

Lecture I

Plasma on a Bulk - of - an - Envelope

⇒ Basic Ideas

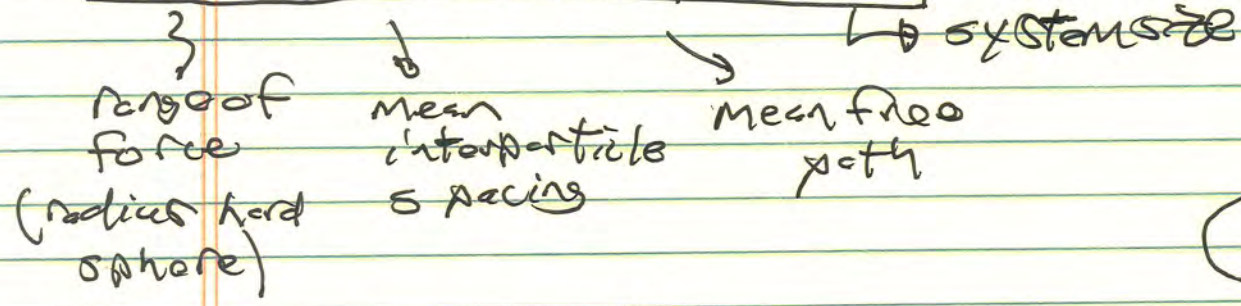
- ① What is a Plasma?
 - dilute gas ($T \gg U_{int}$)
 - net neutral ensemble of charged particles
 - inter-particle interaction Coulombic
 - long range
 - screening

②

Contrast: (Collisional) dilute gas

Scale ordering:

$$d < \bar{r} < \lambda_{mfp} < L$$



long mean free path:

$$d < \bar{r} < L \ll \lambda_{mfp}$$

Key Orderings:

① $nd^3 \ll 1 \Leftrightarrow (d/\bar{r})^3 \ll 1$
 → diluteness - particles free, mostly non-interacting

$$\textcircled{b} \quad l_{\text{mfp}} = 1/n\sigma$$

$$\approx \bar{r} (\bar{r}/d)^2$$

so

$$l_{\text{mfp}}/\bar{r} \approx (\bar{r}/d)^2 \gg 1$$

- collisions rare \Leftrightarrow diluteness

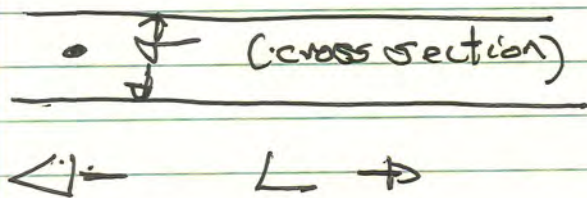
- contrast liquid: $l_{\text{mfp}} \sim \bar{r}$



$$l_{\text{mfp}}/d \sim (\bar{r}/d)^3 \rightarrow l_{\text{mfp}} \text{ exceeds force range}$$

Aside: Why $l_{\text{mfp}} \approx 1/n\sigma$?

\rightarrow define interaction cylinder for particle:



~~particle~~
particle sweeps
out cylinder of
area σ , length L

$$V_{\text{int}} = \sigma L$$

if $\alpha = \#$ collisions in cylinder of length L

then $\alpha = n \bar{V}_{int} = n \sigma L$

then mean length between collisions is

$l_{mp} = L / \alpha = 1 / n \sigma$

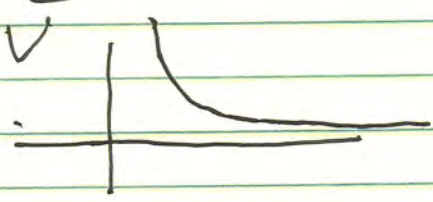
$l_{mp} = 1 / n \sigma$

② $\nu_c \sim v_{th} / l_{mp} \sim v_{th} n \sigma$
collision frequency

① basic diffusivity (characteristic)

$D \sim v_{th} l_{mp} \sim v_{th}^2 / \nu_c$

③ For plasmas:

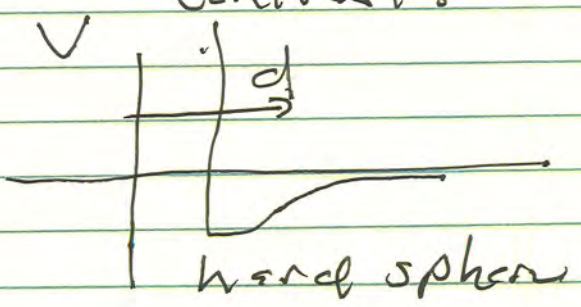


- force is Coulomb

$V = e\phi \sim 1/r$

scale free
long range

Contrast:

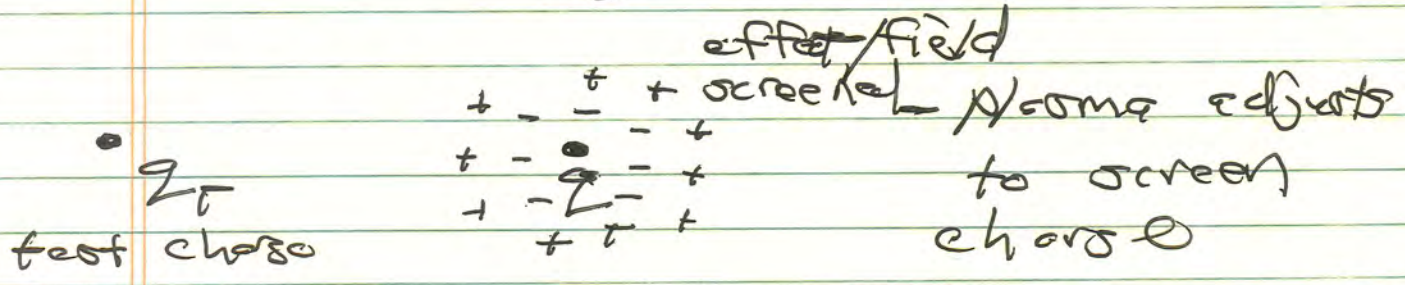


no "d" to characterize interaction.

- but screening occurs:

Screening? - Enter the Debye length!

↳



So $1/r \rightarrow e^{-r/\lambda_D}/r$

How? Solve Poisson's Equation with Screening Response!

$$\nabla^2 \phi = -4\pi \rho$$

$$= -4\pi \left[\rho_{\text{screening}} + 4\pi q_T \delta(\underline{x} - \underline{x}_T) \right]$$

$$= -4\pi n_0 e |e| \left[\frac{\delta n_i}{n_0} - \frac{\delta n_e}{n_0} \right] + 4\pi q_T \delta(\underline{x} - \underline{x}_T)$$

medic/plasma response

For density perturbation (response to test charge):

$$dn_i/n_0 = \exp[-|e|\phi/T_i] \quad v_{i0} \rightarrow 1$$

$$dn_e/n_0 = \exp[+|e|\phi/T_e]$$

charge neutrality

$$\nabla^2 \phi = -4\pi n_0 |e| \left[\cancel{1} - \frac{|e|\phi}{T_i} - \cancel{1} + \frac{|e|\phi}{T_e} \right] + 4\pi \rho_T \delta(x-x_0)$$

$$\nabla^2 \phi = 4\pi n_0 |e|^2 \left(\frac{1}{T_i} + \frac{1}{T_e} \right) \phi + 4\pi \rho_T \delta(x-x_0)$$

$x \rightarrow 0$

note: T needed to screen

$$\phi = \exp[-r/\lambda_D] / r$$

→ screened Coulomb field

$$1/\lambda_D^2 = 4\pi n_0 |e|^2 \left(1/T_e + 1/T_i \right)$$

screening, Debye length.

$$\lambda_D^2 = \left[4\pi n_0 e^2 \left(1/T_i + 1/T_e \right) \right]^{-1}$$

~~Debye~~ Debye length scale is key for plasma!

