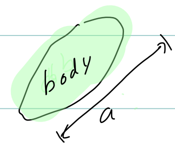
Quasistatic phenomena in conductors

Quasistatic Fields



in time dependent field

study $a \ll \lambda$ = wavelength of $\vec{B}$ or $\vec{E}$ field

ie $\omega a \ll c$ eg $\omega < 1\,GHz$ for $a = 1\,cm$.

Recall (Drude's model) $\tilde{\sigma}(\omega) = \sigma_0 \dfrac{1}{1 - i\omega\tau}$ $\left(\sigma_0 = \dfrac{nq^2\tau}{m}\right)$

So for $\omega \ll \tau^{-1}$, $\tilde{\sigma}(\omega) \simeq \sigma_0$ = constant (independent of ω).

For good conductors $\tau^{-1} \gg \frac{a}{c}$ (unless a is tiny) so we will be in the regime where we can take $\tilde{\sigma}(\omega) = \sigma_0$.

Moreover, for good conductors $\sigma_0 \sim 10^{18}\,Hz$ so $\omega \sigma_0 \lll 1$.

The problem we want to solve is this: put a conductor in an external time dependent magnetic field, $\vec{H}_0(t)$. What are the fields (both magnetic and electric) inside the conductor? Is there a resulting electric field outside the conductor? How is $\vec{H}$ modified outside the conductor? What currents are produced in the conductor?

That $a \ll \lambda \Rightarrow$ working in "near zone", so there are no retardation effects to worry about.

Typical situation: conductor placed inside coil generating $\vec{H}_0(t)$. Also conductor moving into (possibly constant) field $\vec{H}_0$.

Simplification of Maxwell's macroscopic equations:

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{stay the same}$$

We want to use Faraday's law to give us $\vec{E}$ from $\vec{B}$. Since ω is small, we expect $|\vec{E}| \sim \omega|\vec{B}| \ll |\vec{B}|$.

Now $|\vec{D}| \sim |\vec{E}| \ll |\vec{B}| \sim |\vec{H}|$ so $\frac{\partial \vec{D}}{\partial t} \sim \omega \vec{D} \sim \omega^2 B$ can be neglected in

Ampere's law: $\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad \Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}$

Note also that

$$\frac{1}{c}\frac{\partial \vec{D}}{\partial t} \sim \frac{\omega}{c}\epsilon\vec{E} = \frac{\omega}{c}\sigma_0\epsilon\vec{J} \ll \frac{4\pi}{c}\vec{J}$$

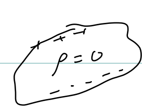
Using Ohm's law

$$\Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} \quad \Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\sigma_0\vec{E}$$

Now $\vec{\nabla}\cdot\left(\frac{4\pi}{c}\sigma_0\vec{E}\right) = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{H}\right) = 0 \quad \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$

and with $\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad \Rightarrow \quad \rho = 0$

$\Rightarrow$ No free charges in bulk of conductor, just as in electrostatics.



(This is not a surprise: we are taking $\omega \approx 0$ in Maxwell equations for conductors).

Summary: $\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial \vec{B}}{\partial t}=0$

$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\sigma_0\vec{E} \qquad \vec{\nabla}\cdot\vec{E}=0$

Take $\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=-\nabla^2\vec{H}$

$\hookrightarrow = \vec{\nabla}\times\left(\frac{4\pi}{c}\sigma_0\vec{E}\right) = \frac{4\pi\sigma_0}{c}\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{4\pi\sigma_0}{c^2}\mu\frac{\partial\vec{H}}{\partial t}$

$$\Rightarrow \boxed{\nabla^2\vec{H}=\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{H}}{\partial t}}$$

Alternatively, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\nabla^2\vec{E}$

$\hookrightarrow \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{1}{c}\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H} = -\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{E}}{\partial t}$

$$\Rightarrow \boxed{\nabla^2\vec{E}=\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

Digression:

Each component of $\vec{H}$ and $\vec{E}$ satisfies the diffusion or heat conduction equation:

$$\nabla^2\psi = \kappa\frac{\partial\psi}{\partial t}$$

If $\psi(\vec{r},t)=\psi(z,t)$ only then $\frac{\partial^2\psi}{\partial z^2}=\kappa\frac{\partial\psi}{\partial t}$

