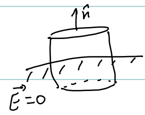
Electrostatics with (around?) conductors ((Garg Chap 14).

We have already covered much of this chapter. These notes focus on new material, and even there, are mostly supplemental → read text!

Summary of main ingredients:

- $\vec{E}=0 \Leftrightarrow \phi =$ constant in conductors
- Charges on conductors live on surface



$E_n \cdot A = 4\pi Q_{enc}$

$E_n = \frac{4\pi Q_{enc}}{A} = 4\pi\sigma$

$$4\pi\sigma = E_n = -\frac{\partial \phi}{\partial n} \qquad \hat{n} = \text{normal to surface}$$

(fields just outside).

- Uniqueness: if ϕ_1 & ϕ_2 solve the boundary value problem (ϕ or $\frac{\partial \phi}{\partial n}$ specified at boundaries) then $\phi_2 = \phi_1$ up to constant (and constant $=0$ if ϕ specified anywhere at a boundary).

Electrostatic energy: From $u = \frac{1}{8\pi}(E^2+B^2)$, for $\vec{B}=0$ and $\vec{E} = -\vec{\nabla}\phi$

one has $$\mathcal{E} = \int_V d^3r \, \frac{1}{8\pi}(\vec{\nabla}\phi)^2 = \int_{\partial V} d^2s \, \frac{1}{8\pi}\phi \frac{\partial \phi}{\partial n} - \frac{1}{8\pi}\int d^3r \, \phi \nabla^2 \phi$$

Then: * if the boundary is at ∞, and the fields vanish there, using $\nabla^2\phi = -4\pi\rho$

$$\mathcal{E} = \frac{1}{2}\int d^3r \, \phi(\vec{r})\rho(\vec{r})$$

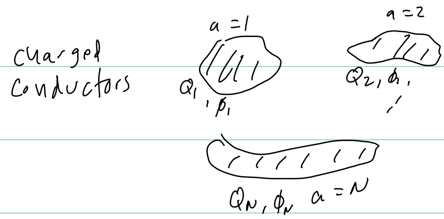
(which you have seen as $\mathcal{E} = \frac{1}{2}\sum_{i\neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_{i\neq j} q_i \phi_i(\vec{r}_i)$ where $\phi_i(\vec{r}) = \sum_{j\neq i} \frac{q_j}{|\vec{r} - \vec{r}_j|}$

is the potential due to all charges but q_i at $\vec{r}$)

* If $\rho = 0$ in V but there are conductors bounding V, then $\phi = \phi_a$ $a = 1, \ldots, N$ in each of the N surfaces bounding V, and (i) $\phi_a =$ constant on the surface $(\partial V)_a$, and (ii) $\frac{\partial \phi}{\partial n} = 4\pi\sigma$ on that surface (the sign change is because this n points <u>into</u> volume)

$$\Longrightarrow \mathcal{E} = \int_{\partial V} d^2s \, \frac{1}{8\pi}\phi \frac{\partial \phi}{\partial n} = \frac{1}{8\pi}\sum_a \phi_a \int_{\partial V_a} d^2s \, (4\pi\sigma) = \frac{1}{2}\sum_a \phi_a Q_a \quad \text{where } Q_a = \int_{\partial V_a} d^2s \, \sigma .$$

Capacitance:



Problem: given ϕ_a's what are Q_a's? OR, given Q_a what are ϕ_a (up to additive constant: assume $\phi(\vec r) \underset{|r|\to\infty}{\to} 0$).

The basic result is: this is a linear relation

$$Q_a = \sum_b C_{ab}\, \phi_b .$$

C_{ab} = capacitance

or C_{aa} = "capacity" or "capacitance"

C_{ab}, $b \neq a$ = coefficient of electrostatic induction.

AND: C_{ab} depend <u>only</u> on geometry (ie, not on ϕ_a nor Q_a).

This is proved in a wishy-washy manner in textbook (and not at all in Jackson). Here is my argument: consider the Green's function for the Poisson eq.

$$\nabla^2 G = -4\pi\, \delta^{(3)}(\vec r - \vec r\,')$$

with appropriate boundary conditions (we need Dirichlet, but keep it general for now). [Note $G = \frac{1}{|\vec r - \vec r\,'|} + F(\vec r, \vec r\,')$ where $\nabla^2 F = 0$ is chosen to fix boundary conditions).]

Then from Green's 2nd identity:

$$\int_V d^3r \left(\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1 \right) = \int_{\partial V} d^2s \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right)$$

with $\psi_1 = \phi$ and $\psi_2 = G$ we have

$$-4\pi\, \phi(\vec r) - \int d^3r'\, G(\vec r, \vec r\,')\, \nabla^2 \phi = \int_{\partial V} d^2s' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

In the case of interest $\nabla^2 \phi = 0$ (no charge in V), $G|_{\partial V} = 0$ (Dirichlet), so

$$\phi(\vec r) = -\frac{1}{4\pi} \int_{\partial V} d^2r'\, \phi(\vec r\,') \frac{\partial G}{\partial n'} = \sum_a \phi_a\, F_a(\vec r)$$

where $F_a(\vec r) = -\frac{1}{4\pi} \int_{\partial V} d^2r' \frac{\partial G}{\partial n'}(\vec r, \vec r\,')$ depends on Geometry but not on ϕ.

From this one can compute $\sigma_b = -\frac{1}{4\pi}\frac{\partial \phi}{\partial n}\Big|_{\partial V_b}$ and $Q_b = \int_{\partial V_b} d^2_S\, \sigma_b$

which gives the C_{ba} in terms of $\int_{\partial V_b} d^2 r \frac{\partial}{\partial n} F_a(\vec r)$ $\Rightarrow$ purely geometric. END of "proof".

