Since $T+U=E$ is conserved, we have

$$\langle T\rangle = \frac{kE}{k+2} \quad , \quad \langle U\rangle = \frac{2E}{k+2}$$

Application: Keplerian orbits, $k=-1$

$$\langle T\rangle = -E \, , \quad \langle U\rangle = 2E \quad ; \quad E<0$$

Note then that a satellite losing energy due to frictional losses as it enters the atmosphere must increase its kinetic energy, i.e. it moves faster! (Think also about angular momentum conservation.)

Noether's Theorem

Lecture 3 (Oct. 12)

"To each independent, continuous one-parameter family of coordinate transformations which leave L invariant there corresponds an associated conserved charge."

(In fact, we only need require S is invariant. See §3.12.4 of the notes.)

Proof: Let $q_\sigma \to \bar{q}_\sigma(q,\zeta)$ be our one-parameter family of transformations with continuous parameter ζ, and with $\bar{q}_\sigma(q,\zeta=0) = q_\sigma \;\forall\sigma$. Invariance of $L \Rightarrow$

$$\frac{d}{d\zeta}\bigg|_{\zeta=0} L(\bar{q},\dot{\bar{q}},t) = \frac{\partial L}{\partial \bar{q}_\sigma}\frac{\partial \bar{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\bar{q}}_\sigma}\frac{\partial \dot{\bar{q}}_\sigma}{\partial\zeta}\bigg|_{\zeta=0}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{\partial \bar{q}_\sigma}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\frac{d}{dt}\left(\frac{\partial \bar{q}_\sigma}{\partial\zeta}\right)_{\zeta=0} = \frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\frac{\partial \bar{q}_\sigma}{\partial\zeta}\right)_{\zeta=0}}^{\text{(conserved charge)}} = 0$$

evaluate along motion of system

Thus, $\Lambda = \sum_{\sigma=1}^{n} \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \bar{q}_\sigma}{\partial \zeta}\Big|_{\zeta=0} = \sum_{\sigma=1}^{n} p_\sigma \frac{\partial \bar{q}_\sigma}{\partial \zeta}\Big|_{\zeta=0}$ is conserved!

Examples

- $L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - U(y)$. Then let

$$\left.\begin{aligned} \bar{x}(x,y,\zeta) &= x + \zeta \\ \bar{y}(x,y,\zeta) &= y \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{\bar{x}} &= \dot{x} \\ \dot{\bar{y}} &= \dot{y} \end{aligned}$$

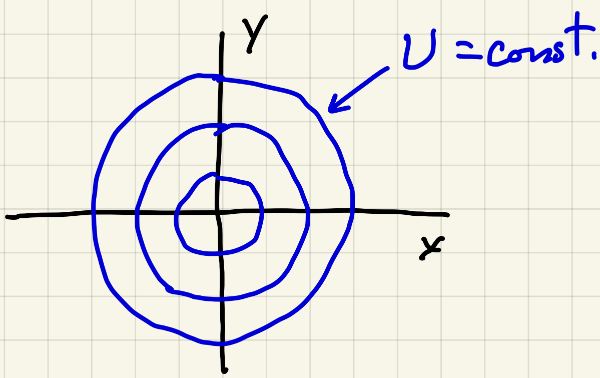
Clearly $\frac{d}{d\zeta} L(\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}) = 0$, and the associated conserved charge is

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{\partial \bar{x}}{\partial \zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \bar{y}}{\partial \zeta}\Big|_{\zeta=0} = \frac{\partial L}{\partial \dot{x}} = p_x$$

i.e. $p_x = m\dot{x}$ is a "constant of the motion".

