Mechanics

Lecture 2 (Oct 7)

Hamilton's principle : $\delta S = 0$ where

$$S[q(t)] = \int_{t_1}^{t_2} dt\, L(q,\dot{q},t) \quad = \text{action functional}$$

with $q = \{q_1, \ldots, q_n\}$ = set of generalized coordinates

The function $L(q,\dot{q},t)$ is the Lagrangian, and is given by $L = T - U$, where T = kinetic energy and U = potential energy. Typically $T = T(q,\dot{q})$ is a quadratic form in the generalized velocities $\{\dot{q}_\sigma\}$, i.e. $T(q,\dot{q}) = T_{\sigma\sigma'}(q)\,\dot{q}_\sigma \dot{q}_{\sigma'}$. For example

$$T = \tfrac{1}{2} m \dot{\vec{x}}^{\,2} = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad \text{Cartesian } (x,y,z)$$

$$= \tfrac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right) \qquad \text{polar } (r,\theta,\phi)$$

The potential energy U is most often a function of q, but $U = U(q,\dot{q})$ applies, e.g., for charged particles in a magnetic field, where

$$U(\vec{x},\dot{\vec{x}}) = \underbrace{q}_{\text{charge}}\, \overbrace{\phi(\vec{x})}^{\text{scalar potential}} - \frac{q}{c} \underbrace{\vec{A}(\vec{x})}_{\text{vector potential}} \cdot \frac{d\vec{x}}{dt}$$

Free particle $\Rightarrow$ $L = \tfrac{1}{2} m \vec{v}^{\,2}$ (§3.6.3)

- NB : In general $L = \tfrac{1}{2} T_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma \dot{q}_{\sigma'} - U(q,\dot{q},t)$

Equations of motion: $\underbrace{F_\sigma}$ $\equiv \dfrac{\partial L}{\partial q_\sigma}$ = generalized force

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q_\sigma}}_{p_\sigma \equiv \frac{\partial L}{\partial \dot q_\sigma} = \text{generalized momentum}}\Big) = \frac{\partial L}{\partial q_\sigma}, \quad \sigma \in \{1, \dots, n\}$$

$y \to q_\sigma$
$y' \to \dot q_\sigma$

Thus, $\dot p_\sigma = F_\sigma$, i.e. Newton's second law.

- Conservation laws:

  Most general setting: to be discussed (Noether's theorem)

  For now, recall results from COV:

① $\dfrac{\partial L}{\partial q_\sigma} = 0 \Rightarrow p_\sigma = \dfrac{\partial L}{\partial \dot q_\sigma}$ = constant ($\dot p_\sigma = 0$)

Momentum p_σ is conserved because the force $F_\sigma = 0$

Example: $T = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$, $U = mgz$

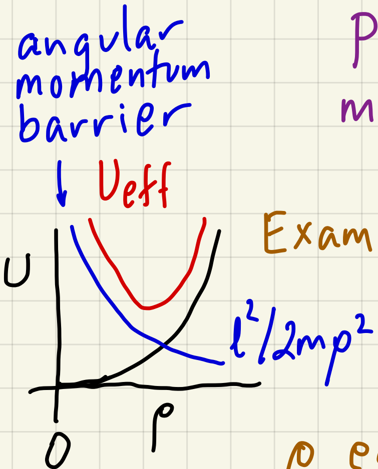
$$\Rightarrow F_x = \frac{\partial L}{\partial \dot x} = 0, \quad F_y = \frac{\partial L}{\partial \dot y} = 0$$

$$p_x = m\dot x \Rightarrow x(t) = x(0) + \frac{p_x}{m}t$$

$$p_y = m\dot y \Rightarrow y(t) = y(0) + \frac{p_y}{m}t$$

$$p_z = m\dot z, \quad F_z = -\frac{\partial U}{\partial z} = -mg$$

$$m\ddot z = -mg \Rightarrow z(t) = z(0) + \dot z(0)t - \tfrac{1}{2}gt^2$$



Example': $L = \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2) - U(\rho)$ (2D polar)

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\rho^2\dot\phi = \text{angular momentum} \equiv \ell$$

ρ eqn: $m\ddot\rho = m\rho\dot\phi^2 - U'(\rho) = \dfrac{\ell^2}{m\rho^3} - U'(\rho) = -U'_{eff}(\rho)$

$$U_{eff}(\rho) = \frac{\ell^2}{2m\rho^2} + U(\rho)$$

Scratch

$$L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) - U(\rho)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} = \ell$$

IMPORTANT: Can substitute $\dot{\phi} = \frac{\ell}{m\rho^2}$ in eqns of motion but <u>not</u> in Lagrangian itself!

