Physics 2D | Quiz 2 Solutions | Fall 2020

Problem 1

momentum conservation: $M v \gamma(v) = 2 m u \gamma(u) - m u \gamma(u)$

$\Rightarrow \quad M v \gamma(v) = m u \gamma(u)$

energy conservation: $M \gamma(v) c^2 = 3 m \gamma(u) c^2$

taking the ratio: $v = \frac{u}{3} \Rightarrow \boxed{u = 3v = 0.9c}$ (a)

(b) $m = \frac{\gamma(v)}{\gamma(u)} \frac{M}{3}$; $\gamma(v) = \frac{1}{\sqrt{1-0.3^2}} = 1.0483$

$\gamma(u) = \frac{1}{\sqrt{1-0.9^2}} = 2.2942$

$\Rightarrow m = \frac{1.0483}{2.2942} \frac{M}{3} \Rightarrow \boxed{m = 0.1523 M}$ (b)(i)

mass converted to energy: $\boxed{\Delta m = M - 3m = 0.5431 M}$ (b)(ii)

(c) Initial kinetic energy:

$K_i = (\gamma(v) - 1) M c^2 = \boxed{0.0483 Mc^2}$

Final kinetic energy: $K_f = 3(\gamma(u) - 1) m c^2 = \boxed{0.5913 Mc^2}$

Difference between final and initial kinetic energies:

$K_f - K_i = (0.5913 - 0.0483) M c^2 = \boxed{0.543 Mc^2}$ (c)

$\Rightarrow \quad \boxed{K_f - K_i = \Delta m c^2}$ as expected

Problem 2

$pc = 15\,MeV$ for electron and proton. $m_e = 0.511\,MeV/c^2$

$m_p = 938.26\,MeV/c^2$

Use $E = \sqrt{p^2c^2 + m^2c^4}$

$E_e = \sqrt{15^2 + 0.511^2}\ MeV = 15.0087\,MeV = K_e + m_ec^2$

$\Rightarrow$ electron kinetic energy: $\boxed{K_e = 14.498\,MeV}$ (a)

$E_p = \sqrt{15^2 + 938.26^2}\ MeV = 938.38\,MeV = K_p + m_pc^2$

$\Rightarrow$ proton kinetic energy: $\boxed{K_p = 0.12\,MeV}$ (a)

(b) From $E = \gamma mc^2$, $p = \gamma m u \Rightarrow \frac{p}{E} = \frac{u}{c^2} \Rightarrow \boxed{\frac{u}{c} = \frac{pc}{E}}$

electron: $\boxed{\frac{u_e}{c} = \frac{15\,MeV}{15.0087\,MeV} = 0.9994}$ (b)

proton: $\boxed{\frac{u_p}{c} = \frac{15\,MeV}{938.38\,MeV} = 0.016}$ (b)

(c) In classical mechanics, $p = mu$

electron: $u_e = \frac{p}{m_e} \Rightarrow \boxed{\frac{u_e}{c} = \frac{pc}{m_ec^2} = \frac{15}{0.511} = 29}$ very ≠ from relativistic

proton: $\boxed{\frac{u_p}{c} = \frac{pc}{m_pc^2} = \frac{15}{938.26} = 0.016}$ same as relativistic

Kinetic energy classically is $K = \frac{1}{2}mu^2 = \frac{(pc)^2}{2mc^2}$

$\boxed{K_e = \frac{(15)^2}{2 \times 0.511}\ MeV = 220\,MeV}$ very ≠ from relativistic

$\boxed{K_p = \frac{15^2}{2 \times 938.26}\ MeV = 0.12\,MeV}$ same as relativistic

$\boxed{\text{Classical result} \approx \text{relativistic result when } pc \ll mc^2}$

Alternative solution

$$p = m u \gamma(u) \Rightarrow u\gamma(u) = \frac{p}{m} \Rightarrow \frac{u/c}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{p}{mc} \Rightarrow$$

$$\Rightarrow \frac{u^2/c^2}{1-u^2/c^2} = \frac{p^2}{m^2c^2} \Rightarrow \frac{u^2}{c^2}\left(1+\frac{p^2}{m^2c^2}\right) = \frac{p^2}{m^2c^2} \Rightarrow$$

$$\Rightarrow \frac{u^2}{c^2} = \frac{p^2/m^2c^2}{1+p^2/m^2c^2} \Rightarrow \frac{u}{c} = \frac{\frac{pc}{mc^2}}{\sqrt{1+(pc/mc^2)^2}}$$

electron:

$$\frac{u_e}{c} = \frac{15/0.511}{\sqrt{1+(15/0.511)^2}} = 0.9994$$

$$\frac{u_p}{c} = \frac{15/938.26}{\sqrt{1+(15/938.26)^2}} = 0.016$$

Then, to get kinetic energy, get $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$,

then $K = (\gamma - 1)mc^2$.