- $L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - U(\sqrt{x^2+y^2})$

$$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) - U(\rho)$$



Define $\bar{\rho}(\rho, \phi, \zeta) = \rho$

$\bar{\phi}(\rho, \phi, \zeta) = \phi + \zeta$

Again $dL/d\zeta = 0$ and we have

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}} \frac{\partial \bar{\rho}}{\partial \zeta}\Big|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{\partial \bar{\phi}}{\partial \zeta}\Big|_{\zeta=0}$$

$$= p_\phi = m\rho^2 \dot{\phi} \quad \text{(angular momentum conserved)}$$

In Cartesian coordinates, this invariance is expressed as

$$\left.\begin{aligned}\bar{x}(\zeta) &= x\cos\zeta - y\sin\zeta \\ \bar{y}(\zeta) &= x\sin\zeta + y\cos\zeta\end{aligned}\right\} \quad \frac{\partial \bar{x}}{\partial \zeta} = -\bar{y}\,, \quad \frac{\partial \bar{y}}{\partial \zeta} = +\bar{x}$$

$$\begin{aligned}\Lambda &= \frac{\partial L}{\partial \dot{x}}\frac{\partial \bar{x}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}}\frac{\partial \bar{y}}{\partial \zeta}\bigg|_{\zeta=0} \\ &= m\dot{x}(-y) + m\dot{y}(+x) = m(x\dot{y} - y\dot{x}) \\ &= \hat{z}\cdot\vec{\rho}\times(m\dot{\vec{\rho}}) = m\rho^2\dot{\phi} = p_\phi\end{aligned}$$

The Hamiltonian

Recall $H(q,p,t) = \sum_\sigma p_\sigma \dot{q}_\sigma - L$. We showed earlier that

$$dH = \sum_\sigma \left(\dot{q}_\sigma \, dp_\sigma - \dot{p}_\sigma \, dq_\sigma\right) - \frac{\partial L}{\partial t}\, dt$$

and therefore

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}\,, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \text{(Hamilton's eqns)}$$

as well as

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

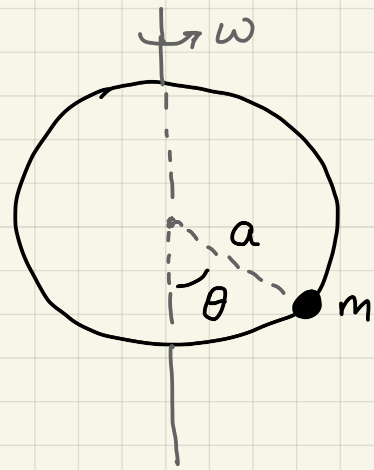
- For $L = \frac{1}{2}m\dot{x}^2 - U(x)$, $p = m\dot{x}$ and $H = \frac{p^2}{2m} + U(x)$
- Read §§3.12.4, 3.13.2

If infinitesimal transformation $\delta t = A(q,t)\,\delta\zeta$, $\delta q_\sigma = B_\sigma(q,t)\,\delta\zeta$ leaves action $\int_{\bar{t}_1}^{\bar{t}_2} d\bar{t}\, L(\bar{q}, \dot{\bar{q}}, \bar{t})$ invariant, then

$\Lambda = -H(q,p,t)\,A(q,t) + p_\sigma B_\sigma(q,t)$ is conserved.

Example : Bead on a rotating hoop

Angular velocity about $\hat{z}$-axis is fixed to be ω. Thus



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$
$$= \frac{1}{2} ma^2(\dot{\theta}^2 + \omega^2\sin^2\theta)$$

$$U = mga\,(1-\cos\theta)$$

Thus, $p_\theta = \dfrac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}$ and

$$H = \dot{\theta}p_\theta - L$$
$$= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\,(1-\cos\theta)$$

NB : $H \neq T+U$ because T not homogeneous of degree 2 in $\dot{\theta}$.