To solve this let $\psi(z,t)=\int\frac{dk}{2\pi}\tilde{\psi}(k,t)\,e^{ikz} \Rightarrow -k^2\tilde{\psi}=\kappa\frac{\partial\tilde{\psi}}{\partial t} \Rightarrow \tilde{\psi}=\tilde{\psi}_0\,e^{-\frac{k^2}{\kappa}t}$

$\Rightarrow \psi(z,t)=\int\frac{dk}{2\pi}\tilde{\psi}_0\,e^{-\frac{k^2}{\kappa}t}e^{ikz}$; the exponent $-\frac{k^2}{\kappa}t+ikz = -\frac{t}{\kappa}\left(k-\frac{i\kappa z}{2t}\right)^2-\frac{\kappa z^2}{4t}$

$\Rightarrow \psi(z,t)=\tilde{\psi}_0\,e^{-\frac{\kappa}{4}z^2/t}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-\frac{t}{\kappa}k^2} = \psi_0\sqrt{\frac{\kappa}{t}}\,e^{-\frac{\kappa}{4}z^2/t}$ (I have absorbed a constant into ψ_0).

Check: $\frac{\partial^2}{\partial z^2}\psi=\psi_0\sqrt{\frac{\kappa}{t}}\frac{\partial}{\partial z}\left(-\frac{\kappa}{2}\frac{z}{t}e^{-\frac{\kappa}{4}z^2/t}\right)=\psi_0\sqrt{\frac{\kappa}{t}}\left(-\frac{\kappa}{2}\frac{1}{t}+\left(\frac{\kappa}{2}\frac{z}{t}\right)^2\right)e^{-\frac{\kappa}{4}z^2/t}$

$\kappa\frac{\partial}{\partial t}\psi=\kappa\,\psi_0\sqrt{\kappa}\left(-\frac{1}{2}t^{-3/2}+\frac{1}{\sqrt{t}}\frac{\kappa}{4}z^2/t^2\right)e^{-\frac{\kappa}{4}z^2/t}=\psi_0\sqrt{\frac{\kappa}{t}}\left(-\frac{\kappa}{2}\frac{1}{t}+\frac{\kappa^2}{4}\frac{z^2}{t^2}\right)e^{-\frac{\kappa z^2}{4t}}$ ✓✓

3D case: using $\psi(\vec{r},t) = X(x)\,Y(y)\,Z(z)$

$$\frac{1}{X}X'' + \frac{1}{Y}Y'' + \frac{1}{Z}Z'' = \kappa\left(\frac{1}{X}\dot{X} + \frac{1}{Y}\dot{Y} + \frac{1}{Z}\dot{Z}\right)$$

$$\Rightarrow \frac{1}{X}X'' = \kappa\frac{1}{X}\dot{X} + f_x(t) \quad \text{etc.} \qquad \text{with } f_x(t) + f_y(t) + f_z(t) = 0$$

For example, if $f_i(t) = 0$ we have three copies of the 1Dim case

$$\psi = \psi_0 \frac{1}{t^{3/2}} e^{-\frac{\kappa}{4} r^2/t}$$

These well known solutions are appropriate for diffusion: as $t \to 0+$
$\psi(z,t) \to \delta(z)$ and $\psi(\vec{r},t) \to \delta(\vec{r})$, with a clear interpretation:
put a pointlike "drop" of fluid and it diffuses out, with distance $\sim \sqrt{t}$.

In the cases we study the problem is different. Imagine starting with a field $\psi_0(\vec{r})$ at $t=0$ (say an external field that is turned off). What happens next? To this end, solve the eigenvalue problem

$$\nabla^2 \psi_n(\vec{r}) = -\gamma_n \psi_n(\vec{r}) \qquad n = 1, 2, \ldots$$

Then $\psi(\vec{r},t) = \sum_n c_n e^{-(\gamma_n/\kappa)t} \psi_n(\vec{r})$ solves $\nabla^2\psi = \kappa\frac{\partial\psi}{\partial t}$

and the c_n's are chosen so that $\psi(\vec{r},0) = \psi_0(\vec{r}) = \sum_n c_n \psi_n(\vec{r})$

(As usual, with eigensystems, $(\psi_n, \psi_m) = 0$ if $\gamma_n \neq \gamma_m$ so one can orthonormalize the solutions so $c_n = (\psi_n, \psi_0)$).

