5-2 The issue is: Can we use the simpler <u>classical</u> expression $p = (2mK)^{1/2}$ instead of the exact

<u>relativistic</u> expression $p = \dfrac{K\left(1 + \frac{2mc^2}{K}\right)^{1/2}}{c}$? As the relativistic expression reduces to

$p = (2mK)^{1/2}$ for $K << 2mc^2$, we can use the classical expression whenever $K << 1$ MeV because mc^2 for the electron is 0.511 MeV.

(a)   Here 50 eV $<< 1$ MeV, so $p = (2mK)^{1/2}$

$$\lambda = \frac{h}{p} = \frac{h}{\left[(2)\left(\frac{0.511 \text{ MeV}}{c^2}\right)(50 \text{ eV})\right]^{1/2}} = \frac{hc}{\left[(2)(0.511 \text{ MeV})(50 \text{ eV})\right]^{1/2}}$$

$$= \frac{1\,240 \text{ eV nm}}{\left[(2)\left(0.511 \times 10^6\right)(50)(\text{eV})^2\right]^{1/2}} = 0.173 \text{ nm}$$

(b)   As 50 eV $<< 1$ MeV , $p = (2mK)^{1/2}$

$$\lambda = \frac{hc}{\left[(2)\left(\frac{0.511 \text{ MeV}}{c^2}\right)\left(50 \times 10^3 \text{ eV}\right)\right]^{1/2}} = 5.49 \times 10^{-3} \text{ nm}.$$

As this is clearly a worse approximation than in (a) to be on the <u>safe</u> side use the

relativistic expression for p: $p = K\dfrac{\left(1 + \frac{2mc^2}{K}\right)^{1/2}}{c}$ so

$$\lambda = \frac{h}{p} = \frac{hc}{\left(K^2 + 2Kmc^2\right)^{1/2}} = \frac{1\,240 \text{ eV nm}}{\left[\left(50 \times 10^3\right)^2 + (2)\left(50 \times 10^3\right)\left(0.511 \times 10^6 \text{ eV}\right)\right]^{1/2}}$$

$$= 5.36 \times 10^{-3} \text{ nm} = 0.005\,36 \text{ nm}$$

5-7   A 10 MeV proton has $K = 10$ MeV $<< 2mc^2 = 1\,877$ MeV so we can use the classical
expression $p = (2mK)^{1/2}$. (See Problem 5-2)

$$\lambda = \frac{h}{p} = \frac{hc}{\left[(2)(938.3 \text{ MeV})(10 \text{ MeV})\right]^{1/2}} = \frac{1\,240 \text{ MeV fm}}{\left[(2)(938.3)(10)(\text{MeV})^2\right]^{1/2}} = 9.05 \text{ fm} = 9.05 \times 10^{-15} \text{ m}$$

5-8 $\lambda = \dfrac{h}{p} = \dfrac{h}{(2mK)^{1/2}} = \dfrac{h}{(2meV)^{1/2}} = \left[\dfrac{h}{(2me)^{1/2}}\right]V^{-1/2}$

$$\lambda = \left[\frac{6.626 \times 10^{-34} \text{ Js}}{\left(2 \times 9.105 \times 10^{-31} \text{ kg} \times 1.602 \times 10^{-19} \text{ C}\right)^{1/2}}\right]V^{-1/2}$$

$$\lambda = \left[\frac{1.226 \times 10^{-9} \text{ kg}^{1/2}\text{m}^2}{sC^{1/2}}\right]V^{-1/2}$$

5-10   As $\lambda = 2a_0 = 2(0.052\,9)$ nm $= 0.105\,8$ nm the energy of the electron is nonrelativistic, so
we can use

$$p = \frac{h}{\lambda} \text{ with } K = \frac{p^2}{2m};$$

$$K = \frac{h^2}{2m\lambda^2} = \frac{\left(6.626 \times 10^{-34} \text{ J} \cdot \text{s}\right)^2}{2\left(9.11 \times 10^{-31} \text{ kg}\right)\left(1.058 \times 10^{-10} \text{ m}\right)^2} = 21.5 \times 10^{-18} \text{ J} = 134 \text{ eV}$$

This is about ten times as large as the ground-state energy of hydrogen, which is 13.6 eV.

