→ Chapman – Enskog Revisited (Moment Hierarchy View)

Recall BGK equation — approximates Boltzmann Eqn.
(Krook/Crook Model)

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f = -\lambda \left(f - f_{eq}\right) \qquad \lambda \sim \nu$$

f_{eq}: local eqbrm (Maxwellian)

For any function A, can define average:

$$\langle A \rangle = \int d\underline{v}\, A\, f(\underline{v}, \underline{x}, t) \Big/ \rho$$

so

$$\partial_t \left(\rho \langle A \rangle\right) = \int A \frac{\partial f}{\partial t} d\underline{v}$$

$$= \int A \left(-\underline{v} \cdot \underline{\nabla} f - \lambda (f - f_{eq})\right)$$

$$= -\underline{\nabla} \cdot \int \underline{v} A f\, d\underline{v} + \int (\underline{v} \cdot \underline{\nabla} A) f\, d\underline{v}$$

$$- \lambda \int A (f - f_{eq})\, d\underline{v}$$

or, using avg notation:

$$\partial_t \rho \langle A \rangle = -\underline{\nabla} \cdot (\rho \langle \underline{v} A \rangle) - \lambda \rho (\langle A \rangle - \langle A \rangle_{eq})$$

with f_{eq}.

[redacted]

so

$A = 1$

$$\partial_t \rho = -\underline{\nabla} \cdot (\rho \underline{V}) \quad + 0$$

continuity ✓

$A =$ [redacted] $\underline{v}$

$$\partial_t (\rho \underline{V}) = -\underline{\nabla} \cdot \int [redacted] \; \underline{v}\,\underline{v}\, f \, d\underline{v}$$

$\langle A \rangle$ kills $\langle \hat{v} \rangle$

$\langle v \rangle = \langle v \rangle_{eq}$

$$\underline{v} = \underline{V} + \hat{\underline{v}}$$

$$\partial_t (\rho \underline{V}) = -\underline{\nabla} \cdot (\rho \underline{V}\,\underline{V}) - \nabla \cdot \underline{\underline{T}}$$

$$\underline{\underline{T}}_{ij} = \int \hat{v}_i \hat{v}_j f \, d\underline{v}$$

stress tensor.

$$\underline{\underline{T}}_{ij} = P \,[redacted]\, \delta_{ij} + T^{visc}_{ij}$$

Pressure.

viscous ⇒ due velocity gradient (vanishes for uniform flow)

How to get $\sigma_{i,j}^{visc}$?

Have:

$$\sigma'_{i,j} = \sigma_{i,j}^{visc} = \sigma_{i,j} - P\delta_{i,j}$$

$$= \rho\left(\langle \tilde{V}_i \tilde{V}_j \rangle - \frac{1}{3} \langle |\tilde{V}|^2 \rangle \delta_{i,j} \right)$$

So how calculate ?

Now can crank out equations for second and third moments: → hierarchy

$$\partial_t \, \rho \langle \underline{V}\underline{V} \rangle - \nabla \cdot \rho \langle \underline{V}\underline{V}\underline{V} \rangle$$

$$- \lambda \rho \left\{ \langle \underline{V}\underline{V} \rangle - \langle \underline{V}\underline{V} \rangle_{eq} \right\}$$

and

$$\partial_t \, \rho \langle \underline{V}\underline{V}\underline{V} \rangle = - \underline{\nabla} \cdot \rho \langle \underline{V}\underline{V}\underline{V}\underline{V} \rangle$$

$$- \lambda \rho \left\{ \langle \underline{V}\underline{V}\underline{V} \rangle - \langle \underline{V}\underline{V}\underline{V} \rangle_{eq} \right\}$$

so have coupled hierarchy → no surprise,

And re-write:

$$\rho\langle V_i V_j\rangle = \rho V_i V_j + P\delta_{ij} + \sigma'_{ij}$$

so

$$\sigma'_{ij} = \rho\langle V_i V_j\rangle - \rho\langle V_i V_j\rangle_{eq}$$

and in stationary state

$$\underline{\nabla}\cdot\rho\langle \underline{V}\underline{V}\underline{V}\rangle = -\lambda\rho\left[\langle\underline{V}\underline{V}\rangle - \langle\underline{V}\underline{V}\rangle_{eq}\right]$$

but $\underline{V} = \overline{\underline{V}} + \tilde{\underline{V}}$, so

$$\rho\langle\underline{V}\underline{V}\underline{V}\rangle_{i,j,k} = \underset{1}{\rho V_i V_j V_k} + \underset{2}{\rho V_i\langle\tilde{V}_j\tilde{V}_k\rangle}$$
$$+ \underset{3}{\rho V_k\langle\tilde{V}_i\tilde{V}_j\rangle} + \underset{4}{\rho V_j\langle\tilde{V}_i\tilde{V}_k\rangle}$$
$$+ \underset{5}{\rho\langle\tilde{V}_i\tilde{V}_j\tilde{V}_k\rangle}$$

At local eqbrm: $5 \to 0$ (odd)
[truncating deviation from feq at higher order]

$$\rho\langle\underline{V}\underline{V}\underline{V}\rangle_{i,j,k} = \rho V_i V_j V_k + \rho V_i\frac{T}{m}\delta_{j,k}$$
$$+ \frac{\rho T}{m}V_j\delta_{i,k} + \frac{\rho T}{m}V_k\delta_{i,j}$$

Now plug into $\rho\langle\underline{V}\underline{V}\rangle$ eqn, after taking out mean. flow (Rey-stress), using E.Eqn.

$$\lambda\rho\left\{\langle \underline{v}\,\underline{v}\rangle - \langle \underline{v}\,\underline{v}\rangle_{eq}\right\}$$

$$= \lambda\, \sigma'_{ij}$$

$$= -\underline{\nabla}\cdot\Big[\rho V_i \frac{T}{m}\delta_{j,k} + \frac{\rho T}{m} V_j\, \delta_{i,k}$$

$$+ \frac{\rho T}{m} V_k\, \delta_{i,j}\Big]$$

all homogeneous but $\underline{V}$

$$\sigma'_{ij} = -\frac{\rho T}{\lambda m}\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right]$$

$$\boxed{\sigma'_{ij} = -\frac{nT}{\lambda}\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\,\underline{\nabla}\cdot\underline{V}\right]}$$

$\underline{\nabla}\cdot\underline{V} = 0.$

$$\boxed{\eta = nT/\lambda} \rightarrow \text{viscosity}$$

if $\lambda \sim n$
$(\nu \sim v_t n \sigma)$

$$\boxed{\eta \sim nT/n\sigma v_T}$$