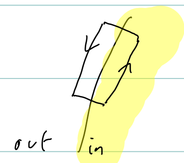
The important point is that ψ_n dies exponentially within a time $\tau \sim \kappa\gamma_1^{-1}$ (assuming $\gamma_1 < \gamma_2 < \ldots$).

Since we expect $\gamma_1 \sim \frac{O(1)}{a^2}$, the typical decay time is $\tau \sim a^2\kappa = \frac{4\pi\mu\sigma_0 a^2}{c^2}$

which for $a \sim 1\,\text{cm}$ and $\sigma_0 \sim 10^{17}\,\text{sec}$, $\mu \sim 1$, gives $\tau \sim 10^{-3}\,\text{sec}$.

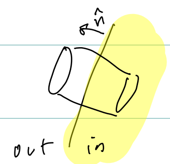
Boundary conditions: (to solve problem fully)

Assume boundaries are between conductor and vacuum.



$$\vec\nabla \times \vec E + \frac{1}{c}\frac{\partial \vec B}{\partial t} = 0 \Rightarrow \vec E_{t_{in}} = \vec E_{t_{out}}$$

$$\vec\nabla \times \vec H = \frac{4\pi}{c}\sigma_0 \vec E \longrightarrow \vec H_{t_{in}} = \vec H_{t_{out}}$$



$$\vec\nabla \cdot \vec B = 0 \longrightarrow B_{n,out} = B_{n,in}$$

Note that with $\mu = 1$ this means $\vec H_{in} = \vec H_{out}$ (or $\vec B_{in} = \vec B_{out}$)

Left with E_n ? L&L says: $\vec\nabla \cdot \vec J = 0$ and $\vec J_{out} = 0 \Rightarrow J_{n,in} = 0$

<u>and since</u> $\vec E_{in} = \sigma_0 \vec J_{in} \Rightarrow E_{n,in} = 0$.

Digression: Garg wants a more refined version. From the previous unit, we had

$$\vec\nabla \cdot \tilde{\vec Z} = 4\pi \tilde\rho' \quad \text{where} \quad \tilde{\vec Z} = \tilde\zeta \tilde{\vec E} \quad \text{and} \quad \tilde\zeta = \tilde\epsilon + i\frac{4\pi\tilde\sigma}{\omega}$$

and $\tilde\rho'$ are charges from currents not subject to Ohm's law (ie, not included in $\tilde{\vec E} = \tilde\sigma \tilde{\vec J}$). From this $\tilde E_{n,out} = \tilde\zeta \tilde E_{n,in}$ (for $\rho' = 0$).

From this we recover $E_{n,in} = 0$ (that is $E_{n,in} \sim \frac{\omega}{\sigma_0} E_{n,out} \to 0$ in the approx).

If Σ = surface charge density (use Σ rather than σ, to avoid confusion with conductivity), then $Z_{n,out} - Z_{n,in} = 4\pi\Sigma$ (sign from $\hat n$ = outward pointing).

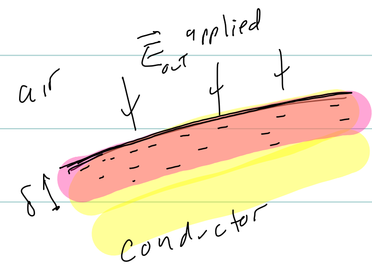
Then, using $E_{n,in} = -i\frac{\omega}{4\pi\sigma_0} E_{n,out} \Rightarrow E_{n,out}\left(1 - i\frac{\omega}{4\pi\sigma_0}\right) = 4\pi\Sigma \Rightarrow E_{n,out} \simeq \left(4\pi + i\frac{\omega}{\sigma_0}\right)\Sigma$

while $E_{n,in} \simeq -i\frac{\omega}{4\pi\sigma_0} E_{n,out} \simeq -i\frac{\omega}{\sigma_0}\Sigma$

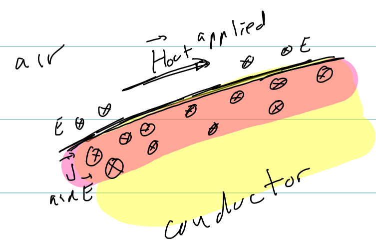
So where are we going with all this?