WRONG:

$$L = \frac{1}{2} m\dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 - U(\rho)$$

$$= \frac{1}{2} m\dot{\rho}^2 + \frac{1}{2} m\rho^2 \left(\frac{\ell}{m\rho^2}\right)^2 - U(\rho)$$

$$= \frac{1}{2} m\dot{\rho}^2 + \frac{\ell^2}{2m\rho^2} - U(\rho)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = m\ddot{\rho} = \frac{\partial L}{\partial \rho} = -\frac{\ell^2}{m\rho^3} - U'(\rho)$$

wrong sign!

RIGHT:

$$L = \frac{1}{2} m\dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 - U(\rho)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} = \ell \quad \text{constant} \quad (\dot{p}_\phi = 0)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = m\ddot{\rho} = \frac{\partial L}{\partial \rho} = m\rho\dot{\phi}^2 - U'(\rho)$$

$$= m\rho\left(\frac{\ell}{m\rho^2}\right)^2 - U'(\rho)$$

$$= +\frac{\ell^2}{m\rho^3} - U'(\rho) = -U'_{eff}(\rho)$$

right sign!

② $\frac{\partial L}{\partial t} = 0 \Rightarrow H = \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} - L = \dot{q}_\sigma p_\sigma - L$ conserved

implied summation on repeated indices

See it again:

$$\frac{dH}{dt} = \cancel{\ddot{q}_\sigma p_\sigma} + \cancel{\dot{q}_\sigma \dot{p}_\sigma} - \cancel{\underbrace{\frac{\partial L}{\partial q_\sigma}}_{F_\sigma} \dot{q}_\sigma} - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} \ddot{q}_\sigma} - \frac{\partial L}{\partial t}$$

$$\boxed{\frac{dL}{dt} = \frac{\partial L}{\partial q_\sigma}\frac{dq_\sigma}{dt} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d\dot{q}_\sigma}{dt} + \frac{\partial L}{\partial t}}$$

Thus, $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$, and for

$$L = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{x}}_j^{\,2} - U(\vec{x}_1, \dots, \vec{x}_N)$$

we have that $H = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{x}}_j^{\,2} + U(\vec{x}_1, \dots, \vec{x}_N)$

is a constant of the motion.

- In general, $H = \dot{q}_\sigma p_\sigma - L(q, \dot{q}, t)$ is a Legendre transform of L:

$$dH = \cancel{p_\sigma d\dot{q}_\sigma} + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \cancel{\frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma} - \frac{\partial L}{\partial t} dt$$

and hence $H = H(q, p, t)$ with

$$\frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -F_\sigma, \quad \frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

We then have Hamilton's equations of motion:

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \dot{\xi}_\alpha = J_{\alpha\beta} \frac{\partial H}{\partial \xi_\beta}$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} q \\ p \end{pmatrix}$$

Procedure

(i) Choose a set of generalized coordinates

(ii) Find KE $T(q,\dot{q},t)$ and PE $U(q,t)$ or $U(q,\dot{q},t)$ and thus the Lagrangian $L(q,\dot{q},t) = T - U$.

(iii) Find the canonical momenta $p_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma}$ and the generalized forces $F_\sigma = \dfrac{\partial L}{\partial q_\sigma}$.

$\longrightarrow p_\sigma = p_\sigma(q,\dot{q},t)$

(iv) Identify any conserved quantities (later: Noether's thm)

(v) Evaluate $\dot{p}_\sigma$ (carefully!) and write $\dot{p}_\sigma = F_\sigma$

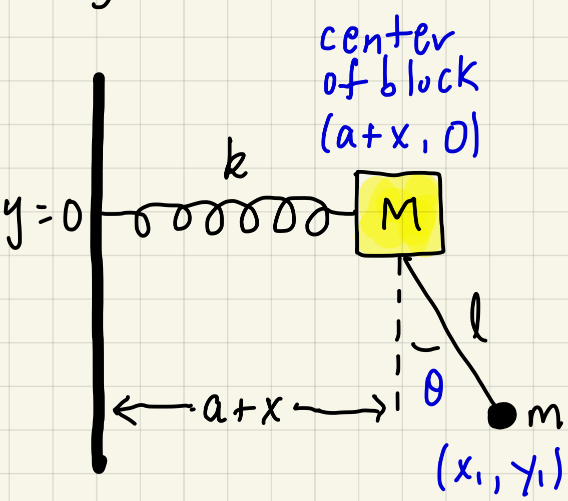
(vi) Integrate the equations of motion to get $\{q_\sigma(t)\}$, the motion of the system.