Which brings us the length scale ordering for plasmas:

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$$\bar{r} \ll \lambda_D \ll \lambda_{MF} \ll L \quad (\text{collisional})$$

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Key Point: A plasma has many particles in Debye sphere ↓

$$\text{de } n \lambda_D^3 \gg 1 \quad \Rightarrow \quad \lambda_D \gg \bar{r}$$

$$\frac{1}{n \lambda_D^3} \ll 1$$

↓
plasma parameter

Why? → Diluteness!

$$\Rightarrow T \gg e^2 / r$$

i.e. - thermal energy must exceed electrostatic energy

- dilute plasma \nleftrightarrow crystal (conceptual opposite)

- Checking:

$$T > e^2 / \bar{r}$$

$$\frac{n T \bar{r}}{n e^2} > 1$$

$$\Lambda_D^2 \bar{r} n > 1$$

$$\boxed{\Lambda_D^2 > \bar{r}^2} \quad \checkmark$$

Ⓟ Plasmas classical !! (but QM tricks handy)

Diluteness \Rightarrow Thermal fluctuations exceed QM fluctuations in energy

Thermal $\rightarrow T$ (energy)

$$\text{QM} \rightarrow E \sim p^2 / 2m$$

$$\sim \hbar^2 k^2 / 2m$$

$$\sim \hbar^2 / 2m \bar{r}^2 \sim \frac{\hbar^2}{m} n^{2/3}$$

so need:

$$T \gg \frac{\hbar^2}{m} n^{2/3}$$

have already for diluteness,

$$T \gg e^2 n^{1/3}$$

so, sufficient to have:

$$e^2 n^{1/3} > \hbar^2 n^{2/3} / m$$

$$\Rightarrow \frac{e^2 n^{1/3}}{\hbar^2 n^{2/3} / m} \approx \frac{me^2}{\hbar^2 n^{1/3}} > 1$$

$$\Rightarrow \frac{\overline{v}}{a_B} \gg 1$$

$$a_B = \hbar^2 / me^2$$

↓
Bohr ($\rightarrow 4\pi\hbar^2 / me^2$)
radius

- for classical plasma, mean interparticle spacing must exceed Bohr radius

- otherwise, quantum or degenerate plasma.

8 Conditions for classical dilute plasma:

$$n \lambda_D^3 \gg 1$$
$$n \lambda_D^3 / a_B \gg 1$$

$$\overline{v} < v_D < l_{max} < L$$

⑤ Frequencies / Resonances

- much of plasma physics deals with collective dynamics of plasmas

⇒ waves / instabilities and their evolution ...

- collective resonances: (electrostatic)
(EM generalization
clear)

$$\underline{D} \cdot \underline{D} = 4\pi \rho_{\text{ext}}$$

$$\underline{D} = \underline{E} + 4\pi \underline{P} = \epsilon(\omega) \underline{E}$$

↓
polarization

↑
dielectric function
$\epsilon(\omega, k)$ more general

$\epsilon(\omega) \rightarrow 0 \Rightarrow$ collective resonance

Now;

- high frequency, so electrons respond (low inertia), ions stationary (high inertia)

$$\underline{P} = n e \underline{dx}$$

$$m_e \frac{d^2 \underline{dx}}{dt^2} = e \underline{E}$$

$$\left(\underline{E} = \underline{E}_0 e^{i\omega t} \right)$$

$$- \omega^2 m_0 d\underline{x} = e \underline{E}$$

$$d\underline{x} = -e \underline{E} / m_0 \omega^2$$

$$4\pi \underline{P} = -\omega p_0^2 / \omega^2$$

$$\underline{D} = \underline{E} + 4\pi \underline{P} = \left(1 - \frac{\omega p_0^2}{\omega^2} \right) \underline{E} = \epsilon(\omega) \underline{E}$$

$$\omega p_0^2 = 4\pi n_0 e^2 / m_0 \rightarrow \text{plasma frequency}$$

\rightarrow space charge oscillation
(no spatial dispersion)

$\rightarrow dn \rightarrow d\underline{E} \rightarrow$ restoring force.