While we are proving things not shown in text nor Jackson: $C_{ab} = C_{ba}$

For this we use "Green's reciprocity": Consider two different charge distributions ρ_1 & ρ_2 and associated potentials $\phi_1(\vec r)$ & $\phi_2(\vec r)$ (for same boundary conditions, including $\phi \to 0$ at ∞).

$$\Rightarrow \int_V d^3 r\, (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) = \int_{\partial V} d^2 r \left(\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n}\right)$$

For our case, $\rho = 0$ in V, and $\frac{\partial \phi}{\partial n} \propto \sigma$. Moreover $\phi_{1,2}$ on ∂V is constant:

$$0 = \sum (\phi_{1a} Q_{2a} - \phi_{2a} Q_{1a})$$

The textbook obtains this in a different way by considering point charges and ignoring singular terms. The rest is as in text:

$$\Rightarrow \quad 0 = \sum_{a,b} \phi_{1a}\phi_{2b}\,(C_{ab} - C_{ba})$$

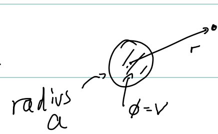
and arbitrariness in ϕ_1 & ϕ_2 $\Rightarrow$ $C_{ab} = C_{ba}$.

Computations: $Q_a = \sum_b C_{ab}\phi_b$, so one may set $\phi_c = 0$ for all c except $c = b$. and then compute $Q_a = C_{ab}\phi_b$. This requires solving the boundary value problem $\nabla^2 \phi = 0$ (with $\phi|_{\partial V}$ as explained); Q_a is computed from $\sigma_a \propto \frac{\partial \phi}{\partial n}\Big|_{\partial V_a}$.

But only simple geometries can be done analytically.

Example.

(a) Single sphere:



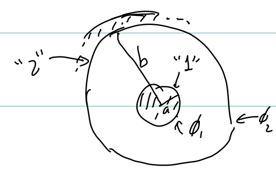
$\phi(\vec r) = \frac{k}{r}$: k set by $\phi\Big|_{\partial V} = \phi(\vec r)\Big|_{|\vec r| = a} = V$ $(= \frac{k}{a})$

$\Rightarrow \phi(\vec r) = V\frac{a}{r}$. Charge on sphere $Q = Va \Rightarrow \boxed{C = a}$

((Can verify $Q = Va$: $\frac{\partial \phi}{\partial n}\Big|_{\partial V} = \frac{\partial \phi}{\partial r}\Big|_{r=a} = -\frac{Va}{a^2} \Rightarrow \sigma = -\frac{1}{4\pi}\left(-\frac{V}{a}\right)$

Then $Q = \int d^2 s\, \sigma = (4\pi a^2)\left(\frac{1}{4\pi}\frac{V}{a}\right) = Va$ ✓✓).

(iii) Concentric spheres: $\nabla^2\phi = 0 \Rightarrow \phi = \frac{k_1}{r} + k_2$



So $\phi_1 = \frac{k_1}{a} + k_2$, $\phi_2 = \frac{k_1}{b} + k_2$

$\Rightarrow k_1 = \frac{\phi_2 - \phi_1}{b^{-1} - a^{-1}}$ $k_2 = \frac{b\phi_2 - a\phi_1}{b-a}$

Now $\sigma_1 = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_a = \frac{1}{4\pi}\frac{k_1}{a^2}$, $\sigma_2 = \frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_{r=b} = -\frac{1}{4\pi}\frac{k_1}{b^2}$

so that $Q_1 = k_1 = -Q_2$

(Note, we knew this all along since $\vec{E} = 0$ in the interior of conductor "2" so that Gauss's law gives charge enclosed in gaussian surface within "2" $= 0$).

Compute: $Q_1 = C_{11}\phi_1$ (set $\phi_2 = 0$). $C_{11} = \frac{Q_1}{\phi_1} = \frac{k_1}{\phi_1} = \frac{1}{\phi_1}\frac{\phi_2 - \phi_1}{b^{-1} - a^{-1}}\Big|_{\phi_2 = 0} = \frac{1}{a^{-1} - b^{-1}} = \frac{ab}{b-a}$

$Q_1 = C_{12}\phi_2$ (set $\phi_1 = 0$) $C_{12} = \frac{Q_1}{\phi_2} = \frac{k_1}{\phi_2} = -\frac{ab}{b-a}$

$C_{21} = C_{12}$

$Q_2 = C_{22}\phi_2$ (set $\phi_1 = 0$) $C_{22} = \frac{Q_2}{\phi_2} = -\frac{Q_1}{\phi_2} = -C_{12}$

$\Rightarrow C_{11} = C_{22} = -C_{12} = -C_{21} = \frac{ab}{b-a}$

Note that there have charges $\pm Q$, so the definition of "capacitance" $Q = C\,\Delta\phi$ applies:

$C = \frac{|Q|}{|\phi_2 - \phi_1|} = \left|\frac{k_1}{\phi_2 - \phi_1}\right| = \frac{ab}{b-a}$.

Some additional comments:

(i) Electrostatic energy $\mathcal{E} = \frac{1}{2}\sum_a Q_a\phi_a = \frac{1}{2}\sum_{a,b} C_{ab}\phi_a\phi_b$

Or with $\phi_a = \sum_b (C^{-1})_{ab} Q_b$, $\mathcal{E} = \frac{1}{2}\sum_{a,b}(C^{-1})_{ab} Q_a Q_b$ (where $C^{-1}\cdot C = \mathbb{1}$ as matrices).

(ii) We have found charges when potentials are specified (Dirichlett problem)

If charges are specified instead, find C_{ab} as before, then $\phi_a = \sum_b (C^{-1})_{ab} Q_b$.

(iii) One can use C_{ab}'s to solve problem with $\rho \neq 0$ and b.c.'s on conducting ∂V. Just solve $\nabla^2\phi = -4\pi\rho$ with grounded conductors 1st, and then add to this $\nabla^2\phi = 0$ with appropriate b.c.'s.

- For two conductors with potential <u>difference</u> V and with charges $\pm Q$ the "capacitance" $\mathcal{C}$ (confusion of terminology!) is $Q = \mathcal{C} V$.