Put a conductor in an external quasistatic field ($\vec{E}$ or $\vec{H}$)

From the difussion/heat transfer equation we expect the fields will not penetrate the conductor much. For $\vec{E}$ it is clear, much like in electrostatic case, charge at surface will screen. But now the charge is spread over some "skin depth" δ fixed by diffusion equation.



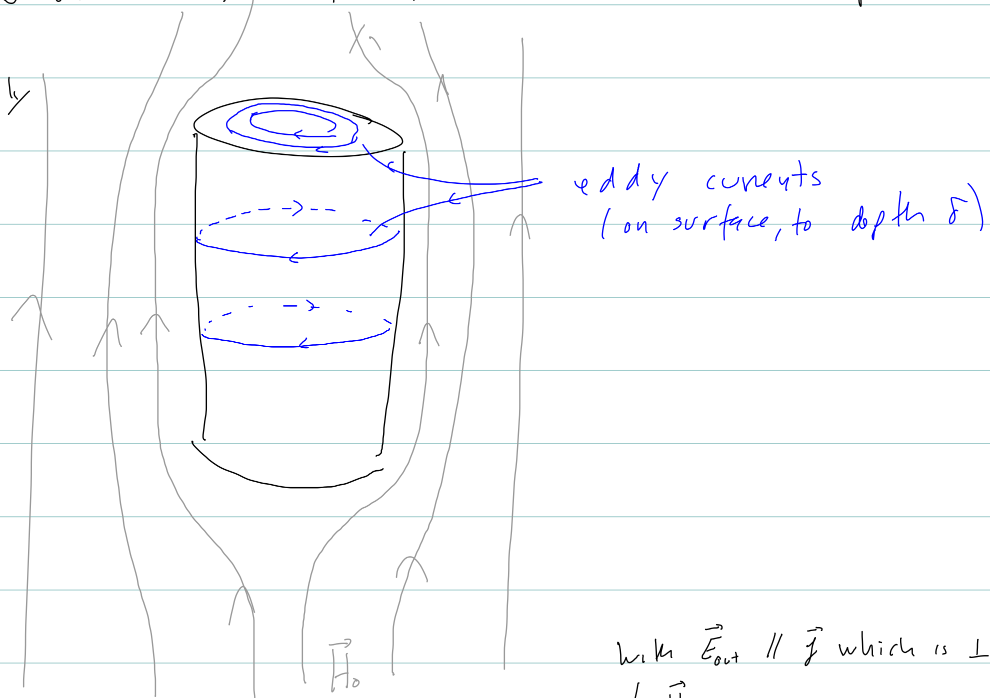
For magnetic field to be screened we need a current on the surface — down to depth δ. Since $\vec{J} = \sigma_0 \vec{E}$ and $J_n = 0$ at boundary, we will have an $\vec{E}_{t,in}$, but then $\vec{E}_{t,out} = \vec{E}_{t,in}$, so also outside



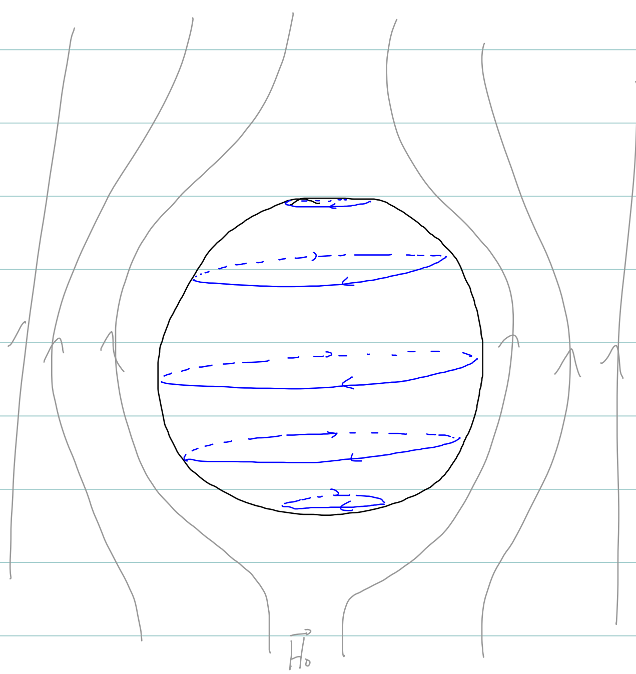
So we want to understand the skin depth and these currents called eddy currents.