Now we express $H(\theta, p_\theta)$:

$$H = \frac{p_\theta^2}{2ma^2} - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta)$$
$$= \frac{p_\theta^2}{2I} + U_{eff}(\theta)$$

where $I = ma^2$ = moment of inertia, and

$$U_{eff}(\theta) = -\frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta)$$

Hamilton's equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \quad , \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial U_{eff}}{\partial \theta}$$

Thus

$$I\ddot{\theta} = -\frac{\partial U_{eff}}{\partial \theta} \Rightarrow \ddot{\theta} = -\frac{1}{I}\frac{\partial U_{eff}}{\partial \theta} = -u'(\theta)$$

Define $\omega_0 \equiv (g/a)^{1/2}$ so

$$u(\theta) \equiv \frac{U_{eff}(\theta)}{I} = (1-\cos\theta)\,\omega_0^2 - \tfrac{1}{2}\sin^2\theta\,\omega^2$$

Equilibrium is achieved when $u'(\theta) = 0$:

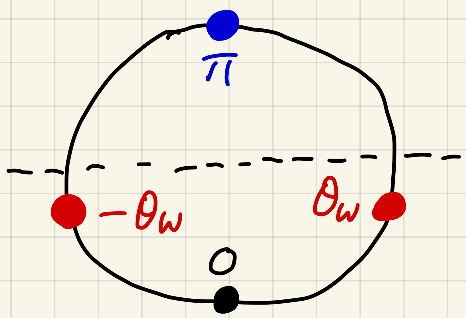
$$u'(\theta) = \omega_0^2 \sin\theta - \omega^2 \sin\theta\cos\theta$$

with solutions

$$\theta^* = 0 \;,\; \theta^* = \pi \;,\; \theta^* = +\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) ,\; \theta^* = -\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right)$$
$$\equiv \pm\theta_\omega \;(\text{if } \omega^2 > \omega_0^2)$$

To assess stability, write $\theta = \theta^* + \delta\theta$, and

$$\delta\ddot{\theta} = -u''(\theta^*)\cdot\delta\theta$$



θ^* stable $\Rightarrow u''(\theta^*) > 0$

θ^* unstable $\Rightarrow u''(\theta^*) < 0$

$$\Omega_{osc} = \sqrt{u''(\theta^*)}$$

$$u''(\theta) = \omega_0^2\cos\theta - \omega^2\cos(2\theta)$$

$$= \begin{cases} \omega_0^2 - \omega^2 \text{ at } \theta^* = 0 \text{ — stable for } \omega^2 < \omega_0^2 \\ -\omega_0^2 - \omega^2 \text{ at } \theta^* = \pi \text{ — always unstable} \\ \omega^2 - \dfrac{\omega_0^4}{\omega^2} \text{ at } \theta^* = \pm\theta_\omega \; (\omega^2 > \omega_0^2) \text{ — stable for } \omega^2 > \omega_0^2 \end{cases}$$

Charged particle in EM fields

charge

Potential energy : $U(\vec{x}, \dot{\vec{x}}) = q\,\phi(\vec{x},t) - \frac{q}{c} A(\vec{x},t)\cdot\dot{\vec{x}}$

Kinetic energy : $T(\dot{\vec{x}}) = \frac{1}{2} m \dot{\vec{x}}^2$ as usual

EM potentials : scalar $\phi(\vec{x},t)$ and vector $\vec{A}(\vec{x},t)$

EM fields :

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Thus the Lagrangian is

$$L(\vec{x},\dot{\vec{x}},t) = \frac{1}{2} m \dot{\vec{x}}^2 - q\,\phi(\vec{x},t) + \frac{q}{c}\vec{A}(\vec{x},t)\cdot\dot{\vec{x}}$$

Canonical momentum : $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + \frac{q}{c}\vec{A}(\vec{x},t)$

NB : the <u>dynamical momentum</u> is $m\dot{\vec{x}} = \vec{p} - \frac{q}{c}\vec{A}$

