J}_m \equiv \int_{-\infty}^{\infty} dt \underbrace{\left\{\frac{\partial S_m(J;\lambda)}{\partial\lambda}\frac{d\lambda}{dt}\right\}}_{f(t)} e^{im\nu t}\, e^{im\phi(0)}$$

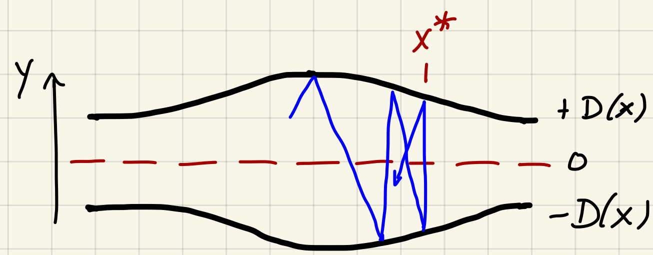
The bracketed term is a smooth function of time t which by assumption varies slowly on the scale ν^{-1}. Call it $f(t)$.

We assume $f(t)$ may be analytically continued off the real t axis, and that its closest singularities in the complex t plane lie at $\mathrm{Im}\,t = \pm\tau$, where $|\nu\tau| \gg 1$. Then $f_m \sim e^{-|m\nu\tau|} = e^{-|m|/\epsilon}$, which is exponentially small in $|\nu\tau| = \frac{1}{\epsilon}$ (hence only $m = \pm 1$ need be considered). Thus, ΔJ may be kept arbitrarily small if $\lambda(t)$ is varied sufficiently slowly.

$$f(t) = \frac{1}{\pi}\frac{\tau}{t^2+\tau^2} \Rightarrow \int_{-\infty}^{\infty}\! dt\, f(t)\, e^{im\nu t} = e^{-|m\nu|\tau} \propto e^{-\frac{|m|}{\epsilon}}$$

- **Examples**

Mechanical mirror : A point particle bounces between two curves $y = \pm D(x)$, with $|D'(x)| \ll 1$. The bounce time is $\tau_\perp / 2v_y$, and we assume $\tau \ll L/v_x$ where L = length.



So there are many bounces, during which the particle samples $D(x)$. The adiabatic invariant is the action,

$$J = \frac{1}{2\pi}\oint dy\, p_y = \frac{2}{\pi} m v_y D(x)$$

The energy is

$$E = \tfrac{1}{2} m (v_x^2 + v_y^2) = \tfrac{1}{2} m v_x^2 + \frac{\pi^2 J^2}{8 m D^2(x)}$$

Thus,

$$v_x^2 = \frac{2E}{m} - \left(\frac{\pi J}{2m D(x)}\right)^2$$

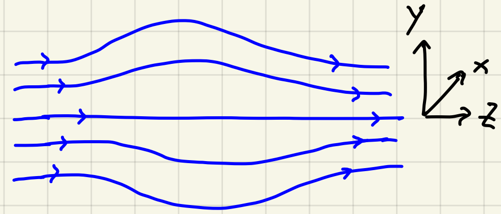
which means the particle turns around when $D(x^*) = \dfrac{\pi J}{\sqrt{8mE}}$.

A pair of such mirrors (when $D(x) = D(-x)$) confines the particle.

Similar physics is present in the magnetic mirror, or "magnetic bottle", discussed in §15.7.3. There the adiabatic invariant is the magnetic moment,

$$M = -\frac{eJ}{mc} = \frac{e^2}{2\pi mc^2}\,\Phi$$

where J = action and Φ = magnetic flux.



magnetic field lines (azimuthally symmetric about the middle line)

- **Resonances**

What happens when $n > 1$? We then have

$$\dot{J}^{\alpha} = -i\dot{\lambda}\sum_{\vec{m}\in\mathbb{Z}^n} m^{\alpha}\,\frac{\partial S_{\vec{m}}(J;\lambda)}{\partial\lambda}\,e^{i\vec{m}\cdot\vec{\phi}}$$

and

$$\Delta J^{\alpha} = -i\sum_{\vec{m}\in\mathbb{Z}^n} m^{\alpha}\int_{-\infty}^{\infty}\!dt\,\frac{\partial S_{\vec{m}}(\vec{J};\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\,e^{i\vec{m}\cdot\vec{\nu}t}\,e^{i\vec{m}\cdot\vec{\beta}}$$

When $\vec{m}\cdot\vec{\nu}(\vec{J}) = 0$, we have a **resonance**, and the integral grows linearly in the time limits, which is a violation of adiabatic invariance. Resonances may result in the breakdown of invariant tori, and provide a route to chaos. Resonances can thus only occur when two or more frequencies $\nu_{\alpha}(\vec{J})$ have a ratio which is a rational number. But even if the frequency ratios are all irrational, any such irrational number may be approximated to arbitrary accuracy by some choice of rational number. To understand how to deal with resonances, we need **canonical perturbation theory.**