(Exercise 88.2)

Relation to C_{ab}: Use $\phi_a = \sum_b (C^{-1})_{ab} Q_b$ and $Q_1 = Q$, $Q_2 = -Q$

so $V = \phi_1 - \phi_2 = (C^{-1})_{11} Q_1 + (C^{-1})_{12} Q_2 - (C^{-1})_{21} Q_1 - (C^{-1})_{22} Q_2$

$= Q\left((C^{-1})_{11} + (C^{-1})_{22} - 2(C^{-1})_{12}\right)$ (used $C^{-1}_{12} = C^{-1}_{21}$).

$\Rightarrow \frac{1}{\mathcal{C}} = C^{-1}_{11} + C^{-1}_{22} - 2C^{-1}_{12}$

To write this in terms of C_{ab},

$$(C^{-1}) = \frac{1}{\det C}\begin{pmatrix} C_{22} & -C_{12} \\ -C_{12} & C_{11} \end{pmatrix}$$

so $\frac{1}{\mathcal{C}} = \frac{1}{\det C}\left(C_{11} + C_{22} + 2C_{12}\right)$

or $\mathcal{C} = \dfrac{C_{11}C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}}$

The energy stored in the capacitor is $\mathcal{E} = \frac{1}{2}\sum_{a,b} (C^{-1})_{ab} Q_a Q_b$

$= \frac{1}{2} Q^2 \left(C^{-1}_{11} + C^{-1}_{22} - 2C^{-1}_{12}\right)$

$= \frac{1}{2}\frac{Q^2}{\mathcal{C}}$

Note added: I just realized C_{ab} is NOT invertible. Why does the text (Garg) as well as the biblical Landau & Lifshitz treat it as such is a mystery. Here is the argument:

We can solve the problem $Q_a = \sum_b C_{ab}\phi_b$ for the case $\phi(\vec{r}) \xrightarrow[r\to\infty]{} \phi_\infty = \text{arbitrary}$. This just correspond to shifting ϕ of the previous, $\phi_\infty = 0$, solution by a constant. This leaves Q_a unaffected, since it is obtained from a derivative, $\left.\frac{\partial\phi}{\partial n}\right|_{\partial V_a}$.

So $Q_a = \sum_b C_{ab}(\phi_b + \phi_\infty)$ is independent of $\phi_\infty \Rightarrow \sum_b C_{ab} = 0 \Rightarrow \det C = 0$

(To see that $\det C = 0$, recall that, considering columns of M as vectors, then $\det M \neq 0 \Leftrightarrow$ the vectors are linearly independent. So the columns of C_{ab} are vectors $(\vec{V}^{(b)})_a = C_{ab}$ then $\sum_b C_{ab} = 0$ is $\sum_b \vec{V}^{(b)} = 0$).

For the 2×2 case $C_{ab} = C_{ba}$ and $\sum_b C_{ab} = 0$ implies

$$(C)_{ab} = c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{for some } c > 0$$

Then $Q = Q_1 = C_{11}\phi_1 + C_{12}\phi_2 = c(\phi_1 - \phi_2) \Rightarrow c = \frac{Q}{\phi_1 - \phi_2}$ $\underline{\text{is}}$ the capacitance

<u>wow!</u>

Methods for solving boundary value problems.

(i) Solve PDE with separation of variables; special functions DONE

(ii) Images

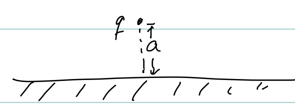
(iii) Green functions (combine the above)

(iv) Numerical

(v) Variational.

Method of Images

By example: point charge with infinite plane conductor:

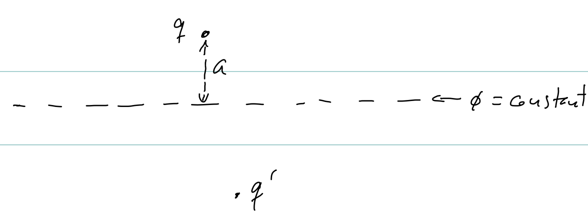


Note: if conductor is finite but ends at distance $L \gg a$, we expect this to be a good approximation

Consider a problem with charge q, a second "image" charge q', and no conductor. We seek to find magnitude of q' and location so that

(i) there exists an equipotential $\phi =$ constant that is a plane a distance a from q

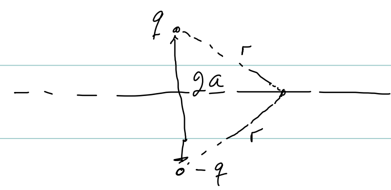
(ii) q' is on the other side of this plane



Then $\phi(\vec{r})$ for this problem is a solution to our problem: it satisfies $\nabla^2\phi = -4\pi\rho$ and the b.c. $\phi =$ const at plane.

In this case the solution is obvious: make $q' = -q$ a distance $2a$ from q

(figure next page)



The points on the mid-plane have potential $\phi = \frac{q}{r} + \frac{(-q)}{r} = 0$

More explicitly, place q at $\vec{r}_0 = (0,0,a)$ and $-q$ at $-\vec{r}_0$. Then

$$\phi(\vec{r}) = q\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{|\vec{r}+\vec{r}_0|}\right) \qquad (A)$$

Then

$$\phi(\vec{r}) = 0$$

determines a surface: $|\vec{r}-\vec{r}_0| = |\vec{r}+\vec{r}_0| \iff x^2+y^2+(z-a)^2 = x^2+y^2+(z+a)^2$

$\iff \boxed{z=0}$

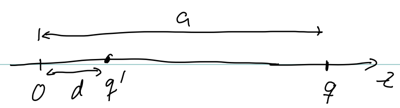
So (A) is a $\phi(\vec{r})$ that gives $\nabla^2\phi = -4\pi\rho$ ($\rho = q\,\delta^3(\vec{r}-\vec{r}_0)$) with

$\phi(\vec{r}) = 0$ on $z=0$.

One can (see text) compute $\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}$ to find charge distribution on conductor. Clearly, $\int d^2s\,\sigma = -q$ (from Gauss's law). One can check this.

One may consider some charges, look for an equipotential of some desired shape, and use the charges on one side as "images".