Let's find the Hamiltonian $H(\vec{x},\vec{p},t)$:

$$\begin{aligned} H(\vec{x},\vec{p},t) &= \vec{p}\cdot\dot{\vec{x}} - L \\ &= \left(m\dot{\vec{x}}^2 + \frac{q}{c}\vec{A}\cdot\dot{\vec{x}}\right) - \left(\frac{1}{2}m\dot{\vec{x}}^2 - q\phi + \frac{q}{c}\vec{A}\cdot\dot{\vec{x}}\right) \\ &= \frac{1}{2}m\dot{\vec{x}}^2 + q\phi \end{aligned}$$

Thus, $H(\vec{x},\vec{p},t) = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2 + q\,\phi(\vec{x},t)$

If $\frac{\partial\phi}{\partial t} = 0$ and $\frac{\partial A}{\partial t} = 0$ then $\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$

and $H(\vec{x}(t), \vec{p}(t))$ is a constant of the motion.

Equations of motion: recall $L = \frac{1}{2} m \dot{\vec{x}}^2 - q\phi + \frac{q}{c} \vec{A} \cdot \dot{\vec{x}}$

EL eqns: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}$

$$\frac{d}{dt}\left(m\dot{x}^\alpha + \frac{q}{c}A^\alpha\right) = m\ddot{x}^\alpha + \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{q}{c}\frac{\partial A^\alpha}{\partial t}$$

$$\frac{\partial L}{\partial x^\alpha} = -q\frac{\partial \phi}{\partial x^\alpha} + \frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\dot{x}^\beta$$

Thus,

$$m\ddot{x}^\alpha + \frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta + \frac{q}{c}\frac{\partial A^\alpha}{\partial t} = -q\frac{\partial \phi}{\partial x^\alpha} + \frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\dot{x}^\beta$$

$$m\ddot{x}^\alpha = -q\frac{\partial \phi}{\partial x^\alpha} - \frac{q}{c}\frac{\partial A^\alpha}{\partial t} - \frac{q}{c}\left(\frac{\partial A^\alpha}{\partial x^\beta} - \frac{\partial A^\beta}{\partial x^\alpha}\right)\dot{x}^\beta$$

Now $B^\gamma = \epsilon_{\mu\nu\gamma}\,\partial_\mu A^\nu$, so

$$\begin{aligned}\epsilon_{\alpha\beta\gamma} B^\gamma &= \epsilon_{\alpha\beta\gamma}\,\epsilon_{\mu\nu\gamma}\,\partial_\mu A^\nu \\ &= \left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right)\partial_\mu A^\nu \\ &= \frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\end{aligned}$$

and we have

$$m\ddot{x}^\alpha = -q\frac{\partial \phi}{\partial x^\alpha} - \frac{q}{c}\frac{\partial A^\alpha}{\partial t} + \frac{q}{c}\epsilon_{\alpha\beta\gamma}\,\dot{x}^\beta B^\gamma$$

or in vector form,

$$\begin{aligned}m\ddot{\vec{x}} &= -q\vec{\nabla}\phi - \frac{q}{c}\frac{\partial \vec{A}}{\partial t} + \frac{q}{c}\dot{\vec{x}} \times \vec{B} \\ &= q\vec{E} + \frac{q}{c}\dot{\vec{x}} \times \vec{B} \qquad \text{(Lorentz force law)}\end{aligned}$$

Hamilton's equations of motion:

$$H(\vec{x},\vec{p},t) = \frac{1}{2m}\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2 + q\phi$$

- $$\dot{x}^\alpha = +\frac{\partial H}{\partial p^\alpha} = \frac{1}{m}\left(p^\alpha - \frac{q}{c}A^\alpha\right)$$
- $$\dot{p}^\alpha = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{m}\left(p^\beta - \frac{q}{c}A^\beta\right)\left(-\frac{q}{c}\frac{\partial A^\beta}{\partial x^\alpha}\right) - q\frac{\partial \phi}{\partial x^\alpha}$$