Once we look at these in general terms, we can look at specific cases. Garg shows two geometries (of conductors) in time varying (harmonic) external $\vec{H}$: cylinder and sphere.

Qualitatively

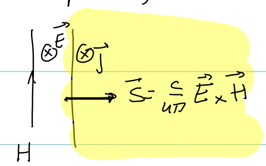


with $\vec{E}_{out} \parallel \vec{J}$ which is $\perp$ to $\vec{H}$



We also want to understand energy conservation:

We see (above pic's)



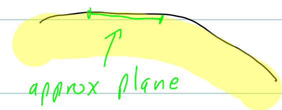
There is energy flow into conductor. Where does it go?

There is also energy dissipation, from $\vec{J} \cdot \vec{E} = \sigma E^2$ in the conductor

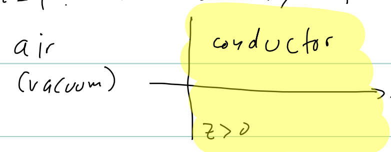
The energy that flows in = energy dissipated.

Plane conductor

While $a \to \infty$ is outside the regime we are studying, we can look at a plane conductor as a local approximation of a large but finite size conductor



Take $\mu = 1$. Set boundary of conductor on xy plane, and conductor on $z>0$.



Assume $\vec{B} (= \vec{H}) = B_0 \hat{x}$ (along plane; $e^{-i\omega t}$ dependence implicit) for $z = 0-$.

$\Rightarrow$ $\vec{B} = B_0 \hat{x}$ for $z = 0+$ because $\vec{B}_{in} = \vec{B}_{out}$.

The difussion equation is

$$\nabla^2 \vec{B} = -i \frac{4\pi \sigma_0 \omega}{c^2} \vec{B}$$

For $B_{y,z}$ with b.c. $B_{y,z} = 0$ at $z=0$ gives $B_{y,z} = 0$.

For B_x, we look for a solution that depends on z only, $B_x = B_0 \hat{x} e^{ikz}$

$$\Rightarrow \quad k^2 = i \frac{4\pi\sigma_0\omega}{c^2} \quad \Rightarrow \quad k = \pm \sqrt{i} \sqrt{\frac{4\pi\sigma_0\omega}{c^2}}$$

With $\sqrt{i} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$ this is $k = \pm (1+i) \sqrt{\frac{2\pi\sigma_0\omega}{c^2}}$

This gives $\sim e^{\pm(i-1)\sqrt{\frac{2\pi\sigma_0\omega}{c^2}} z}$. The $-$ sign solution gives B_x increasing with z, which is unphysical. So keep only $+$ sign:

$$B_x = B_0 e^{-(1-i)\sqrt{\frac{2\pi\sigma_0\omega}{c^2}} z} = B_0 \, e^{-(1-i) z/\delta}$$

where $\delta = \frac{c}{\sqrt{2\pi\sigma_0\omega}}$ is the "skin depth".

For Cu @ 300°K, $\delta(60\,\text{Hz}) \simeq 8.5\,\text{mm}$, $\delta(100\,\text{MHz}) = 7\,\mu\text{m}$

From $\vec{\nabla}\times\vec{H} = \frac{4\pi\sigma_0}{c}\vec{E}$ we can compute $\vec{E}$:

$$\epsilon_{ijk}\,\partial_j B_k = \epsilon_{izx}\frac{\partial B_x}{\partial z} = \delta_{iy}\frac{\partial}{\partial z}\left(B_0 e^{-(1-i)z/\delta}\right) = \delta_{iy}\left[-(1-i)\frac{1}{\delta}B_0 e^{-(1-i)z/\delta}\right]$$

$$\Rightarrow \quad \vec{E} = -\frac{cB_0}{4\pi\sigma_0\delta}(1-i)\,\hat{y}\;e^{-(1-i)z/\delta} = E_0\,\hat{y}\,e^{-(1-i)z/\delta}$$

where $E_0 = -\dfrac{cB_0}{4\pi\sigma_0\delta}(1-i)$

Writing the fields as 'real part of' and restoring ω-dependence we discover phase shift:

$$\vec{B} = B_0\hat{x}\,e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$$

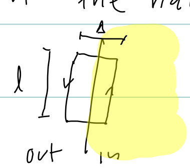
$$\vec{E} = -|E_0|\,\hat{y}\,e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t-\frac{\pi}{4}\right)$$

(The phase shift is from $1-i = \sqrt{2}\,e^{-i\pi/4}$).