Example:

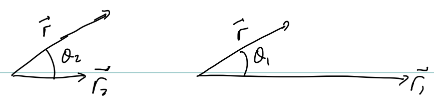


$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}_1|} + \frac{q'}{|\vec{r}-\vec{r}_2|} \qquad \vec{r}_1 = (0,0,a) \qquad \vec{r}_2 = (0,0,d)$$

On $|\vec{r}| = R$ (a sphere about origin) we have $\phi = 0$ if

$$q|\vec{r}-\vec{r}_2| = -q'|\vec{r}-\vec{r}_1| \quad \text{ie} \quad q\sqrt{R^2+d^2-2Rd\cos\theta_2} = -q'\sqrt{R^2+a^2-2Ra\cos\theta_1}$$

where $\theta_{1,2}$ are



$\Rightarrow \theta_2 = \theta_1 \equiv \theta$

Take, say $\theta = 0$. Then $-\frac{q'}{q} = \frac{R-d}{a-R}$. If $\theta = \pi$ $-\frac{q'}{q} = \frac{R+d}{R+a}$

$$\Rightarrow \frac{R-d}{a-R} = \frac{R+d}{R+a} \Rightarrow R^2 + R(a-d) - ad = -R^2 + R(a-d) + ad \Rightarrow R^2 = ad \Rightarrow -\frac{q'}{q} = \frac{R - R^2/a}{a-R} = \frac{R}{a}$$

Does this work for general θ? Squaring $q\sqrt{\ } = -q'\sqrt{\ }$:

$$a^2(R^2+d^2-2Rd\cos\theta) \overset{?}{=} R^2(R^2+a^2-2Ra\cos\theta)$$

$$\Rightarrow a^2R^2 + R^4 - 2\frac{R^3}{a}\cos\theta \overset{?}{=} R^4 + R^2a^2 - 2R^3 a\cos\theta \quad \underline{\text{Yes}}!$$

So $\phi(\vec{r}) = q\left[\frac{1}{|\vec{r}-\vec{r}_1|} - \frac{R/a}{|\vec{r}-\vec{r}_2|}\right]$ has $\phi = 0$ on $|\vec{r}| = R$ and satisfies

$$\nabla^2\phi = -4\pi q\,\delta^{(3)}(\vec{r}-\vec{r}_1)$$

Third example: conducting sphere in uniform external field $\vec{E}_0$

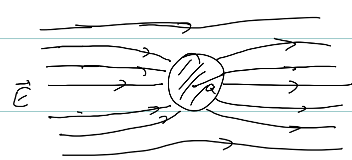
Consider a dipole $\vec{d} = d\hat{z}$ plus a field (superposition) $\vec{E}_0 = E_0\hat{z}$ so that

$$\phi(\vec{r}) = \frac{dz}{|\vec{r}|^3} - E_0 z \qquad \text{(we have put } \vec{d} \text{ at the origin).}$$

Then the surface $|\vec{r}| = a$ (a sphere of radius a) has $\phi = 0$ if $\frac{d}{a^3} = E_0$

So with our image "charge" being a dipole ($\vec{d}$) we have a conducting sphere of radius a in a field $\vec{E}_0 = E_0\hat{z}$ has potential

$$\phi(\vec{r}) = E_0 z\left(\frac{a^3}{|\vec{r}|^3} - 1\right) = -\vec{E}_0\cdot\vec{r}\left(1 - \frac{a^3}{|\vec{r}|^3}\right) \qquad (|\vec{r}| \geq a).$$



The charges have redistributed themselves, $\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}$, to create a dipole. The dipole moment above is $\vec{d} = a^3\vec{E}_0$.

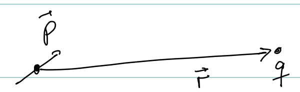
More generally, the σ on a conductor placed in an external field $\vec{E}_0$ produces an induced field that can be expanded in a multipole expansion. The leading term is the dipole (the charge on the conductor is assumed to vanish). The corresponding dipole moment $\vec{\mathcal{P}}$ is linear in $\vec{E}_0$, but in general geometries the linearity means $\mathcal{P}_i = \alpha_{ij} E_{0j}$, α_{ij} = "polarizability" tensor

In the case above $\alpha_{ij} = \delta_{ij} a^3$.

Moreover, the potential energy of the uncharged conductor in the external field, in the dipole approximation, is

$$\mathcal{E} = -\frac{1}{2}\vec{P}\cdot\vec{E}_0$$

To see this, consider the uncharged conductor in the presence of a point charge q at $\vec{r}$ in a frame with $\vec{P}$ at the origin.



For large $\vec{r}$, the field at the conductor is approximately uniform, $\vec{E}_0 = -\frac{q\vec{r}}{r^3}$

Now $\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, \nabla\phi\cdot\nabla\phi = \frac{1}{8\pi}\int_{\partial V} d^2r\, \phi\frac{\partial\phi}{\partial n} - \frac{1}{8\pi}\int_V d^3r\, \phi\nabla^2\phi$

Assuming fields vanish at infinity, and using $\nabla^2\phi = -4\pi\rho$

$$\mathcal{E} = \frac{1}{2}\sum_a Q_a\phi_a + \frac{1}{2}q\phi(\vec{r})$$

a generalization of our previous expression for $\mathcal{E}$ that now includes q. Now, we are assuming 1 conductor, with $Q=0 \Rightarrow \mathcal{E} = \frac{1}{2}q\phi(\vec{r}) = \frac{1}{2}q\frac{\vec{P}\cdot\vec{r}}{r^3} = -\frac{1}{2}\vec{P}\cdot\left(-\frac{q\vec{r}}{r^3}\right) = \underline{-\frac{1}{2}\vec{P}\cdot\vec{E}_0}$.

NOTE:

This derivation, taken from L&L, and in Exercise 87.5 of Garg, (implicitly) subtracts a divergence from ϕ of q at q, ie, $\frac{q}{0}$ (coming from $\int\phi\nabla\phi$.)

This is why the result is negative even if $u = \frac{1}{8\pi}E^2 > 0$.