Thus,

$$m\dot{x}^\alpha = p^\alpha - \frac{q}{c}A^\alpha$$

$$\dot{p}^\alpha = \frac{q}{mc}\overbrace{\left(p^\beta - \frac{q}{c}A^\beta\right)}^{m\dot{x}^\beta}\frac{\partial A^\beta}{\partial x^\alpha} - q\frac{\partial \phi}{\partial x^\alpha}$$

Take the time derivative of the first equation:

$$\begin{aligned}
m\ddot{x}^\alpha &= \dot{p}^\alpha - \frac{q}{c}\frac{dA^\alpha}{dt} \\
&= \left(\frac{q}{c}\dot{x}^\beta\frac{\partial A^\beta}{\partial x^\alpha} - q\frac{\partial \phi}{\partial x^\alpha}\right) - \left(\frac{q}{c}\frac{\partial A^\alpha}{\partial x^\beta}\dot{x}^\beta - \frac{q}{c}\frac{\partial A^\alpha}{\partial t}\right) \\
&= q\left(-\frac{\partial \phi}{\partial x^\alpha} - \frac{1}{c}\frac{\partial A^\alpha}{\partial t}\right) + \frac{q}{c}\dot{x}^\beta\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\right)
\end{aligned}$$

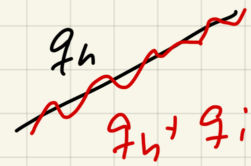
$$\Rightarrow \quad m\ddot{\vec{x}} = q\vec{E} + \frac{q}{c}\dot{\vec{x}}\times\vec{B}$$

Again, we obtain the Lorentz force law.

Fast Perturbations : Rapidly Oscillating Fields

Consider an oscillating force $F(t) = F_0 \sin\omega t$. Newton's 2^{nd} law then says $m\ddot{q} = F\sin\omega t$, the solution of which is

$$q(t) = \underbrace{a + bt}_{\substack{q_h(t) \\ \text{(homogeneous)}}} - \underbrace{\omega^{-2} F_0 \sin\omega t}_{\substack{q_i(t) \\ \text{(inhomogeneous)}}}$$



Note that $q_i(t) \propto \omega^{-2}$ is very small as $\omega \to \infty$.
Now consider the time-dependent Hamiltonian

$$H(q,p,t) = H^0(q,p,t) + \tilde{V}(q)\cos(\omega t)$$

The external force is then $F(q,t) = -\tilde{V}'(q)\cos(\omega t)$. We now separate the motion $\{q(t), p(t)\}$ into slow components $\{Q(t), P(t)\}$ and fast components $\{\xi(t), \pi(t)\}$:

$$q(t) = Q(t) + \xi(t)$$
$$p(t) = P(t) + \pi(t)$$

$$H = H^0(Q+\xi, P+\pi) + \tilde{V}(Q+\xi)\cos(\omega t)$$

We further assume that ξ and π are small, and we expand in these quantities:

$$\dot{Q} + \dot{\xi} = \frac{\partial H}{\partial P} = \frac{\partial H^0}{\partial P} + \left(\frac{\partial^2 H^0}{\partial P^2}\pi + \frac{\partial^2 H^0}{\partial Q \partial P}\xi\right) + \frac{1}{2}\left(\frac{\partial^3 H^0}{\partial P^3}\pi^2 + 2\frac{\partial^2 H^0}{\partial Q \partial P^2}\xi\pi + \frac{\partial^2 H^0}{\partial Q^2 \partial P}\xi^2\right) + \dots$$

$$H^0(Q+\xi, P+\pi) = H^0 + \frac{\partial H^0}{\partial Q}\xi + \frac{\partial H^0}{\partial P}\pi + \frac{1}{2}\frac{\partial^2 H^0}{\partial Q^2}\xi^2 + \dots$$

$$\dot{P} + \dot{\pi} = -\frac{\partial H}{\partial Q} = -\frac{\partial H^0}{\partial Q} - \left(\frac{\partial^2 H^0}{\partial Q^2} \zeta + \frac{\partial^2 H^0}{\partial Q \partial P} \pi \right) - \frac{1}{2} \left(\frac{\partial^3 H^0}{\partial Q^3} \zeta^2 \right.$$

$$\left. + 2 \frac{\partial^3 H^0}{\partial Q^2 \partial P} \zeta \pi + \frac{\partial^3 H^0}{\partial Q \partial P^2} \pi^2 \right) - \frac{\partial \tilde{V}}{\partial Q} \cos(\omega t) - \frac{\partial^2 \tilde{V}}{\partial Q} \zeta \cos(\omega t) + \dots$$

