10-6   (a) $n_1 + n_2 \approx 10^{20}$

$$\frac{n_2}{n_1} = \exp\left(\frac{-4.86 \text{ eV} \times 1.602 \times 10^{-19} \text{ J/eV}}{1.38 \times 10^{-23} \text{ J/K} \times 1.600 \times 10^3 \text{ K}}\right) = 4.98 \times 10^{-16}.$$

Assuming $n_1 \approx 10^{20}$,

$$n_2 \approx n_1 \left(4.98 \times 10^{-16}\right) = \left(10^{20}\right)\left(4.98 \times 10^{-16}\right) = 4.98 \times 10^4 .$$

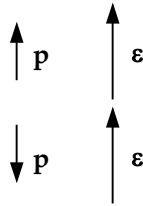(b)   Power emitted = number of photons emitted/s × (energy/photon)

$$= \left(\frac{1}{\tau}\right) \times n_2 \times 4.86 \text{ eV}$$

$$= 10^7 \text{ s}^{-1} \times 4.88 \times 10^4 \times 4.86 \text{ eV} \times 1.602 \times 10^{-19} \text{ J/eV}$$

$$= 3.88 \times 10^{-7} \text{ J/s}$$

$$= 0.388 \ \mu\text{W}$$

10-7   (a)

$$E_U = -\mathbf{p} \cdot \boldsymbol{\varepsilon} = -\varepsilon \cos 0° = -p\varepsilon$$
$$E_D = -\mathbf{p} \cdot \boldsymbol{\varepsilon} = -\varepsilon \cos 180° = +p\varepsilon$$

so $\Delta E = E_D - E_U = 2p\varepsilon$.

(b)   Let $n(2p\varepsilon)$ be the number of molecules in the excited state.

$$\frac{n(2p\varepsilon)}{n(0)} = \frac{g(2p\varepsilon) A e^{-2p\varepsilon/k_B T}}{g(0) A e^0} = 2 e^{-2p\varepsilon/k_B T}$$

(c) $\dfrac{1.90}{1} = \dfrac{n(2p\varepsilon)}{n(0)} = 2 e^{-2p\varepsilon/k_B T}$. For $p = 1.0 \times 10^{-30}$ Cm and $\varepsilon = \left(1.0 \times 10^6 \text{ V/m}\right)$,

$$\frac{2p\varepsilon}{k_B T} = \frac{(2)\left(1.0 \times 10^{-30} \text{ Cm}\right)\left(1.0 \times 10^6 \text{ V/m}\right)}{\left(1.38 \times 10^{-23} \text{ J/K}\right) T} = \frac{0.144 \ 9}{T}$$

so $1.90 = 2 e^{-0.1449/T}$ or $0.95 = e^{-0.1449/T}$. Solving for T, $\ln(0.95) = \dfrac{-0.149 \ 9}{T}$ or

$T = 2.83$ K .

(d) $\overline{E} = [n(2p\varepsilon)][2p\varepsilon] + \dfrac{[n(0)][0]}{n(2p\varepsilon) + n(0)} = \dfrac{[n(2p\varepsilon)/n(0)](2p\varepsilon)}{[n(2p\varepsilon) + n(0)] + 1} = \dfrac{\left[2e^{-2p\varepsilon/k_B T}\right][2p\varepsilon]}{2e^{-2p\varepsilon/k_B T} + 1}$

$$= \frac{2p\varepsilon}{1 + (1/2) e^{2p\varepsilon/k_B T}} .$$

As $T \to 0$, $\overline{E} \to 0$ and as $T \to \infty$, $E \to \dfrac{2p\varepsilon}{3/2} = \dfrac{4p\varepsilon}{3}$.