In addition $\vec{J} = \sigma_0\vec{E}$ is now determined. Note that for $\delta \ll a$ the current is confined to the "surface" of the conductor, and can be modeled by a surface current density $\vec{K} = \int_0^\infty dz\,\vec{J} = \sigma_0 E_0\,\hat{y}\,\frac{\delta}{1-i} = -\frac{cB_0}{4\pi}\,\hat{y}$

In $\vec{K} = -\frac{cB_0}{4\pi}\hat{y}$ there is no σ_0 ! $\vec{K}$ is there to shield $\vec{B}_{in}$:

In the naive approach one has, from Ampere's law



$$\int d^2a\,\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\int d^2a\,\vec{J}$$

$$\oint \vec{B}\cdot d\vec{l} = (B_{int} - B_{out})\,l = \frac{4\pi}{c}\,l\,K_\perp$$

ie $\quad B_{in} - B_{out} = \frac{4\pi}{c}K_\perp$.

We have $K_\perp = -\frac{c}{4\pi}B_{out}$ so it must be that $B_{in} = 0$. In this naive approximation B_{in} corresponds to the our B_{in} at $z \gg \delta$, hence vanishingly small.

Define "surface impedance" Z_s : $\vec{E} = Z_s \vec{K}$

So that in the present case $Z_s = \dfrac{(1-i)}{\sigma_0 \delta}$

Note also that, as expected $|\vec{E}|/|\vec{B}| \ll 1$:

$$\frac{|\vec{E}|}{|\vec{B}|} = \frac{c}{4\pi\sigma_0}\frac{\sqrt{2}}{\delta} = \frac{c}{4\pi\sigma_0}\sqrt{2}\,\frac{\sqrt{2\pi\sigma_0\omega}}{c} = \sqrt{\frac{\omega}{4\pi\sigma_0}} \ll 1$$

<u>Energetics</u>: Compute $\vec{S}_{out}$; $\vec{E}_{t\,in} = \vec{E}_{t\,out}$ gives $\vec{E}$ outside conductor ($E_{n,out} = \tilde{\varepsilon} E_{n,in} = 0$). Then

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{c}{4\pi}\left(E_0 B_0\, \hat{y}\times\hat{x}\right) = \frac{c}{4\pi} E_0 \left(-\frac{4\pi\sigma_0\delta}{c(1-i)}E_0\right)(-\hat{z})$$

$$= \frac{\sigma_0\delta}{1-i} E_0^2\, \hat{z}$$

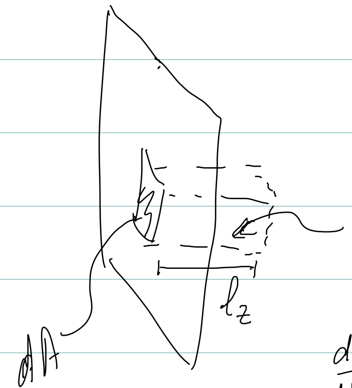
Brief review of averaging over time: complex fields $a e^{-i\omega t}$ are really $\frac{1}{2}(a e^{-i\omega t} + a^* e^{i\omega t})$. Then $\overline{ab} = \frac{1}{T}\int_0^T dt\, \frac{1}{4}(a e^{-i\omega t} + c.c)(b e^{-i\omega t} + c.c)$

$$= \frac{1}{4}(ab^* + a^* b) = \frac{1}{2}\mathrm{Re}(ab^*)$$

Time average $\vec{S}$:

$$\overline{\vec{S}} = \frac{1}{2}\mathrm{Re}\left(\frac{\sigma_0\delta}{1-i} E_0 E_0^*\, \hat{z}\right) = \frac{1}{4}\sigma_0\delta|E_0|^2\, \hat{z}$$

Let's compare with the energy dissipated. Work done per unit volume per unit time: $\vec{j}\cdot\vec{E}$. Time averaged: $\frac{1}{2}\,\mathrm{Re}(\vec{j}\cdot\vec{E}^*)$.