Fundamental
characteristic
frequency,

$$\frac{v_{th}^2}{\omega p_0^2} = \frac{1}{2}$$

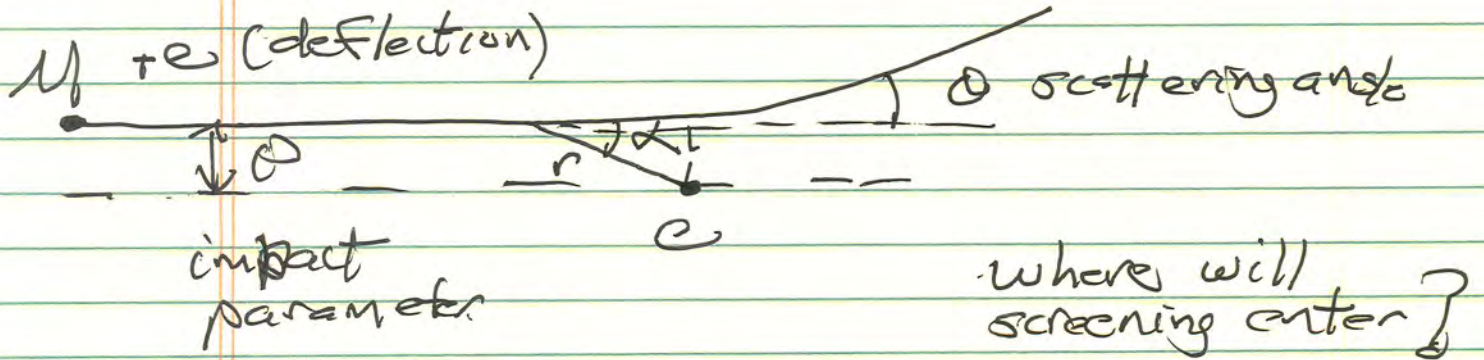
$$\text{Now, } \underline{D} \cdot \underline{D} = \left(1 - \omega p_0^2 / \omega^2 \right) \underline{D} \cdot \underline{E} = 4\pi \rho_{ext}$$

$$- \rho_{ext} = \rho_{ext} (\omega \omega p_0) \neq 0$$

$\Rightarrow E$ response large $\Rightarrow \epsilon \rightarrow \infty$
collective resonance \downarrow

⑥ Transport \leftrightarrow Coulomb Collisions

- back to Rutherford scattering;



- Cross-section?

- interaction is weak deflection
"momentum transfer cross section"

- many glancing collisions occur

- central force, so ~~momentum~~
 \vec{p} conserved, but direction changed.

for deflection:

$$M \Delta v_{\perp} = \Delta p_{\perp} = \int_{-\infty}^{\infty} dt \underline{F}_{\perp}$$

$$= \int_{-\infty}^{\infty} dt \frac{e^2}{r^2} \sin \theta$$

$$= \int_{-\infty}^{\infty} dt \frac{e^2}{r^2} \frac{b}{r}$$

$$\Delta p_{\perp} = e^2 \int_{-\infty}^{+\infty} \frac{\rho dt}{(\rho^2 + v^2 t^2)^{3/2}}$$

$$\sim e^2 \rho \int_{-\infty}^{+\infty} \frac{1}{\rho^3} \frac{dt}{(1 + v^2/\rho^2 t^2)^{3/2}}$$

$$\sim e^2 / \rho v$$

but $\Delta p_{\perp} \sim \mu v \sin \theta$ (defn.)
 $\sim \mu v \theta$ weak defl.

$$\theta \sim e^2 / \mu v^2 \rho$$

scattering angle
(weak deflection)

Now,

$$dV = \rho d\rho = d \left(\frac{e^2}{\mu v^2 \theta} \right)^2$$

interaction cylinder

$N_i B_i$ - Cross section heavily weights weak deflections
 = long range character of force (weak, but many interactions)
 \rightarrow cylinder.

10/11

$$d\sigma = \left(\frac{e^2}{uv^2} \right)^2 \frac{d\Omega}{\Omega^3}$$

↓
weak deflection
divergence.

n.b. small $\Omega \leftrightarrow$ large σ (more cylinder volume).