2n constants of integration $\{q_\sigma(0), \dot{q}_\sigma(0)\}$

§3.8 : Cartesian, cylindrical, and polar coordinates

§3.10 : Examples

§3.10.4 : Pendulum attached to mass on a spring



a = unstretched length of spring

coordinates of mass m : (x_1, y_1)

$$x_1 = a + x + l\sin\theta \ , \quad y_1 = -l\cos\theta$$

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$$
$$= \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\dot{\theta}$$

$$U = \tfrac{1}{2}kx^2 + mgy_1$$
$$= \tfrac{1}{2}kx^2 - mgl\cos\theta$$

Lagrangian:

$$L = T - U$$
$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mg\ell\cos\theta$$

Generalized momenta:

- $p_x = \dfrac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\ell\cos\theta\,\dot{\theta}$
- $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = m\ell\cos\theta\,\dot{x} + m\ell^2\dot{\theta}$

Generalized forces:

- $F_x = \dfrac{\partial L}{\partial x} = -kx$
- $F_\theta = \dfrac{\partial L}{\partial \theta} = -m\ell\sin\theta\,\dot{x}\dot{\theta} - mg\ell\sin\theta$

Equations of motion:

- $\dot{p}_x = F_x \Rightarrow (M+m)\ddot{x} + m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,\dot{\theta}^2 = -kx$
- $\dot{p}_\theta = F_\theta \Rightarrow m\ell\cos\theta\,\ddot{x} + m\ell^2\ddot{\theta} \cancel{- m\ell\sin\theta\,\dot{x}\dot{\theta}}$
$$= \cancel{- m\ell\sin\theta\,\dot{x}\dot{\theta}} - mg\ell\sin\theta$$

Conserved quantities:

Only $H = \dot{x}p_x + \dot{\theta}p_\theta - L$

$$= \left[(M+m)\dot{x}^2 + m\ell\cos\theta\,\dot{x}\dot{\theta}\right] + \left[m\ell\cos\theta\,\dot{x}\dot{\theta} + m\ell^2\dot{\theta}^2\right]$$
$$- \frac{1}{2}(M+m)\dot{x}^2 - \frac{1}{2}m\ell^2\dot{\theta}^2 - m\ell\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}kx^2 - mg\ell\cos\theta$$
$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}kx^2 - mg\ell\cos\theta$$
$$= T + U \equiv E$$

Small oscillations: linearize the equations of motion

- $(M+m)\ddot{x} + m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\,\dot{\theta}^2 = -kx$
- $m\ell\cos\theta\,\ddot{x} + m\ell^2\ddot{\theta} = -mg\ell\sin\theta$

$$\Rightarrow \quad \begin{array}{l} \cdot\ (M+m)\ddot{x} + m\ell\ddot{\theta} = -kx \\ \cdot\ \ddot{x} + \ell\ddot{\theta} = -g\theta \end{array} \qquad \left(\begin{array}{l}\text{expand about } x=\theta=0 \\ \text{assume } x,\theta,\dot{x},\dot{\theta} \text{ small}\end{array}\right)$$

The five parameters (M, m, ℓ, k, g) may be reduced to three:

$$u \equiv \frac{x}{\ell}\ , \quad \alpha \equiv \frac{m}{M}\ , \quad \omega_0^2 \equiv \frac{k}{M}\ , \quad \omega_1^2 \equiv \frac{g}{\ell}$$

Then we have

- $(1+\alpha)\ddot{u} + \alpha\ddot{\theta} + \omega_0^2 u = 0$
- $\ddot{u} + \ddot{\theta} + \omega_1^2\theta = 0$

This linear system may be solved by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{-i\omega t} \qquad \frac{d^2}{dt^2} \longrightarrow -\omega^2$$

$$\Rightarrow \begin{pmatrix} \omega_0^2 - (1+\alpha)\omega^2 & -\alpha\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} = 0$$

A nontrivial sol$^{\underline{n}}$ requires that the determinant vanish:

$$\omega^4 - \left[\omega_0^2 + (1+\alpha)\omega_1^2\right]\omega^2 + \omega_0^2\omega_1^2 = 0$$

$$\omega_\pm^2 = \tfrac{1}{2}\left[\omega_0^2 + (1+\alpha)\omega_1^2\right] \pm \tfrac{1}{2}\sqrt{\left[\omega_0^2 - (1+\alpha)\omega_1^2\right]^2 + 4\alpha\,\omega_0^2\omega_1^2}$$

