6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$ with $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator. Differentiating $\psi(x) = Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx} = -2\alpha x\psi(x) + C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2} = -\frac{2\alpha x d\psi}{dx} - 2\alpha\psi(x) - (2\alpha x)Ce^{-\alpha x^2} = (2\alpha x)^2\psi(x) - 6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires $(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$.

Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and $E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2C^2 \int x^2 e^{-2\alpha x^2} dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-25 At its limits of vibration $x = \pm A$ the classical oscillator has all its energy in potential form: $E = \frac{1}{2}m\omega^2 A^2$ or $A = \left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, then the corresponding amplitudes are $A_n = \left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$.

For the calculation of the average position $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi_0(x)|^2 dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand $x^2|\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\langle x^2\rangle = 2C_0^2\int_0^{\infty} x^2 e^{-ax^2} dx = 2C_0^2\left(\frac{1}{4a}\right)\left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\langle x^2\rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting $\langle p_x \rangle = 0$.

(b) To find $\langle p_x^2 \rangle$ we express the average energy as the sum of its kinetic and potential energy contributions: $\langle E \rangle = \left\langle \frac{p_x^2}{2m} \right\rangle + \langle U \rangle = \frac{\langle p_x^2 \rangle}{2m} + \langle U \rangle$. But energy is sharp in the oscillator ground state, so that $\langle E \rangle = E_0 = \frac{1}{2}\hbar\omega$. Furthermore, remembering that $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator, and using $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$ from Problem 6-32, gives $\langle U \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle = \frac{1}{4}\hbar\omega$. Then

$\langle p_x^2 \rangle = 2m(E_0 - \langle U \rangle) = 2m\left(\frac{\hbar\omega}{4}\right) = \frac{m\hbar\omega}{2}$.

(c) $\Delta p_x = \left(\langle p_x^2 \rangle - \langle p_x \rangle^2\right)^{1/2} = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$

6-34 From Problems 6-32 and 6-33, we have $\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$ and $\Delta p_x = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$. Thus,

$\Delta x \Delta p_x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\frac{m\hbar\omega}{2}\right)^{1/2} = \frac{\hbar}{2}$ for the oscillator ground state. This is the minimum uncertainty product permitted by the uncertainty principle, and is realized only for the ground state of the quantum oscillator.

6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i}\right)\frac{d}{dx}$ to each of the candidate functions yields

(a) $[p_x]\{A\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx)\}$

(b) $[p_x]\{A\sin(kx) - A\cos(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx) + A\sin(kx)\}$

(c) $[p_x]\{A\cos(kx) + iA\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{-A\sin(kx) + iA\cos(kx)\}$

(d) $[p_x]\{e^{ik(x-a)}\} = \left(\frac{\hbar}{i}\right)ik\{e^{ik(x-a)}\}$

In case (c), the result is a multiple of the original function, since

$$-A\sin(kx) + iA\cos(kx) = i\{A\cos(kx) + iA\sin(kx)\}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10$ MeV and $m = \frac{938.28\ \text{MeV}}{c^2}$. Using $\hbar = 197.3\ \text{MeV fm}/c\left(1\ \text{fm} = 10^{-15}\ \text{m}\right)$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[(2)(938.28\ \text{MeV}/c^2)(10\ \text{MeV})\right]^{1/2}} = 1.44\ \text{fm}.$$

7-2 (a) To the left of the step the particle is free with kinetic energy E and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x \le 0$$

To the right of the step the kinetic energy is reduced to $E - U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x \ge 0$$

with $D = 0$ for waves incident on the step from the left. At $x = 0$ both ψ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0) = A + B = C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A-B) = ik_2C.$$

(b) Eliminating C gives $A+B=\frac{k_1}{k_2}(A-B)$ or $A\left(\frac{k_1}{k_2}-1\right)=B\left(\frac{k_1}{k_2}+1\right)$. Thus,

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

$$T=1-R=\frac{4k_1k_2}{(k_1+k_2)^2}$$

(c) As $E\to U$, $k_2\to 0$, and $R\to 1$, $T\to 0$ (no transmission), in agreement with the result for any energy $E<U$. For $E\to\infty$, $k_1\to k_2$ and $R\to 0$, $T\to 1$ (perfect transmission) suggesting correctly that very energetic particles do not *see* the step and so are unaffected by it.