We can pick out from these equations the fast dynamics:

$$\dot{\zeta} = H^0_{QP} \zeta + H^0_{PP} \pi + \dots$$

$$\dot{\pi} = -H^0_{QQ} \zeta - H^0_{QP} \pi - \tilde{V}_Q \cos(\omega t) + \dots$$

where $H^0_{QQ} \equiv \dfrac{\partial^2 H^0}{\partial Q^2}$, $H^0_{QP} \equiv \dfrac{\partial^2 H^0}{\partial Q \partial P}$, etc.

We have ignored terms oscillating with frequencies near $0, 2\omega, 3\omega$, etc. The slow dynamics are obtained by <u>averaging</u> over the fast dynamics, viz.

$$\dot{Q} = H^0_P + \frac{1}{2} H^0_{QQP} \langle \zeta^2 \rangle + H^0_{QPP} \langle \zeta \pi \rangle + \frac{1}{2} H^0_{PPP} \langle \pi^2 \rangle + \dots$$

$$\dot{P} = -H^0_Q - \frac{1}{2} H^0_{QQQ} \langle \zeta^2 \rangle - H^0_{QQP} \langle \zeta \pi \rangle - \frac{1}{2} H^0_{QPP} \langle \pi^2 \rangle - \tilde{V}_{QQ} \langle \zeta \cos(\omega t) \rangle + \dots$$

We solve the fast dynamics by writing $\tilde{V}_Q \cos(\omega t) = \operatorname{Re} \tilde{V}_Q e^{-i\omega t}$, $\zeta(t) = \operatorname{Re} \zeta_0 e^{-i\omega t}$, $\pi(t) = \operatorname{Re} \pi_0 e^{-i\omega t}$ and inverting

$$\begin{pmatrix} H^0_{QP} + i\omega & H^0_{PP} \\ -H^0_{QQ} & -H^0_{QP} + i\omega \end{pmatrix} \begin{pmatrix} \zeta_0 \\ \pi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{V}_Q \end{pmatrix}$$

We obtain

$$\zeta(t) = \omega^{-2} H^0_{PP} \tilde{V}_Q \cos\omega t + O(\omega^{-4})$$

$$\pi(t) = -\omega^{-2} H^0_{QP} \tilde{V}_Q \cos\omega t - \omega^{-1} \tilde{V}_Q \sin\omega t + O(\omega^{-3})$$

Now we average, using $\langle \cos^2\omega t \rangle = \langle \sin^2 \omega t \rangle = \frac{1}{2}$ and $\langle \cos\omega t \sin\omega t \rangle = 0$. We obtain

$$\langle \zeta^2(t) \rangle = \frac{1}{2}\omega^{-4}\left(H^0_{PP}\tilde{V}_Q\right)^2 + \dots \quad \langle \zeta(t)\pi(t) \rangle = \frac{1}{2}\omega^{-4} H^0_{PP} H^0_{QQ} \tilde{V}_Q^2 + \dots$$

$$\langle \pi^2(t) \rangle = \frac{1}{2}\omega^{-2}\tilde{V}_Q^2 + \frac{1}{2}\omega^{-4}\left(H^0_{QP}\tilde{V}_Q\right)^2 + \dots$$

$$\langle \zeta(t) \cos\omega t \rangle = \frac{1}{2}\omega^{-2} H^0_{PP} \tilde{V}_Q + \dots$$