work done in volume: $\frac{1}{2}\mathrm{Re}(\vec{j}\cdot\vec{E}^*)\, l_z\, dA$

$\Rightarrow$ work done / unit surface area

$$\frac{dQ}{dt\,dA} = \int_0^\infty dz\ \frac{1}{2}\mathrm{Re}(\vec{j}\cdot\vec{E}^*)$$

$$= \int_0^\infty dz\ \frac{1}{2}\mathrm{Re}(\sigma_0\,\vec{E}\cdot\vec{E}^*)$$

Use $\vec{E} = E_0\,\hat{y}\,e^{-(1-i)z/\delta}$ $\longrightarrow$

$$= \frac{1}{2}\sigma_0|\vec{E}_0|^2\int_0^\infty dz\ e^{-2z/\delta}$$

$$= \frac{1}{4}\,\sigma_0\,\delta\,|E_0|^2$$

Same as $|\vec{S}|$! Energy flows in = energy dissipated. ☺

Note one can also write $Z_s^{-1} = \frac{\sigma\delta}{1-i} = \frac{\sigma\delta}{2}(1+i)$ so

$$\frac{dQ}{dt\,dA} = \frac{1}{2}\mathrm{Re}\left(\frac{1}{Z_s}\right)|E_0|^2 = \frac{1}{2}\mathrm{Re}\left(\frac{1}{Z_s}\vec{E}_0\cdot\vec{E}_0^*\right) = \frac{1}{2}\mathrm{Re}(\vec{K}\cdot\vec{E}_0)$$

Or since $\left|\frac{\vec{E}}{\vec{B}}\right|^2 = \frac{\sqrt{2}\,c}{4\pi}\frac{1}{\sigma_0\delta}$, $\quad \frac{dQ}{dt\,dA} = \frac{1}{4}\,2\left(\frac{c}{4\pi}\right)^2\frac{1}{\sigma_0\delta}|\vec{B}_0|^2 = \frac{1}{2}\left(\frac{c}{4\pi}\right)^2\mathrm{Re}\,Z_s\,|\vec{B}_0|^2$

$\vec{E}$ outside conductor? (ie for $z<0$).

As we said in the introduction, it is given by Faraday's law

$$\vec{\nabla}\times\vec{E} - i\frac{\omega}{c}\vec{B} = 0$$

with b.c. $\vec{E}(z=0-) = E_0\,\hat{y}$

By symmetry $\vec{E} = \hat{y}\, E_y(z)$ only, and recall $\vec{B} = \hat{x} B_0$

so $(\vec{\nabla}\times\vec{E})_x = -\frac{\partial E_y}{\partial z} = i\frac{\omega}{c} B_0$

$$\Rightarrow \quad E_y = E_0 - i\frac{\omega}{c} B_0 z \qquad \text{(disagree with sign in Garg).}$$

$$\lambda \frac{\omega}{2\pi} = c \qquad = E_0 - 2\pi i \left(\frac{z}{\lambda}\right) B_0$$

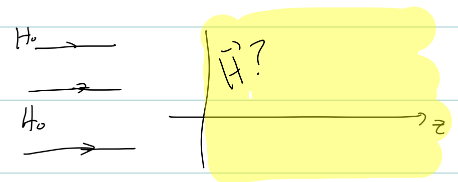
So $E_y(z)$ seems to increase without bounds as $z\to-\infty$. But this is not so: we are assuming distance scales are $\ll \lambda$ so the solution is a good approximation only close to the conductor. For example if conductor is inside a solenoid, $z\to-\infty$ will take us outside this region. Moreover, as $|\vec{E}|$ increases, it cannot be neglected in Ampere's law.

$\delta(\omega)$: Note $\delta(\omega) = \frac{c}{\sqrt{2\pi\sigma_0\omega}} \sim \frac{1}{\sqrt{\omega}} \to \infty$ as $\omega\to 0$! It would appear that in static case $\vec{E}$ penetrates the whole conductor! But wait

$|\vec{E}|/|\vec{B}| = \sqrt{\frac{\omega}{4\pi\sigma_0}} \to 0$ as $\omega\to 0$. So there is no field.

$\vec{H} = H_0 \hat{z}$?