7-3 With $E=25\text{ MeV}$ and $U=20\text{ MeV}$, the ratio of wavenumber is $\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{25}{25-20}\right)^{1/2}=\sqrt{5}=2.236$. Then from Problem 7-2 $R=\frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)^2}=0.146$ and $T=1-R=0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$R=\left|\frac{B}{A}\right|^2=\frac{(k_1/k_2-1)^2}{(k_1/k_2+1)^2}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}.$$

The wavenumbers are those for electrons with kinetic energies $E=54.0\text{ eV}$ and $E-U=54.0\text{ eV}+10.0\text{ eV}=64.0\text{ eV}$:

$$\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{54\text{ eV}}{64\text{ eV}}\right)^{1/2}=0.918\,6.$$

Then, $R=\frac{(0.918\,6-1)^2}{(0.918\,6+1)^2}=1.80\times10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is $\frac{1}{T(E)}=1+\left[\frac{U^2}{4E(U-E)}\right]\sinh^2\alpha L$ with $\alpha=\frac{[2m(U-E)]^{1/2}}{\hbar}$. For $E<<U$, we find $(\alpha L)^2\approx\frac{2mUL^2}{\hbar^2}>>1$ by hypothesis. Thus, we may write $\sinh\alpha L\approx\frac{1}{2}e^{\alpha L}$. Also $U-E\approx U$, giving $\frac{1}{T(E)}\approx 1+\left(\frac{U}{16E}\right)e^{2\alpha L}\approx\left(\frac{U}{16E}\right)e^{2\alpha L}$ and a probability for transmission $P=T(E)=\left(\frac{16E}{U}\right)e^{-2\alpha L}$.

(b) Numerical Estimates: $(\hbar=1.055\times10^{-34}\text{ Js})$

1) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-21}$ J, $L = 10^{-10}$ m;

$$\alpha = \frac{\left[2m(U-E)\right]^{1/2}}{\hbar} = 5.12 \times 10^{8}\ \text{m}^{-1} \text{ and } e^{-2\alpha L} = 0.90$$

2) For $m = 9.11 \times 10^{-31}$ kg, $U - E = 1.60 \times 10^{-19}$ J, $L = 10^{-10}$ m;

$\alpha = 5.12 \times 10^{9}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.36$

3) For $m = 6.7 \times 10^{-27}$ kg, $U - E = 1.60 \times 10^{-13}$ J, $L = 10^{-15}$ m;

$\alpha = 4.4 \times 10^{14}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.41$

4) For $m = 8$ kg, $U - E = 1$ J, $L = 0.02$ m; $\alpha = 3.8 \times 10^{34}\ \text{m}^{-1}$ and

$e^{-2\alpha L} = e^{-1.5\times 10^{33}} \approx 0$

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$, the first approximation to the decay length δ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(30\ \text{MeV})\right]^{1/2}} = 0.415\,6\ \text{fm}.$$

This gives an effective width for the (infinite) well of $R + \delta = 9.415\,6$ fm, and a ground state energy $E_1 = \dfrac{\pi^2 (197.3\ \text{MeV fm}/c)^2}{2\left(3\,755.8\ \text{MeV}/c^2\right)(9.415\,6\ \text{fm})^2} = 0.577$ MeV. From this E we calculate $U - E = 29.42$ MeV and a new decay length

$$\delta = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(29.42\ \text{MeV})\right]^{1/2}} = 0.419\,7\ \text{fm}.$$

This, in turn, increases the effective well width to 9.419 7 fm and lowers the ground state energy to $E_1 = 0.576$ MeV. Since our estimate for E has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of E_1, the alpha particle in the ground state has speed $v_1 = \left(\dfrac{2E_1}{m}\right)^{1/2} = \left[\dfrac{2(0.576\ \text{MeV})}{\left(3\,755.8\ \text{MeV}/c^2\right)}\right]^{1/2} = 0.017\,5c$. In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \dfrac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E - U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9$ fm, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2\ \text{eV})\left(\frac{5.29 \times 10^4\ \text{fm}}{9\ \text{fm}}\right) = 28.14\ \text{MeV}.$$

From this we conclude that $U = -1.86$ MeV to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E = 4.05$ MeV has kinetic energy $4.05 + 1.86 = 5.91$ MeV inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m}\right)^{1/2} = \left[\frac{2(5.91)}{3\,755.8\ \text{MeV}/c^2}\right]^{1/2} = 0.056c\,.$$

Thus, we find for the collision frequency $f = \frac{v}{2R} = \frac{0.056c}{2(9\ \text{fm})} = 9.35\times10^{20}\ \text{Hz}$.