Plugging into the slow equations for $\dot{Q}$ and $\dot{P}$, we have

$$\dot{Q} = H^0_P + \frac{1}{4}\omega^{-2} H^0_{PPP} \tilde{V}_Q^2 + \dots$$

$$\dot{P} = -H^0_Q - \frac{1}{4}\omega^{-2} H^0_{QPP} \tilde{V}_Q^2 - \frac{1}{2}\omega^{-2} H^0_{PP} \tilde{V}_Q \tilde{V}_{QQ} + \dots$$

which may be written as

$$\dot{Q} = \frac{\partial K}{\partial P} \quad , \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

where the effective Hamiltonian is

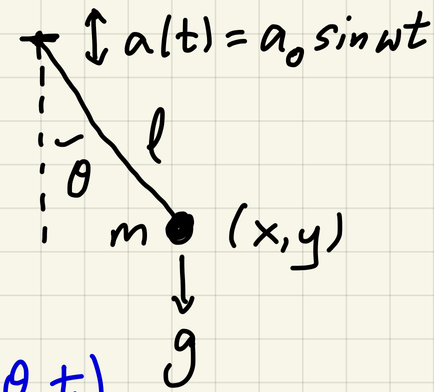
$$K(Q,P) = H^0(Q,P) + \frac{1}{4\omega^2}\frac{\partial^2 H^0}{\partial P^2}\left(\frac{\partial \tilde{V}}{\partial Q}\right)^2 + O(\omega^{-4})$$

Example: pendulum with oscillating support

Coordinates of mass m:

$$x = \ell \sin\theta$$

$$y = a(t) - \ell\cos\theta$$



The Lagrangian is

$$L = \frac{1}{2} m\ell^2\dot{\theta}^2 + m(g+\ddot{a})\ell\cos\theta + \frac{d}{dt}G(\theta,t)$$

we may ignore this

From this, we obtain the Hamiltonian

$$H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta - m\ell\ddot{a}\cos\theta$$

With $a(t) = a_0 \sin\omega t$, the perturbing potential is

$$\tilde{V}(\theta) = m\ell a_0\omega^2\cos\theta$$

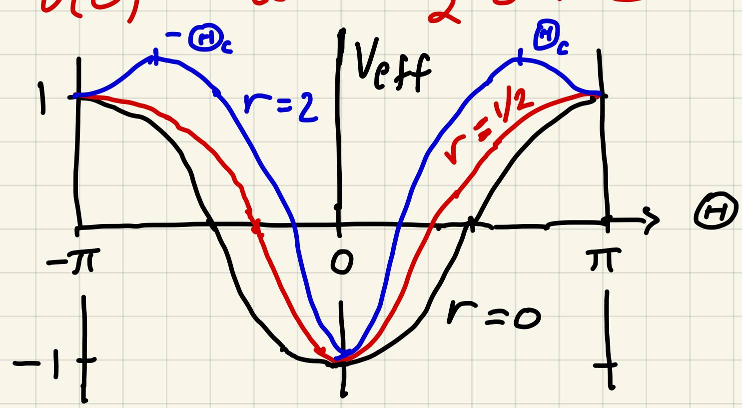
We write $\theta = \Theta + \xi$, $p_\theta = L + \pi$ and compute $K(\Theta, L)$:

$$K(\Theta, L) = \frac{L^2}{2m\ell^2} - mg\ell\cos\Theta + \frac{1}{4} m a_0^2\omega^2\sin^2\Theta$$

Thus, the effective potential is

$$V_{\text{eff}}(\Theta) = mg\ell\, v(\Theta) \quad , \quad v(\Theta) = -\cos\Theta + \frac{r}{2}\sin^2\Theta$$

with $r = \omega^2 a_0^2 / 2g\ell$.



$r<1$: $\Theta = 0$ stable, $\Theta = \pi$ unstable

$r>1$: $\Theta = 0, \pi$ stable, $\pm\Theta_c$ unstable