Solution through Taylor expansion:

electron: $m_e c^2 \ll pc \Rightarrow$

$$E_e = \sqrt{p^2c^2+m_e^2c^4} = pc\sqrt{1+\frac{m_e^2c^4}{p^2c^2}} \simeq pc\left(1+\frac{m_e^2c^4}{2p^2c^2}\right) = pc + \frac{m_e^2c^4}{2pc}$$

$$\Rightarrow K_e = pc + \frac{(m_ec^2)^2}{2pc} - m_ec^2 = \left(15 + \frac{0.511^2}{30} - 0.511\right) \text{MeV} = \boxed{14.498 \text{ MeV}}$$

proton: $m_p c^2 \gg pc \Rightarrow$

$$E_p = \sqrt{p^2c^2+m_p^2c^4} = m_pc^2\sqrt{1+\frac{p^2c^2}{m_p^2c^4}} = m_pc^2\left(1+\frac{p^2c^2}{2m_p^2c^4}\right) \Rightarrow$$

$$\Rightarrow K_p = \cancel{m_pc^2} + \frac{p^2c^2}{2m_pc^2} - \cancel{m_pc^2} = \frac{15^2}{2\times 938.26} \text{MeV} = \boxed{0.12 \text{ MeV}}$$

Problem 3

$\lambda_m = 10{,}000\,\mathring{A}$ where maximum power is emitted

Wien's law: $\lambda_m T = \frac{hc}{4.96 k_B} \Rightarrow T = \frac{hc}{4.96\, k_B \lambda_m} \Rightarrow$

$$\Rightarrow \boxed{T = \frac{12{,}400 \times 11{,}600}{4.96 \times 10{,}000} K = 2{,}900\,K} \text{ (a)}$$

(b) Total power is $P = \sigma T^4 . A$, where A = surface area

For sphere, $A = 4\pi R^2 \Rightarrow P = \sigma T^4 \times 4\pi R^2 \Rightarrow$

$$\Rightarrow R = \left(\frac{P}{4\pi \sigma T^4}\right)^{1/2} \;;\; P = 5{,}000 W,\ \sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$$

$$\Rightarrow R = \left(\frac{5{,}000 W \times 10^8 m^2 K^4}{4\pi \times 5.67 \times W \times 2{,}900^4 K^4}\right)^{1/2} = 0.01 m \Rightarrow \boxed{R = 1 cm} \text{ (b)}$$

(c) Power is proportional to energy density u_λ:

$$u_\lambda = \frac{8\pi}{\lambda^4}\left(\frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1}\right) = \frac{8\pi}{\lambda^4} \bar{E}(\lambda, T) \Rightarrow$$

$$\Rightarrow \frac{u_\lambda(\lambda_1)}{u_\lambda(\lambda_2)} = \left(\frac{\lambda_2}{\lambda_1}\right)^4 \frac{\bar{E}(\lambda_1, T)}{\bar{E}(\lambda_2, T)} \qquad \boxed{\lambda_1 = 10\lambda_m,\ \lambda_2 = \frac{\lambda_m}{10}}$$

$\boxed{u_\lambda(\lambda_1) >> u_\lambda(\lambda_2) \text{ because } \bar{E}(\lambda_1, T) >> \bar{E}(\lambda_2, T)}$

$\bar{E}(\lambda_1, T) \approx k_B T$ classical result, since $\lambda_1 >> \lambda_m$

$$\bar{E}(\lambda_2, T) \approx \frac{hc}{\lambda_2} e^{-hc/\lambda_2 k_B T} = 49.6\, k_B T\, e^{-49.6} \Rightarrow$$

$$\frac{u_\lambda(\lambda_1)}{u_\lambda(\lambda_2)} = \left(\frac{\lambda_m/10}{10\lambda_m}\right)^4 \frac{k_B T}{49.6\, k_B T e^{-49.6}} = 10^{-8} \cdot \frac{e^{49.6}}{49.6} \sim 7 \times 10^{11}$$

$$\Rightarrow \boxed{\frac{u_\lambda(100{,}000\,\mathring{A})}{u_\lambda(1{,}000\,\mathring{A})} \sim 10^{12}} \text{ (c)}$$