I believe this makes this general sounding argument somewhat questionable. In Appendix B of this Unit I compute the total energy of grounded sphere in $\vec{E}_0$ take away the energy of the no-conductor case. The result is

$$\mathcal{E} = \frac{1}{6}a^3E_0$$

Variational Method

Good for analytic approximation, but for precision look at numerical methods.
It is used in other areas of physics → worth taking a look.

Consider the functional

$$W[\psi] = \int_V d^3r \left[\frac{1}{8\pi} (\vec{\nabla}\psi(\vec{r}))^2 - \psi(\vec{r})\rho(\vec{r}) \right]$$

where $\psi(\vec{r})$ is piecewise smooth, satisfying Dirichlet b.c. on ∂V.

Then W is minimized by the solution to Poisson, $\nabla^2\psi = -4\pi\rho$ satisfying the b.c.'s.

Trivial to show

$$\delta W = \int d^3r \left[\frac{1}{4\pi} \vec{\nabla}\psi \cdot \vec{\nabla}\delta\psi(\vec{r}) - \rho(\vec{r})\,\delta\psi(\vec{r}) \right]$$

$$= \underbrace{\int_{\partial V} d^2s \, \frac{1}{4\pi} \hat{n}\cdot(\delta\psi \vec{\nabla}\psi)}_{=0 \text{ since } \delta\psi = 0 \text{ on } \partial V} - \int_V d^3r \, \delta\psi(\vec{r}) \left[\frac{1}{4\pi} \nabla^2\psi + \rho \right]$$

at extremum $\Rightarrow \frac{1}{4\pi}\nabla^2\psi + \rho = 0$, as advertised

To see that it is a minimum (not a maximum or saddle point)
expand $W[\psi+\delta\psi]$ to order $\delta\psi^2$

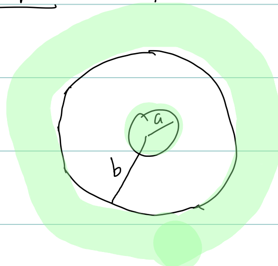
$$\delta^{(2)} W = \int d^3r \frac{1}{8\pi} (\vec{\nabla}\delta\psi)^2 \geq 0 \quad . \quad \underline{\text{DONE}}!$$

Notes

- To use this, find some functions $\psi_1, \psi_2, \psi_3, \ldots$ possibly with adjustable parameters, that satisfy the b.c.'s. Then minimize $W[\alpha_1\psi_1 + \alpha_2\psi_2 + \cdots]$ w.r.t. $\alpha_1, \alpha_2, \alpha_3, \ldots$ and adjustable parameters.
- If $\rho = 0$ in V, then $W[\psi] = \frac{1}{8\pi}\int d^3r \, \vec{E}\cdot\vec{E}$ = electrostatic energy.
- For 2 conductors, if $\phi_1 = 0$, $\phi_2 = 1 \Rightarrow W[\psi_{min}] = \frac{1}{2}C(\Delta\phi)^2 = \frac{1}{2}C$

Exercise 91.1

Example: Cylindrical capacitor (circular cross section):



Some trial functions

(i) $\alpha(r-a)$

(ii) $\alpha(r-a)+\beta(r-a)^2$

We need to satisfy $\phi(r=b)=V$.

(i) $\alpha(b-a)=V \Rightarrow \alpha = V/(b-a) \Rightarrow$ no freedom for variation

$$\frac{1}{2}\frac{C}{l} = \frac{W}{l}\left[\frac{V}{b-a}(r-a)\right] = \frac{2\pi}{8\pi}\int_a^b r\,dr\left[\frac{1}{b-a}\right]^2 = \frac{1}{8}\frac{b+a}{b-a} \Rightarrow \frac{C}{l} = \frac{1}{4}\frac{b+a}{b-a}$$

(ii) $\alpha(b-a)+\beta(b-a)^2=V \Rightarrow \alpha = \frac{V}{b-a}-\beta(b-a)$

So Now $(\vec{\nabla}\psi)^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 = \left(\alpha+2\beta(r-a)\right)^2$

$$W[\beta] = \frac{1}{8\pi}\int_V d^3r\,\left(\alpha+2\beta(r-a)\right)^2 = \frac{1}{8\pi}\int dz\cdot 2\pi\cdot\int_a^b r\,dr\,\left(\alpha+2\beta(r-a)\right)^2$$

$$= \frac{1}{4}\int dz\left[\alpha^2\frac{1}{2}(b^2-a^2) + 4\alpha\beta\left(\frac{1}{3}(b^3-a^3)-\frac{1}{2}a(b^2-a^2)\right) + 4\beta^2\left(\frac{1}{4}(b^4-a^4)-\frac{2a}{3}(b^3-a^3)+a^2\frac{1}{2}(b^2-a^2)\right)\right]$$

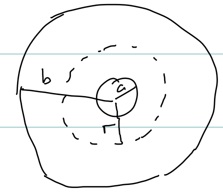
$$\frac{dW}{d\beta} = \frac{\partial W}{\partial\alpha}(-(b-a)) + \frac{\partial W}{\partial\beta} = 0 \Leftrightarrow -(b-a)(b^2-a^2)\left(\frac{V}{b-a}-\beta(b-a)\right) + 4\left(-(b-a)\beta+\frac{V}{b-a}-(b-a)\beta\right)\left(\frac{1}{3}(b^3-a^3)-\frac{1}{2}a(b^2-a^2)\right)$$
$$+ 8\beta\left[\frac{1}{4}(b^2-a^2)\left(b^2+\frac{3}{2}a^2\right)\right]$$

Solving $\beta = -\frac{V}{b^2-a^2} \Rightarrow \alpha = \frac{2bV}{b^2-a^2}$

$$\Rightarrow \psi(r) = \frac{V}{b^2-a^2}(r-a)\left[2b-r+a\right]$$

$$\frac{1}{2}\frac{C}{l} = \frac{W}{l} = \frac{2\pi}{8\pi}\int_a^b r\,dr\left[\frac{2(b+a-r)}{b^2-a^2}\right]^2 = \frac{1}{12}\frac{a^2+4ab+b^2}{b^2-a^2} \qquad \frac{C}{l} = \frac{1}{6}\frac{a^2+4ab+b^2}{b^2-a^2}$$