- long range character of Coulomb force ↓

- screening will enter as cut-off

Now seek:

Momentum transfer cross section

i.e. take out forward scattering, as no transfer.

10/11

$$d\sigma_{+} = (1 - \cos\theta) d\sigma$$

↓
removed forward scatt

$$\approx \Omega^2 \left(\frac{e^2}{uv^2} \right)^2 \frac{1}{\Omega^3}$$

$$d\sigma \approx \left(\frac{e^2}{uv^2} \right)^2 \frac{1}{\Omega}$$

$\frac{1}{\theta_0} \sim \frac{1}{\theta_0}$

- $\theta_0 \rightarrow$ low angle cut-off
- \rightarrow large θ
- \rightarrow screening

$$\theta_0 \sim e^2 / \mu v^2 \theta \sim e^2 / \mu v^2 \lambda_D$$

and:

max range force,
Debye Length.

$$\nabla_t \approx \left(\frac{e^2}{\mu v^2} \right)^2 \ln(1/\theta_0)$$

(resolved systematically by dynamic θ in orbits)

$$\ln \Lambda = \ln(1/\theta_0) = \ln(T \lambda_D / e^2)$$

$$\nabla_t \sim \left(\frac{e^2}{T} \right)^2 \ln \Lambda$$

screened
Coulomb
cross-section

Coulomb
Logarithm

Notes:

$$\nabla_t \sim \bar{r}^2 (e^2 / \bar{r} T)^2 \ln \Lambda$$

And now:

$$\boxed{\nabla_t \sim \bar{r}^2 \left(e^2 / \bar{r} T \right)^2 \ln \Delta}$$

$$\left(e^2 / \bar{r} T \right)^2 \rightarrow \left(\left[\frac{1}{n \lambda_D^3} \right]^{2/3} \right)^2$$

$$\sim \bar{r}^4 / \lambda_D^4$$

$$\sim \left(\frac{1}{n \lambda_D^3} \right)^{4/3}$$

$$\boxed{\nabla_t \sim \bar{r}^2 \left(\frac{1}{n \lambda_D^3} \right)^{4/3} \ln \Delta}$$

Check:

$$- \text{Im} \omega_p \sim 1 / n \nabla_t$$

so

$$\text{Im} \omega_p \approx \bar{r} \left(\lambda_D / \bar{r} \right)^4 / \ln \Delta$$

$$\text{Im} \omega_p / \omega_D \sim \left(\lambda_D / \bar{r} \right)^3 / \ln \Delta$$

$$\sim \left(n \lambda_D^3 \right) / \ln \Delta$$

$$n \lambda_D^3 \gg \ln A$$

$$\rightarrow \boxed{\lambda_{mp} \gg \lambda_D}$$

consistent
with screening

$$\Rightarrow \boxed{\bar{r} \ll \lambda_D \ll \lambda_{mp} \ll L}$$

collisional plasma
ordering

Further points about transport:

- no mass μ scaling in ∇_T , λ_{mp}
(except $\ln A$)

$$- \text{so } v_{col} \sim \mu^{-1/2} \rightarrow v_{th}$$

$$\tau_{col} \sim \mu^{1/2}$$

$$\tau_{ec} / \tau_{ic} \sim (m_e / m_i)^{1/2} \ll 1.$$

electron collisions
(e^+e^-) more rapid.

things as before:

→ thermal conductivity

$$\lambda \sim n v_{th} l_{mp}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ e_V & \text{bigger} & \text{under } \mu \\ & \text{electrons} & \end{matrix}$$

⇒ electrons control thermal conductivity

→ viscosity

$$\eta_i \sim M_i n v_{thi} l_{mpi}$$

$$l_{mpi} \sim \tau / n$$

$$(\eta_i \sim m_i n v_{thi} l_{mpi})$$

$$\eta \sim M v_{thi} / \tau_f$$

$$\eta \sim (M v_0 T)^{1/2} / \tau_f$$

ions control flow.