5-11   (a)   In this problem, the electron must be treated relativistically because we must use relativity when $pc \approx mc^2$. (See problem 5-5). the momentum of the electron is

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{10^{-14} \text{ m}} = 6.626 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

and $pc = 124$ MeV $\gg mc^2 = 0.511$ MeV. The energy of the electron is

$$E = \left(p^2 c^2 + m^2 c^4\right)^{1/2}$$

$$= \left[\left(6.626 \times 10^{-20} \text{ kg} \cdot \text{m/s}\right)^2 \left(3 \times 10^8 \text{ m/s}\right)^2 + \left(0.511 \times 10^6 \text{ eV}\right)^2 \left(1.602 \times 10^{-19} \text{ J/eV}\right)^2 \right]^{1/2}$$

$$= 1.99 \times 10^{-11} \text{ J} = 1.24 \times 10^8 \text{ eV}$$

so that $K = E - mc^2 \approx 124$ MeV.

(b)   The kinetic energy is too large to expect that the electron could be confined to a region the size of the nucleus.

5-12   Using $p = \frac{h}{\lambda} = mv$, we find that $v = \frac{h}{m\lambda} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{\left(9.11 \times 10^{-31} \text{ kg}\right)\left(1 \times 10^{-10} \text{ m}\right)} = 7.27 \times 10^6 \text{ m/s}$.

From the principle of conservation of energy, we get

$$eV = \frac{mv^2}{2} = \frac{\left(9.11 \times 10^{-31} \text{ kg}\right)\left(7.27 \times 10^6 \text{ m/s}\right)^2}{2} = 2.41 \times 10^{-17} \text{ J} = 151 \text{ eV}.$$

Therefore $V = 151$ V.

5-15   For a free, non-relativistic electron $E = \frac{m_e v_0^2}{2} = \frac{p^2}{2m_e}$. As the wavenumber and angular frequency of the electron's de Broglie wave are given by $p = \hbar k$ and $E = \hbar \omega$, substituting these results gives the dispersion relation $\omega = \frac{\hbar k^2}{2m_e}$. So $v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m_e} = \frac{p}{m_e} = v_0$.

5-17 $E^2 = p^2 c^2 + \left(m_e c^2\right)^2$

$E = \left[p^2 c^2 + \left(m_e c^2\right)^2\right]^{1/2}$. As $E = \hbar \omega$ and $p = \hbar k$

$$\hbar\omega = \left[\hbar^2 k^2 c^2 + \left(m_e c^2\right)^2\right]^{1/2} \text{ or}$$

$$\omega(k) = \left[k^2 c^2 + \frac{\left(m_e c^2\right)^2}{\hbar^2}\right]^{1/2}$$

$$v_p = \frac{\omega}{k} = \frac{\left[k^2 c^2 + \left(m_e c^2/\hbar\right)^2\right]^{1/2}}{k} = \left[c^2 + \left(\frac{m_e c^2}{\hbar k}\right)^2\right]^{1/2}$$

$$v_g = \frac{d\omega}{dk}\bigg|_{k_0} = \frac{1}{2}\left[k^2 c^2 + \left(\frac{m_e c^2}{\hbar}\right)^2\right]^{-1/2} 2kc^2 = \frac{kc^2}{\left[k^2 c^2 + \left(m_e c^2/\hbar\right)^2\right]^{1/2}}$$

$$v_p v_g = \left\{\frac{\left[k^2 c^2 + \left(m_e c^2/\hbar\right)^2\right]^{1/2}}{k}\right\}\left\{\left[k^2 c^2 + \left(m_e c^2/\hbar\right)^2\right]^{1/2}\right\} = c^2$$

Therefore, $v_g < c$ if $v_p > c$.