There are two eigenvalues for ω^2, given by

$$\omega_\pm^2 = \frac{1}{2}\left[\omega_0^2 + (1+\alpha)\omega_1^2\right] \pm \frac{1}{2}\sqrt{\left[\omega_0^2 - (1+\alpha)\omega_1^2\right]^2 + 4\alpha\,\omega_0^2\omega_1^2}$$

The general sol$\underline{^n}$ is then

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} u_0^+ \\ \theta_0^+ \end{pmatrix} e^{-i\omega_+ t} + \begin{pmatrix} u_0^- \\ \theta_0^- \end{pmatrix} e^{-i\omega_- t}\right]$$

solution must be real

where

$$\begin{pmatrix} \omega_0^2 - (1+\alpha)\omega_\pm^2 & -\alpha\omega_\pm^2 \\ -\omega_\pm^2 & \omega_1^2 - \omega_\pm^2 \end{pmatrix}\begin{pmatrix} u_0^\pm \\ \theta_0^\pm \end{pmatrix} = 0$$

normal modes

$$\begin{pmatrix} u_0^\pm \\ \theta_0^\pm \end{pmatrix} e^{-i\omega_\pm t}$$

This fixes the ratios $\dfrac{u_0^\pm}{\theta_0^\pm} = \left(\dfrac{\omega_1^2}{\omega_\pm^2} - 1\right) \in \mathbb{R}$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad \begin{matrix} u_2/u_1 \\ \vdots \\ u_n/u_1 \end{matrix}$$

Thus, we are free to choose $\theta_0^\pm$, which are two <u>complex</u> constants $\Rightarrow$ four real parameters.
We fix them via the initial conditions,

$$\begin{pmatrix} u(0) \\ \theta(0) \end{pmatrix} \text{ and } \begin{pmatrix} \dot{u}(0) \\ \dot{\theta}(0) \end{pmatrix} \Rightarrow \text{four real pieces of initial data}$$

Here we have used the fact that if $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{-i\omega t}$ is a sol$\underline{^n}$, then so is $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix} e^{+i\omega t}$. In this sense, we might speak of <u>four</u> eigenfrequencies $\{\omega_+, \omega_-, -\omega_+, -\omega_-\}$ of which two are positive and two are negative.

Scratch

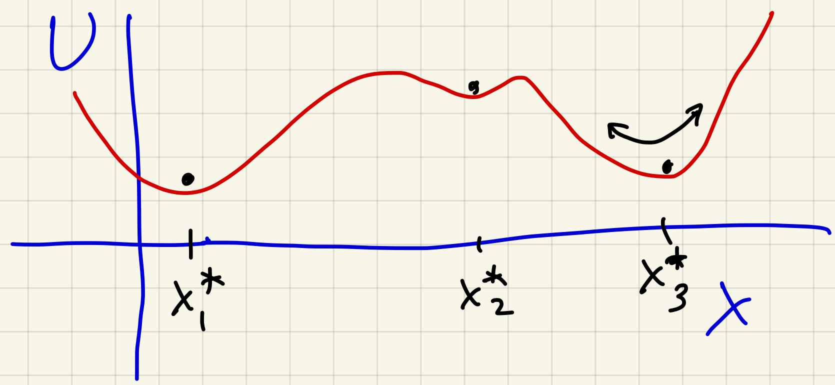
$$U(x) = \frac{1}{2}kx^2 + \frac{1}{4}bx^4$$

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - \frac{1}{4}bx^4$$

$$m\ddot{x} = -kx - \cancel{bx^3}$$

Eqbm @ $x = 0$, $\dot{x} = 0$



expand about $x = x_j^*$ (sol$^{\text{ns}}$ to $U'(x^*) = 0$)

$$\Rightarrow \quad x = x_j^* + \delta x$$

$$\Rightarrow \quad m\ddot{\delta x} = -U''(x_j^*)\,\delta x$$

$$\omega_j^2 = \sqrt{U''(x_j^*)/m}$$

U'' small $\qquad\qquad$ U'' big

w_j small $\qquad\qquad$ w big

Virial Theorem

- formula describing time-averaged motion of a mechanical system

Define the virial $G(q,p) = \sum_\sigma q_\sigma p_\sigma$, for which

$$\frac{dG}{dt} = \sum_\sigma (\dot{q}_\sigma p_\sigma + \dot{p}_\sigma q_\sigma) = \sum_\sigma \left\{ \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} + q_\sigma \frac{\partial L}{\partial q_\sigma} \right\}$$