The exact solution is elementary: use a gaussian surface



$$E(r)\, l\, 2\pi r = 4\pi Q_{enc} = 4\pi \lambda l$$

$$\Rightarrow \quad E(r) = \frac{2\lambda}{r} \quad \Rightarrow \quad \phi = -2\lambda \ln(r/a)$$

$$Q = C\,\Delta\phi = C\left(-2\lambda \ln(b/a)\right) \qquad (Q = -\lambda l)$$

$$C = \frac{1}{2\ln(b/a)}$$

One may compare the approximate solutions to the exact one by, say, plotting $\frac{C_{approx}}{C_{exact}}$ as a function of $x = \frac{b}{a}$

For $x = 1+\epsilon$, $C_{exact} = \frac{1}{2\ln(1+\epsilon)} \simeq \frac{1}{2\epsilon}$

while $C^{(i)}_{approx} = \frac{1}{4}\frac{x+1}{x-1} = \frac{1}{4}\frac{2}{\epsilon} = \frac{1}{2\epsilon}$

and $C^{(ii)}_{approx} = \frac{1}{6}\frac{6}{(1+\epsilon)^2-1} = \frac{1}{2\epsilon}$

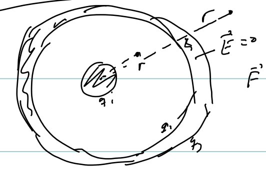
but as $x \gg 1$, $C_{exact} = \frac{1}{2\ln(x)}$

while $C^{(i)}_{approx} \simeq \frac{1}{4}$ and $C^{(ii)}_{approx} = \frac{1}{6}$

Next: Appendices

Not for lecture: unfinished business

Appendix A | Exercise 88.7

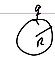


$\vec{E}=0$

$\vec{F}=-\vec{\nabla}\phi$

$\phi = r^{-k} \quad \vec{\nabla}\phi = -k\frac{\vec{r}}{r^{k+2}}$

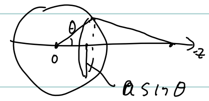
Concentric conducting spheres.



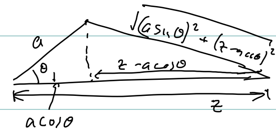
q on surface.

$$\vec{F} = \underset{\substack{\uparrow \\ \text{depends on } R}}{K}\frac{q\vec{r}}{r^{3+\eta}} = K_0\left(\frac{R}{r}\right)^{\eta}\frac{q\vec{r}}{r^3}$$

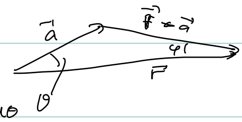
Force from spherical shell with uniform density



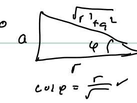
$a\sin\theta$



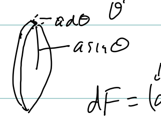
$\sqrt{(a\sin\theta)^2+(z-a\cos\theta)^2} = \sqrt{z^2+a^2-2az\cos\theta}$



$$\cos\varphi = \frac{(\vec{r}-\vec{a})\cdot\vec{r}}{|\vec{r}-\vec{a}|\,r} = \frac{r-a\cos\theta}{\sqrt{r^2+a^2-2ar\cos\theta}}$$



$\cos\theta=0$, $\cos\varphi = \frac{r}{\sqrt{\ }}$ ✓



$$dF = (\overset{2\pi}{\downarrow}a^2\sin\theta\, d\theta)\,\sigma\left(\frac{r-a\cos\theta}{\sqrt{r^2+a^2-2ar\cos\theta}}\right) f\left(\sqrt{r^2+a^2-2ar\cos\theta}\right)$$

$$F = 2\pi a^2\sigma\int_{-1}^{1}dx\,\frac{r-ax}{\sqrt{r^2+a^2-2arx}}\,f\left(\sqrt{r^2+a^2-2arx}\right)$$

$$d\phi = (2\pi a^2\sin\theta\, d\theta)\,\sigma\, V(\sqrt{\ ---\ })$$

$$\phi = 2\pi a^2\sigma\int_{-1}^{1}dx\, V(\sqrt{\ \ })$$

$$= \frac{2\pi a^2\sigma}{ar}\int_{r-a}^{r+a}du\, u\, V(u)$$

$$= \frac{2\pi a^2\sigma}{ar}\int du\,\frac{u}{u^{1+\eta}}$$

$\frac{u^{1-\eta}}{1-\eta}$

$$= \frac{2\pi a\sigma}{r}\frac{1}{1-\eta}\left[(r+a)^{1-\eta}-(r-a)^{1-\eta}\right]$$

For $\eta=0$

$$= \frac{4\pi a^2\sigma}{r} \checkmark$$

$$u = \sqrt{r^2+a^2-2arx}$$

$$du = \frac{1}{2}\frac{1}{\sqrt{r^2+a^2-2arx}}\,(-2ar\,dx) \qquad \frac{dx}{\sqrt{r^2+a^2-2arx}} = -\frac{du}{ar} \quad \text{nice.}$$

$$u^2 = r^2+a^2-2arx \implies x = \frac{r^2+a^2-u^2}{2ar}$$

$$F = \frac{\overset{2\pi}{\vee}a^2\sigma}{ar}\int_{u_{min}}^{u_{max}}du\left(r-\frac{1}{2r}(r^2+a^2-u^2)\right)f(u)$$

Need limits $U_\pm = \sqrt{r^2+a^2\pm 2ar} = (r\pm a)$

$$F = \frac{2\pi a\sigma}{2r^2}\int_{r-a}^{r+a}du\,(r^2-a^2+u^2)\,f(u)$$

If $f(u) = \frac{1}{u^p}$

$$F = \frac{2\pi a\sigma}{2r^2}\int_{r-a}^{r+a}du\left[\frac{r^2-a^2}{u^p}+\frac{1}{u^{p-2}}\right] = \frac{a\sigma}{4r^2}\left[(r^2-a^2)\frac{u^{1-p}}{1-p}+\frac{u^{3-p}}{3-p}\right]_{r-a}^{r+a}$$

$$= \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{1-p}\left((r+a)^{1-p}-(r-a)^{1-p}\right)+\frac{1}{3-p}\left((r+a)^{3-p}-(r-a)^{3-p}\right)\right]$$

