Coleman, Acausality
Barut, Chaps V, VI

Self-Field of Electron & the Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than infering the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of $\vec{F}_{RR}$ ("RR" = radiation reaction) is not completely justified, can we obtain $\vec{F}_{RR}$ directly?

The program should be clear:

(i) Compute $A_\mu \to F_{\mu\nu}$ due to electron

(ii) Compute $\vec{F}_{RR}$ due to $F_{\mu\nu}$ → give motion of electrons

Of course, there is no ordering here (which is first, the chicken of the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (ie, mass). We should regulate this (ie, cut-off the integral near $\vec{x} = \vec{x}_{electron}$), then subtract it using a "bare" mass (i.e, a contribution to the energy which is not of electromagnetic origin). Call this m_0.
* The radiation field should give rise to a force responsible for energy loss: it should be T-odd (dissipation! think air drag $\vec{F} \propto \vec{v}$) and we expect $\vec{F}_{RR} \propto \dddot{\vec{x}}$.

The two equations are

- Field due to point charge (electron):

$$A_\mu(x) = 4\pi q \int d\lambda\, u_\mu\, G_{ret}(x - y(\lambda))$$

Notes from (PHYS 203A, p.1 of chapter 4 "Fields of Moving charges")

and

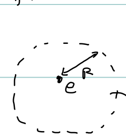
- Equation of motion

$$\frac{dp_\alpha}{d\lambda} = \frac{q}{c} F_{\alpha\beta} U^\beta$$

(PHYS 203A, chap 2., p.6)

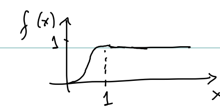
And we take $p^\alpha = m_0 U^\alpha$ with m_0 as explained above.

The integral giving A_μ will diverge at the electron. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) It only affects $|\vec{x} - \vec{x}_{el}| < r_e$ and (ii) I has a paramter that removes the cutting-off in some limit. For example



A_μ = unregulated

$$A_\mu \to f\left(\frac{r}{R}\right) A_\mu^{unregulated}$$



Remove cut-off by $R \to 0$.

Our choice of cut-off is in wave-number space: recall

$$G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2}$$

(PHYS203A chap. 2, p.13 of revised notes).

The $x \to 0$ region corresponds to $k \to \infty$. So we take

$$G_\Lambda(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left[\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \right]$$

(cut-off removed by $\Lambda \to \infty$).

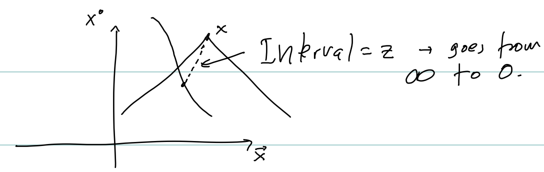
It is $k^2 - \Lambda^2$ rather than $k^2 + \Lambda^2$ so that poles are at $k^0 = \pm\sqrt{\vec{k}^2 + \Lambda^2}$, real.

We are ready to compute. We need $F_{\mu\nu}$ so take $\partial_\mu A_\nu$ above:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 4\pi q \int d\lambda \, U_\nu \, \partial_\mu G_\Lambda^{ret}(x - y(\lambda)) \; - (\mu \leftrightarrow \nu)$$

Here $U^\nu = \frac{dy^\nu}{d\lambda}$. The integral runs over $-\infty < \lambda < \lambda_0$, where λ_0 solves the retarded condition $x - y(\lambda_0) = 0$. We choose as parameter

$$\lambda = z^2 = (x-y)^2$$



This is useful because $G_\Lambda^{ret}(x)$ is a scalar function with dimensions of L^{-2}, ie, of wave-vector, so it depends on x and Λ only through the combination $\Lambda^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G_\Lambda^{ret}(x-y) = \Lambda^2 f(\Lambda z)$$

for some function f. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and <u>not</u> necessary.

Use $\partial_\mu^x G_\Lambda^{ret}(x-y) = \Lambda^2 \partial_\mu^x z \frac{df}{dz}$ and $2z\,\partial_\mu^x z = 2(x-y)^\lambda \, \partial_\mu (x-y)_\lambda$

Recall

$$\partial_\mu^x y_\lambda = \frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \qquad \text{(PHYS703A, chap 4, p. 2)}$$

so $z\,\partial_\mu^x z = (x-y)^\lambda \left[\frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \right] = (x-y)_\mu$ so we have

$$\partial_\mu G_\Lambda^{ret} = \Lambda^2 \frac{(x-y)_\mu}{z} \frac{df}{dz} \quad \text{and} \quad F_{\mu\nu} = 4\pi q \Lambda^2 \int_0^\infty dz \, \frac{dy_\nu}{dz} \frac{(x-y)_\mu}{z} \frac{df}{dz} \; - (\mu \leftrightarrow \nu)$$

Integrate by parts

$$F_{\mu\nu}(x) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \frac{d}{dz}\left[\frac{(x-y)_\mu}{z}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

$$= 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z}\frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

Now, we are interested in $F_{\mu\nu}(x)$ for x = location of charge q. So at some time x^0, we want $x = y(\lambda_*)$ with λ_* determined by $x^0 = y^0(\lambda_*)$. In terms of z, λ_* is $z=0$. So $x-y = y(0)-y(z)$.