Suppose $T = \frac{1}{2} T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k=2$ in the generalized velocities, and that $\partial U / \partial \dot{q}_\sigma = 0$. Then

$$\sum_\sigma \dot{q}_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma \frac{\partial T}{\partial \dot{q}_\sigma} = 2T$$

Now consider the time average of $\dot{G}$ over $[0, \tau]$:

$$\left\langle \frac{dG}{dt} \right\rangle_\tau = \frac{1}{\tau} \int_0^\tau dt\, \frac{dG}{dt} = \frac{G(\tau) - G(0)}{\tau}$$

If G is bounded, then we have $\langle \dot{G} \rangle_\tau \to 0$ as $\tau \to \infty$. This is the case for any bounded motion, such as planetary orbits. In such cases,

$$2\langle T \rangle = - \left\langle \sum_{\sigma=1}^{n} q_\sigma F_\sigma \right\rangle \qquad n = \underset{\substack{\uparrow \\ \text{dim}^{\text{n}} \text{ of space}}}{d} \cdot N$$

$$= \left\langle \sum_{j=1}^{N} \vec{x}_j \cdot \frac{\partial}{\partial \vec{x}_j} U(\vec{x}_1, \ldots, \vec{x}_N) \right\rangle = k \langle U \rangle$$

if $U(\vec{x}_1, \ldots, \vec{x}_N)$ homogeneous of degree k in $\{ x_j^\alpha \}$.

Scratch

Euler's thm for homogeneous functions:

$f(x_1, \dots, x_n)$ homogeneous of degree k if

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^k f(x_1, \dots, x_n)$$

examples

$$f(x,y) = x^5 + a x^4 y + b \frac{y^6}{x} \qquad k=5$$

$$T(\dot{q}_1, \dots, \dot{q}_n) = \frac{1}{2} T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'} \qquad k=2$$

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=1} f(\lambda x_1, \dots, \lambda x_n) = x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n}$$

$$= \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=1} \lambda^k f(x_1, \dots, x_n)$$

$$= \left. k \lambda^{k-1} f(x_1, \dots, x_n) \right|_{\lambda=1}$$

$$\therefore \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = kf$$

Check: $\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\left(x^5 + a x^4 y + b \frac{y^6}{x}\right) =$

$$x \cdot 5x^4 + x \cdot 4a x^3 y - x \cdot b \frac{y^6}{x^2} + y \cdot 0 + y \cdot a x^4 + y \cdot 6b \frac{y^5}{x}$$

$$= 5x^5 + 5a x^4 y + 5b \frac{y^6}{x} = 5\left(x^5 + a x^4 y + b \frac{y^6}{x}\right)$$

Since $T + U = E$ is conserved, we have

$$\langle T \rangle = \frac{kE}{k+2} \quad , \quad \langle U \rangle = \frac{2E}{k+2}$$

Application: Keplerian orbits, $k = -1$

$$\langle T \rangle = -E \, , \quad \langle U \rangle = 2E \quad ; \quad E < 0$$

Note then that a satellite losing energy due to frictional losses as it enters the atmosphere must <u>increase</u> its kinetic energy, i.e. it moves faster! (Think also about angular momentum conservation.)

<u>Noether's Theorem</u>

<u>Lecture 3 (Oct. 12)</u>

"To each independent, continuous one-parameter family of coordinate transformations which leave L invariant there corresponds an associated <u>conserved charge</u>."

(In fact, we only need require S is invariant. See § 3.14 of the notes.)

<u>Proof</u>: Let $q_\sigma \to \bar{q}_\sigma(q, \zeta)$ be our one-parameter family of transformations with continuous parameter ζ, and with $\bar{q}_\sigma(q, \zeta = 0) = q_\sigma \;\forall \sigma$. Invariance of $L \Rightarrow$

$$\frac{d}{d\zeta}\bigg|_{\zeta=0} L(\bar{q}, \dot{\bar{q}}, t) = \frac{\partial L}{\partial \bar{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\bar{q}}_\sigma} \frac{\partial \dot{\bar{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \frac{\partial \bar{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{d}{dt}\left(\frac{\partial \bar{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = \frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\right)_{\zeta=0}}^{\text{(conserved charge)}} = 0$$

evaluate along motion of system