Now suppose the external applied field is perpendicular to the surface $z=0$, $\vec{H}_0 = H_0 \hat{z}$



We want to find $\vec{H}$ inside. But importantly, $\vec{H}$ outside is not $\vec{H}_0$. Still as a 1st guess we can assume $\vec{H} \simeq \vec{H}_0$ at $z=0-$ (just outside) $\Rightarrow \vec{H}_{in} = \vec{H}_{out}$ means $\vec{H}(z=0+) = H_0 \hat{z}$. Let's use the diffusion equation to find $\vec{H}$ ($= \frac{1}{\mu}\vec{B} = \vec{B}$ since we are using $\mu=1$): but this is the same as before, except for $H_z = B_z$ instead of B_x:

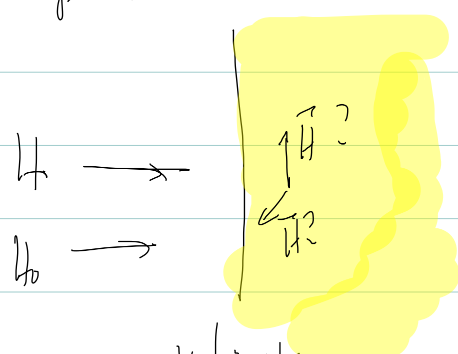
$$H_z = H_0 \hat{z}\, e^{-(1-i)z/\delta}$$

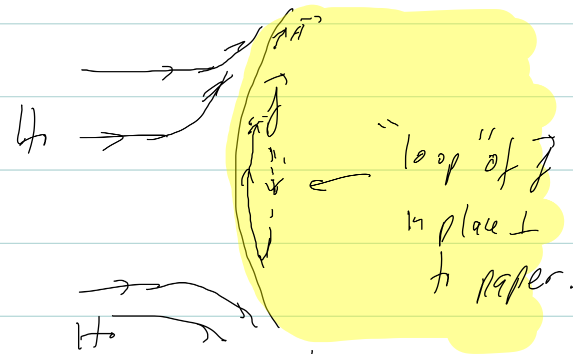
The problem here is $\vec{\nabla}\cdot\vec{B} = \overset{1}{\cancel{\mu}}\, \vec{\nabla}\cdot\vec{H} = \partial_z H_z \neq 0$ in violation of $\vec{\nabla}\cdot\vec{B} = 0$.

To understand what happens we cannot continue to take an infinite plane approximation to the finite size body, of size $a \ll \lambda$. A field $\vec{H}_{\parallel}$ must be produced that is parallel to the surface (Garg calls it $B_\perp$ because it is perpendicular to $\vec{H}_0$ – I find his nomenclature confusing). With this one can satisfy $\vec{\nabla}\cdot\vec{B} = 0$ since $\vec{\nabla}\cdot\vec{B} = \frac{\partial B_{\parallel}}{\partial x_{\parallel}} + \frac{\partial B_z}{\partial z}$

Note that $\vec{B}$ is then confined to a region of depth δ in the conductor.

Let's assume $\delta \ll a$ (the opposite limit $\delta \gg a$ is basically that of $\omega = 0$, i.e., magnetostatics). Then, in order to shield the bulk of the conductor from $\vec{B}$ we need a current $\vec{J}$ in the skin. What breaks the symmetry in the xy plane if $\vec{H}_0 = H_0 \hat{z}$, i.e., is $H_\parallel$ along $\hat{x}$ or $\hat{y}$? The answer is the finite size, as is easily seen from the picture:





And, of course that means there is an $\vec{E}$ field $\left(\vec{E} = \frac{1}{\sigma}\vec{J}\right)$. Note that $H_\parallel$ changes on scale of curvature, which itself is the scale of the size a of the body, while H_z changes over scale δ. Since $\vec{\nabla}\cdot\vec{B} = 0$ we have $\frac{H_\parallel}{a} \sim \frac{H_0}{\delta}$ or $H_\parallel = \frac{a}{\delta} H_0 \gg H_0$: The $\parallel$-field dominates and again $\vec{H}_{in} = \vec{H}_{out}$ means now that close to the body $\vec{H}$ is nothing like the uniform applied $\vec{H}_0$ — on surfaces that are not parallel to $\vec{H}_0$.

One can see this explicitly in analytic solutions of the cylinder and sphere problems (those shown in p. 7 of these notes), but we will not go through those calculations. The general principles is what we are after, and that is enough to figure out generally what happens in other geometries, e.g.