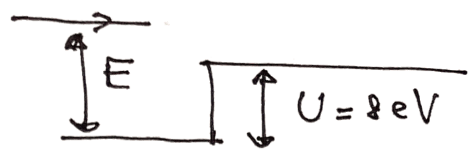
Physics 2D — Quiz 5 Solutions — Fall 2020

Problem 1



reflection coefficient is

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$R = \frac{1}{4} \Rightarrow \qquad k_2 = \sqrt{\frac{2m(E-U)}{\hbar^2}}$$

$$(k_1 + k_2)^2 = 4(k_1 - k_2)^2 \Rightarrow k_1 + k_2 = 2(k_1 - k_2) \Rightarrow$$

$$\Rightarrow \boxed{k_1 = 3k_2} \qquad k_1^2 = \frac{2mE}{\hbar^2} = 9k_2^2 = 9 \cdot \frac{2m}{\hbar^2}(E-U) \Rightarrow$$

$$E = 9(E-U) \Rightarrow 8E = 9U \Rightarrow E = \frac{9}{8}U \Rightarrow \boxed{E = 9\,eV}$$

$\Rightarrow$ kinetic energy of reflected electrons = 9 eV; kinetic energy of transmitted electrons = 1 eV (a)

(b) the transmitted electrons move 3 times slower than the reflected ones

(c) Electrons are incident from right to left.
Formula for R is the same. Therefore,

kinetic energy of reflected electrons = 1 eV; kinetic energy of transmitted electrons = 9 eV (c)

Problem 2

energy in 3D box with $L_3 = 2L$, $L_1 = L_2 = L$:

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_1^2}{L^2} + \frac{n_2^2}{L^2} + \frac{n_3^2}{4L^2}\right)$$

$$= \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2 + \frac{n_3^2}{4}\right)$$

Lowest energy: $E_{1,1,1} = \frac{\hbar^2\pi^2}{2mL^2}\cdot\left(1+1+\frac{1}{4}\right) = 2.25\,eV$

$\Rightarrow \boxed{\frac{\hbar^2\pi^2}{2mL^2} = 1eV \equiv E_0}$ $\boxed{E_{n_1n_2n_3} = E_0\left(n_1^2+n_2^2+\frac{n_3^2}{4}\right)}$

n_1, n_2, n_3	$n_1^2+n_2^2+\frac{n_3^2}{4}$	energy	degeneracy	
1, 1, 1	2.25	2.25eV	1	4 lowest energy states
1, 1, 2	1+1+1	3eV	1	
1, 1, 3	$1+1+\frac{9}{4}$	4.25eV	1	
2, 1, 1	$4+1+\frac{1}{4}$	5.25eV	2	
1, 1, 4	1+1+4	6eV		

(c) If we make L_3 smaller the energies of the states changes. The state (1, 1, 3) increases its energy faster than the state (2, 1, 1) so at some point they will become equal and there is triple degeneracy

i.e. the condition is $E_{1,1,3} = E_{2,1,1} \Rightarrow$

with $E_{n_1n_2n_3} = E_0\left(n_1^2 + n_2^2 + \frac{n_3^2}{(L_3/L)^2}\right) \Rightarrow$

$$1+1+\frac{9}{(L_3/L)^2} = 4+1+\frac{1}{(L_3/L)^2} \Rightarrow \frac{8}{(L_3/L)^2} = 3 \Rightarrow$$

$$\left(\frac{L_3}{L}\right)^2 = \frac{3}{8} \Rightarrow \frac{L_3}{L} = 1.63 \Rightarrow \boxed{L_3 = 1.63L} \text{ (c)}$$

Problem 3

$$\psi(r,\theta,\phi) = C \cdot r \cdot e^{-r/a_0} \cdot \cos\theta$$

(a) since m_ℓ appears as $e^{im_\ell\phi}$ $\Rightarrow$ $\boxed{m_\ell = 0}$

since radial function has no nodes, $l = n-1$

since the power of r is 1 $\Rightarrow$ $l=1 \Rightarrow n=2$

since the exponential is e^{-Zr/na_0} $\Rightarrow$ $Z=2$

(b) (i) there is no uncertainty in L_z

because $[L_z] = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ $\Rightarrow$ $[L_z]\psi = 0 \Rightarrow L_z = 0$ is sharp

$\Rightarrow \langle L_z\rangle = 0$, $\langle L_z^2\rangle = 0$, and $\boxed{\Delta L_z = 0}$

(ii) For L_x, $\langle L_x\rangle = 0$

$\langle L_x^2\rangle = \frac{1}{3}\langle L^2\rangle$ by symmetry

$\langle L^2\rangle = \hbar^2 l(l+1) = 2\hbar^2 \Rightarrow \langle L_x^2\rangle = \frac{2}{3}\hbar^2$

$\Rightarrow \Delta L_x = \sqrt{\langle L_x^2\rangle - \langle L_x\rangle^2} = \sqrt{\frac{2}{3}}\hbar = 0.82\hbar$

$\boxed{\Delta L_x = 0.82\hbar}$

(c) $R(r) = C r e^{-r/a_0} \Rightarrow P(r) = r^2 R^2(r)$

$\Rightarrow P(r) = C' r^4 e^{-2r/a_0}$

$\boxed{\text{Use } \int_0^\infty dr\, r^n e^{-\lambda r} = \frac{n!}{\lambda^{n+1}}}$

$$\langle r\rangle = \frac{\int_0^\infty dr\, r^5 e^{-2r/a_0}}{\int_0^\infty dr\, r^4 e^{-2r/a_0}} = \frac{5!}{4!}\frac{a_0}{2} = \frac{5}{2}a_0$$

$$\Rightarrow \boxed{\langle r\rangle = \frac{5}{2}a_0 = 2.5a_0}$$