5-24   (a) $\Delta x \Delta p = \hbar$ so if $\Delta x = r$, $\Delta p \approx \dfrac{\hbar}{r}$

(b) $K = \dfrac{p^2}{2m_e} \approx \dfrac{(\Delta p)^2}{2m_e} = \dfrac{\hbar^2}{2m_e r^2}$

$U = -\dfrac{ke^2}{r}$

$E = \dfrac{\hbar^2}{2m_e r^2} - \dfrac{ke^2}{r}$

(c)   To minimize E take $\dfrac{dE}{dr} = -\dfrac{\hbar^2}{m_e r^3} + \dfrac{ke^2}{r^2} = 0 \Rightarrow r = \dfrac{\hbar^2}{m_e ke^2}$ = Bohr radius = a_0 . Then

$E = \left(\dfrac{\hbar}{2m_e}\right)\left(\dfrac{m_e ke^2}{\hbar^2}\right)^2 - ke^2 \left(\dfrac{m_e ke^2}{\hbar^2}\right) = \dfrac{m_e k^2 e^4}{2\hbar^2} = -13.6 \text{ eV}$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k)e^{ikx}\,dk = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha^2(k-k_0)^2} e^{ikx}\,dk = \frac{A}{\sqrt{2\pi}} e^{-\alpha^2 k_0^2} \int_{-\infty}^{+\infty} e^{-\alpha^2\left(k^2 - (2k_0 + ix/\alpha^2)k\right)}\,dk$$

. Now complete the square in order to get the integral into the standard form
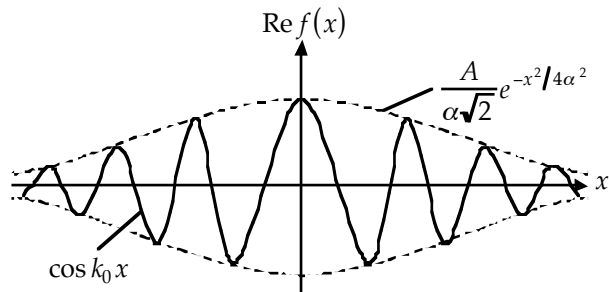
$$\int_{-\infty}^{+\infty} e^{-az^2}\,dz:$$

$$e^{-\alpha^2\left(k^2 - (2k_0 + ix/\alpha^2)k\right)} = e^{+\alpha^2\left(k_0 + ix/2\alpha^2\right)^2} e^{-\alpha^2\left(k - (k_0 + ix/2\alpha^2)\right)^2}$$

$$f(x) = \frac{A}{\sqrt{2\pi}} e^{-\alpha^2 k_0^2} e^{\alpha^2\left(k_0 + ix/2\alpha^2\right)^2} \int_{k=-\infty}^{+\infty} e^{-\alpha^2\left(k - (k_0 + ix/2\alpha^2)\right)^2}\,dk$$

$$= \frac{A}{\sqrt{2\pi}} e^{-x^2/4\alpha^2} e^{ik_0 x} \int_{z=-\infty}^{+\infty} e^{-\alpha^2 z^2}\,dz$$

where $z = k - \left(k_0 + \dfrac{ix}{2\alpha^2}\right)$. Since $\int_{z=-\infty}^{+\infty} e^{-\alpha^2 z^2}\,dz = \dfrac{\pi^{1/2}}{\alpha}$, $f(x) = \dfrac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2} e^{ik_0 x}$. The

real part of $f(x)$, $\mathrm{Re}\,f(x)$ is $\mathrm{Re}\,f(x) = \dfrac{A}{\alpha\sqrt{2}} e^{-x^2 4\alpha^2} \cos k_0 x$ and is a gaussian

envelope multiplying a harmonic wave with wave number k_0. A plot of $\mathrm{Re}\,f(x)$
is shown below:

Comparing $\dfrac{A}{\alpha\sqrt{2}}e^{-x^2 4\alpha^2}$ to $Ae^{-(x/2\Delta x)^2}$ implies $\Delta x = \alpha$.

(c)   By same reasoning because $\alpha^2 = \dfrac{1}{4\Delta k^2}$, $\Delta k = \dfrac{1}{2\alpha}$. Finally $\Delta x \Delta k = \alpha\left(\dfrac{1}{2\alpha}\right) = \dfrac{1}{2}$.

6-2   (a)   Normalization requires

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{A^2}{2}\right) \int_{-\frac{L}{4}}^{\frac{L}{4}} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

so $A = \dfrac{2}{\sqrt{L}}$.