Since the divergences are associated with the field at $x = x_{\text{electron}} = y(0)$, we expand the integrand in powers of z.

Note that $$\int_0^\infty dz\, f(\Lambda z)\, z^n = \frac{1}{\Lambda^{n+1}} \underbrace{\int_0^\infty d\xi\, f(\xi)\, \xi^n}_{\text{some pure number}} \equiv c_n \frac{1}{\Lambda^{n+1}}$$

So only a finite number of terms need be retained: beyond some power the expansion terms vanish as $\Lambda \to \infty$. This will leave us with some divergent terms (expected, like self-energy), and some Λ-independent terms, the big pay-off of this long computation.

In fact, since there is a Λ^2 in front we need include only $n=1$ above.

So we have

$$F_{\mu\nu}(x) = 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z} \frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2} \frac{dy_\nu}{dz} \right] - (\mu \leftrightarrow \nu)$$

Use $y^\mu(z) = y^\mu(0) + z \left.\frac{dy^\mu}{dz}\right|_0 + \frac{z^2}{2} \left.\frac{d^2 y^\mu}{dz^2}\right|_0$ and $\frac{d^2 y_\nu}{dz^2} = \left.\frac{d^2 y_\nu}{dz^2}\right|_0 + z \left.\frac{d^3 y_\nu}{dz^3}\right|_0$

and let dots denote derivatives at current time, ie, $\dot{y}^\mu = \left.\frac{dy^\mu}{dz}\right|_0$, etc.

$$F_{\mu\nu}(y(0)) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(z\Lambda) \left[(\dot{y}_\mu + \tfrac{1}{2} z \ddot{y}_\mu)(\ddot{y}_\nu + z \dddot{y}_\nu) - \mu \leftrightarrow \nu \right] \quad + \mathcal{O}(1/\Lambda)$$

↳ ignore henceforth

$$= -4\pi q \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}_\nu - \dot{y}_\nu \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}_\nu - \dot{y}_\nu \dddot{y}_\mu) \right]$$

Postpone determination of c_0 & c_1. Instead, we are ready to compute $\vec{F}_{RR}$

$$\frac{d}{dz} m_0 u_\alpha = \frac{q}{c} F_{\alpha\beta} u^\beta \quad \text{or}$$

$$m_0 \ddot{y}_\mu = \frac{q}{c} F_{\mu\nu} \dot{y}^\nu = -\frac{4\pi q^2}{c} \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}\cdot\dot{y} - \dot{y}^2 \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}\cdot\dot{y} - \dot{y}^2 \dddot{y}_\mu) \right]$$

Now, as $z \to 0$, $\frac{dz}{ds} \to 1$, so we can interpret the derivatives as w.r.t s

so $\dot{y}^2 = 1$ and $\dot{y}\cdot\ddot{y} = 0$ (and $\dot{y}\cdot\dddot{y} + \ddot{y}^2 = 0$). So

$$\left(m_0 - \frac{4\pi q^2}{c} c_0 \Lambda\right) \ddot{y}_\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}^2\right)$$

The divergent self-energy can be combined with a divergent bare mass $m_0(\Lambda)$ to leave a finite mass, the physical electron mass $m_e = m_0 - \frac{4\pi q^2}{c} c_0 \Lambda$ (so we don't much care what c_0 is). So we have

$$m_e \ddot{y}^\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2\right)$$

In the nonrelativistic limit, $\dot{\vec{y}} \ll 1$ and we recognize the NR version of $\vec{F}_{RR}$, proportional to $\dddot{\vec{y}}$. Comparing with our simplistic energy conservation-on-average argument we can read off the constant c_1:

$\vec{F}_{RR} = \frac{2}{3}\frac{q^2}{c^3}\frac{d^3\vec{y}}{dt^3} \Rightarrow 4\pi c_1 = \frac{2}{3}$ $\left(c_1 = \frac{1}{6\pi}\right)$. So finally

$$\boxed{m\ddot{y}^\mu = \frac{2}{3}\frac{q^2}{c}\left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2\right)}$$