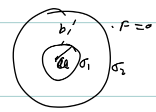
$$F = \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{1-p}\left((r+a)^{1-p}-(r-a)^{1-p}\right) + \frac{1}{3-p}\left((r+a)^{3-p}-(r-a)^{3-p}\right)\right]$$

For $p=2$

$$F = \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{-1}\left(\frac{1}{r+a}-\frac{1}{r-a}\right) + \frac{1}{1}\left((r+a)-(r-a)\right)\right]$$

$$= \frac{2\pi a\sigma}{2r^2}\left[2a+2a\right] = \frac{4\pi a^2\sigma}{r^2} = \frac{Q}{r^2} \checkmark$$

So



For $V = \frac{q}{r^{1+\eta}} \Rightarrow F = \frac{(1+\eta)q}{r^{2+\eta}}$ $\quad p = 2+\eta, \quad \sigma \to \sigma(1+\eta)$ (irrelevant in ratio)

$$F = \frac{\pi a\sigma}{r^2}\left[\frac{(r^2-a^2)}{-1-\eta}\left[(r+a)^{-1-\eta}-(r-a)^{-1-\eta}\right] + \frac{1}{1-\eta}\left((r+a)^{1-\eta}-(r-a)^{1-\eta}\right)\right]$$

$$\phi = \frac{q}{r^k} \qquad F = -\frac{d\phi}{dr} = \frac{kq}{r^{k+1}}$$

$$d\phi = 2\pi a^2\sigma \sin\theta\, d\theta\, f\left(\sqrt{a^2+r^2-2ar\cos\theta}\right)$$

$$\phi = \frac{2\pi a\sigma}{r}\frac{1}{1-\eta}\left[(r+a)^{1-\eta}-(r-a)^{1-\eta}\right) \simeq \frac{2\pi a\sigma}{r}(1+\eta)\left[(r+a)(1-\eta\ln(r+a)) - (r-a)(1-\eta\ln(r-a))\right]$$

$$= \frac{2\pi a\sigma}{r}\left[2a + \eta\left[2a - (r+a)\ln(r+a) + (r-a)\ln(r-a)\right]\right]$$

$$= \frac{2\pi a\sigma}{r}\left[2a + \eta\left[2a - r\ln\frac{r+a}{r-a} - a\ln(r^2-a^2)\right]\right]$$

$2a\ln(2b) - (b+\epsilon)\ln(b+\epsilon) + (b-\epsilon)\ln(b-\epsilon)$ $\qquad b = a+\epsilon$

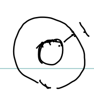
$2\epsilon\ln(2a+2\epsilon) - (2a+\epsilon)\ln(2a+\epsilon) + \epsilon\ln\epsilon$

$2a\left[\cancel{\ln 2a} + \frac{\epsilon}{a}\right] - \cancel{2a\ln 2a} - \epsilon\ln 2a - 2a\frac{\epsilon}{2a} + \epsilon\ln\epsilon$

$= \epsilon(1-\ln 2\epsilon) + \epsilon\ln\epsilon \qquad \times\frac{1}{\epsilon} \to \ln\epsilon + 1 - \ln 2a$!

Set $\phi_{inner} = \phi_{outer}$ ("connected by a wire")

$$\phi(r=a) = \frac{2\pi a\sigma}{a}\left[2a + \eta\left[2a - (a+\epsilon)\ln\frac{2a}{\epsilon} - a\ln(2a\epsilon)\right]\right]$$



Appendix B: Energy of conducting sphere in $\vec{E}_0$ relative to no sphere.

The energy density with conductor relative to that without it is

$$u = \frac{1}{8\pi}\left[(\vec{E}_0 + \vec{E}_{ind})^2 - \vec{E}_0^{\,2}\right] \quad \text{(where } \vec{E}_{ind} \text{ is the induced field)}$$

$$= \frac{1}{8\pi} \vec{E}_{ind} \cdot (2\vec{E}_0 + \vec{E}_{ind})$$

Integrating over all space $\int d^3r\, \vec{E}_0 \cdot \vec{E}_{ind} = \vec{E}_0 \cdot \int d^3r\, \vec{E}_{ind} = 0$

Since $\vec{E}_{ind} = \frac{3\vec{p}\cdot\vec{r}\,\vec{r} - r^2\vec{p}}{r^5}$ $\quad \int \vec{E}_{ind} = 0$ (see spherical symmetry argument below)

We are left with $= \frac{1}{8\pi}\int_V d^3r\, \vec{E}_{ind}^{\,2}$

Now $\vec{E}_{ind}^{\,2} = \frac{1}{r^{10}}\left[(\vec{p}\cdot\vec{r})^2 r^2 (9-3-3) + r^4 p^4\right] = \frac{1}{r^8}\left(3(\vec{p}\cdot\vec{r})^2 + r^2 p^2\right)$

Now the volume V is the space exterior to $|\vec{r}| = a$, so it is spherically symmetric around $\vec{r} = 0$; so that

$$\int_V d^3r\, f(r)\, r_i r_j = \int d^3r\, f(r)\, \frac{1}{3}\delta_{ij} r^2$$

Using this, $\frac{1}{8\pi}\int_V d^3r\, \vec{E}_{ind}^{\,2} = \frac{1}{8\pi}\int_V d^3r\, \frac{1}{r^8}\left(3 p_i p_j \frac{1}{3}\delta_{ij} r^2 + r^2 p^2\right) = \frac{p^2}{4\pi}\int_V d^3r\, \frac{1}{r^6}$

$$= \frac{p^2}{4\pi} \cdot 4\pi \int_a^\infty r^2 dr\, \frac{1}{r^6} = p^2 \frac{1}{3}\frac{1}{a^3} = \frac{1}{3}\vec{p}\cdot\vec{E}_0$$