(b) $$P = \int_{0}^{\frac{L}{8}} |\psi|^2 dx = A^2 \int_{0}^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right) dx = \left(\frac{4}{L}\right)\left(\frac{1}{2}\right) \int_{0}^{\frac{L}{8}} \left(1 + \cos\left(\frac{4\pi x}{L}\right)\right) dx$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{8}\right) + \left(\frac{2}{L}\right)\left(\frac{L}{4\pi}\right) \sin\left(\frac{4\pi x}{L}\right)\Big|_{0}^{\frac{L}{8}} = \frac{1}{4} + \frac{1}{2\pi} = 0.409$$

6-3   (a) $A \sin\left(\dfrac{2\pi x}{\lambda}\right) = A \sin\left(5 \times 10^{10} x\right)$ so $\left(\dfrac{2\pi}{\lambda}\right) = 5 \times 10^{10} \text{ m}^{-1}$, $\lambda = \dfrac{2\pi}{5 \times 10^{10}} = 1.26 \times 10^{-10} \text{ m}$.

(b) $p = \dfrac{h}{\lambda} = \dfrac{6.626 \times 10^{-34} \text{ Js}}{1.26 \times 10^{-10} \text{ m}} = 5.26 \times 10^{-24} \text{ kg m/s}$

(c) $\quad K = \dfrac{p^2}{2m} \quad m = 9.11 \times 10^{-31}$ kg

$$K = \frac{\left(5.26 \times 10^{-24} \text{ kg m/s}\right)^2}{\left(2 \times 9.11 \times 10^{-31} \text{ kg}\right)} = 1.52 \times 10^{-17} \text{ J}$$

$$K = \frac{1.52 \times 10^{-17} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 95 \text{ eV}$$

6-4   The time development of Ψ is given by Equation 6.8 or

$$\Psi(x, t) = \int a(k) e^{i\{kx - \omega(k)t\}} dk = \left(\frac{C\alpha}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty} e^{\{ikx - i\omega(k)t - \alpha^2 k^2\}} dk \, ,$$

with $\omega(k) = \dfrac{\hbar k^2}{2m}$ for a free particle of mass m. As in Example 6.3, the integral may be reduced

to a recognizable form by completing the square in the exponent. Since $\omega(k)t = \left(\dfrac{\hbar t}{2m}\right)k^2$, we

group this term together with $\alpha^2 k^2$ by introducing $\beta^2 = \alpha^2 + \dfrac{i\hbar t}{2m}$ to get

$$ikx - \omega(k)t - \alpha^2 k^2 = -\left(\beta k - \frac{ix}{2\beta}\right)^2 - \frac{x^2}{4\beta^2} \, .$$

Then, changing variables to $z = \beta k - \dfrac{ix}{2\beta}$ gives

$$\Psi(x, t) = \left(\frac{C\alpha}{\beta\sqrt{\pi}}\right) e^{-x^2/4\beta^2} \int_{-\infty}^{\infty} e^{-z^2} = \left(\frac{C\alpha}{\beta}\right) e^{-x^2/4\beta^2} \, .$$

     To interpret this result, we must recognize that β is complex and separate real and

imaginary parts. Thus, $\left|\beta^2\right|^2 = \left|\alpha^2 + \dfrac{i\hbar t}{2m}\right|^2 = \alpha^4 + \left(\dfrac{\hbar t}{2m}\right)^2$ and the exponent for Ψ is

$$\frac{x^2}{4\beta^2} = \frac{x^2\left(\alpha^2 - \frac{i\hbar t}{2m}\right)}{4\left|\beta^2\right|^2} = \frac{x^2}{4\left[\alpha^2 + \left(\frac{\hbar t}{2m\alpha}\right)^2\right]} + (\text{imaginary terms})$$

then

$$|\Psi(x, t)| = \frac{C\alpha}{\left(\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}} e^{\{-x^2/\left[4\{\alpha^2 + (\hbar t/2m\alpha)^2\}\right]\}} \, .$$

We see that apart from a phase factor, $\Psi(x, t)$ is still a gaussian but with amplitude diminished by $\dfrac{\alpha}{\left(\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}}$ and a width $\Delta x(t) = \left[\alpha^2 + \left(\dfrac{\hbar t}{2m\alpha}\right)^2\right]^{1/2}$ where $\alpha = \Delta x(0)$ is the initial width.