In subtracting $\vec{E}_0^{\,2}$ from $(\vec{E}_0 + \vec{E}_{ind})^2$ we must also subtract $\int u$ <u>inside</u> the ball (which vanishes for conductor):

$$\int_{|\vec{r}|<a} d^3r\, \frac{1}{8\pi}E_0^2 = \frac{4\pi}{3}a^3\left(\frac{1}{8\pi}E_0^2\right) = \frac{1}{6}a^3 E_0^2$$

$$\Rightarrow \quad \Delta\mathcal{E} = \left(\frac{1}{3} - \frac{1}{6}\right)\vec{p}\cdot\vec{E}_0 = \frac{1}{6}\vec{p}\cdot\vec{E}_0$$

This disagrees with textbook and Landau & Lifshitz. Ugh!

Furthermore...

The textbook models $\vec{E}_0$ as a charge q at $z = s$ with $q \to \infty$ and $s \to \infty$ keeping $q/s^2 = E_0$ fixed, and including an image charge $q' = -\frac{a}{s} q$ at $z = \frac{a^2}{s}$.

So

$$\phi(\vec{r}) = q \left[\frac{1}{|\vec{r} - s\hat{z}|} - \frac{a/s}{|\vec{r} - \frac{a^2}{s}\hat{z}|} \right]$$

On surface:

$$\sigma = -\frac{1}{4\pi} \frac{\partial \phi}{\partial r}\bigg|_{r=a} = \frac{q}{4\pi} \left[\frac{a - s\cos\theta}{(a^2+s^2-2as\cos\theta)^{3/2}} - \frac{\frac{q}{s}(a - \frac{a^2}{s}\cos\theta)}{(a^2 + \frac{a^4}{s^2} - 2\frac{a^3}{s}\cos\theta)^{3/2}} \right]$$

$$= \frac{q}{4\pi} \frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}} \left[a - s\cos\theta - \frac{s^2}{a^2}(a - \frac{a^2}{s}\cos\theta) \right]$$

$$= -\frac{q}{4\pi} \frac{(s^2-a^2)}{a} \frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}}$$

Potential energy, interference term $= \int d^2r' \frac{\sigma(\vec{r}')\, q}{|\vec{r}' - s\hat{z}|} = -\frac{q^2}{4\pi} \frac{(s^2-a^2)}{a} \int_{-1}^{1} 2\pi a^2 \, d\cos\theta \frac{1}{(a^2+s^2-2as\cos\theta)^2}$

$$= -\frac{1}{2} q^2 a (s^2-a^2) \cdot \frac{1}{2as} \cdot \left[\frac{1}{(s-a)^2} - \frac{1}{(s+a)^2} \right]$$

$$= -\frac{1}{2} q^2 a (s^2-a^2) \frac{1}{2as} \frac{4as}{(s^2-a^2)^2} = -\frac{q^2 a}{s^2-a^2}$$

Note that this is the same as for the two point charges, $\frac{q q'}{s - a^2/s} = -\frac{q^2 \frac{a}{s}}{s - a^2/s}$.

with $q = E_0 s^2$, $\mathcal{E} = -\frac{E_0^2 s^4 a}{s^2 - a^2} = -E_0^2 s^2 a - E_0^2 a^3 - O(\frac{a^2}{s^2})$.

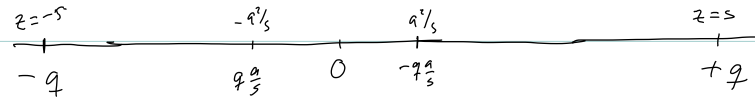
This is not $-\frac{1}{2}\vec{P}\cdot\vec{E}_0$ of exercise 89.5, nor energy in previous exercise.

It diverges. It has not $\frac{1}{2}$. It is negative.

The self-interaction term $\int d^2r\, d^2r' \frac{\sigma(\vec{r})\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{q^2}{(4\pi)^2} \frac{(s^2-a^2)^2}{a^2} a^4 \int d\Omega\, d\Omega' \frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}} \frac{1}{(a^2+s^2-2as\cos\theta')^{3/2}} \frac{1}{|\vec{r}-\vec{r}'|}$

seems convergent and positive, and should be added. No time to compute right now

Another approach: this also has $\phi = 0$ on $|\vec{r}| = a$ and $\vec{E} \simeq$ constant as $s \to \infty$ & $q \to \infty$.



Take $E_0 = \frac{2q}{s^2}$ Then energy to bring in images:

$$\mathcal{E} = \frac{2q\left(-q\frac{a}{s}\right)}{s - a^2/s} + \frac{2q\left(q\,a/s\right)}{s + a^2/s} + \frac{\left(q\frac{a}{s}\right)\left(-q\frac{a}{s}\right)}{2a^2/s}$$

$$= q^2 a\left[\frac{-4a^2}{s^4 - a^4} - \frac{1}{2}\frac{1}{sa}\right]$$

$$= -\frac{1}{4}E_0^2 s^4 a\left[\frac{4a^2}{s^4}\left(1 + \frac{a^4}{s^4} + \cdots\right) + \frac{1}{2sa}\right]$$

$$= -\frac{1}{8}E_0^2 s^3 - E_0^2 a^3$$

Check sign: is $\vec{p} = a^3\vec{E}_0$ or $-a^3\vec{E}_0$?

$$\phi = -E_0 z\left(1 - \frac{a^3}{r^3}\right) \qquad \vec{E}_{ind} = -\vec{\nabla}\, E_0 a^3 \frac{z}{r^3} = -E_0 a^3\left(\frac{-3zx}{r^5}, \frac{-3zy}{r^5}, \frac{-3z^2}{r^5} + \frac{1}{r^3}\right)$$

$$= \frac{3\vec{p}\cdot\vec{r}\,\vec{r} - r^2\vec{p}}{r^5} \quad \text{with } \vec{p} = a^3\vec{E}_0$$