(e) $dE_{\text{total}} = \bar{N}E = \dfrac{2p\varepsilon N}{1+(1/2)e^{2p\varepsilon/k_B T}}$

$C = \dfrac{dE_{\text{total}}}{dT} = \dfrac{(Nk_B/2)(2p\varepsilon/k_B T)^2\, e^{2p\varepsilon/k_B T}}{\left[1+(1/2)e^{2p\varepsilon/k_B T}\right]^2}$

(f)   By expanding e^x where $x = \dfrac{2p\varepsilon}{k_B T}$ one can show that $C \to 0$ for $T \to \infty$ as

$C = \left(\dfrac{8N}{9}\right)\left(\dfrac{p^2\varepsilon^2}{k_B^2}\right)\left(\dfrac{1}{T^2}\right)$ and $C \to 0$ for $T \to 0$ as $C = \dfrac{(2Nk_B)[2p\varepsilon/(k_B T)]^2}{e^{2p\varepsilon/k_B T}}$. To find

the maximum in $C = \left(\dfrac{Nk_B}{2}\right)(x^2)\left\{\dfrac{e^x}{\left[1+(1/2)e^x\right]^2}\right\}$ set $\dfrac{dC}{dT} = 0$ or $\left(\dfrac{dC}{dx}\right)\left(\dfrac{dx}{dT}\right) = 0$.

Taking derivatives we get:

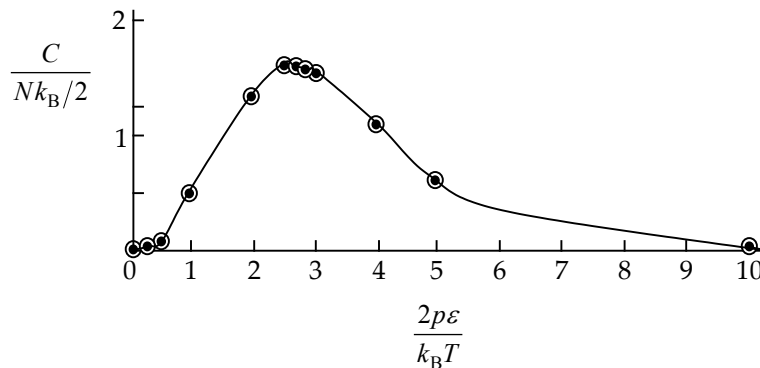$$\left[\dfrac{-x^3 e^x}{\left(1+(1/2)e^x\right)^2}\right]\left[\dfrac{2+x-xe^x}{1+(1/2)e^x}\right] = 0.$$

Setting the first factor equal to 0 yields the minima in C at $T = 0$ and $T = \infty$, while the second factor yields a maximum at the solution of the transcendental equation,

$\dfrac{x+2}{x-2} = \dfrac{e^x}{2}$. This transcendental equation has a solution at $x \approx 2.65$, which

corresponds to a temperature of $\dfrac{2p\varepsilon}{k_B T} = 2.65$ or $T = \dfrac{2p\varepsilon}{2.65 k_B} = \dfrac{0.144\,9}{2.65} = 0.054\,7\ \text{K}$. The

expression for heat capacity can be rewritten as $C = A\left[e^x \dfrac{x}{\left(1+(1/2)e^x\right)^2}\right]$ where

$A = \dfrac{Nk_B}{2}$ and $x = \dfrac{2p\varepsilon}{k_B T}$. Below is the sketch of C as a function of $\dfrac{2p\varepsilon}{k_B T}$.

The heat capacity is the change of internal energy with temperature. For both large temperature $(T \rightarrow \infty)$ and low temperature $(T \rightarrow 0)$ the internal energy is constant and so the heat capacity is zero. At T approximately equal to 0.0547 K there is a rapid change of energy with temperature; so the heat capacity becomes large and reaches its maximum value.

10-10   (a)   Using Equation 10.18 $\dfrac{(N/V)\hbar^3}{(8)(k_{\mathrm{B}}mT)^{3/2}} \ll 1$ for MB, we get

$$\frac{N}{V} = \frac{6.02 \times 10^{23}}{22.4\ \mathrm{m}^3} = 2.69 \times 10^{22}\ \mathrm{m}^{-3}$$

$$\hbar^3 = 1.174 \times 10^{-102}\ (\mathrm{Js})^3$$

$$k_{\mathrm{B}}T = \left(1.38 \times 10^{-23}\ \mathrm{J/K}\right)(300\ \mathrm{K}) = 4.14 \times 10^{-21}\ \mathrm{J}$$

$$m_{\mathrm{He}} = \frac{4\ \mathrm{g}}{6.02 \times 10^{23}} = 6.64 \times 10^{-27}\ \mathrm{kg}$$

So $\dfrac{(N/V)\hbar^3}{8(k_{\mathrm{B}}mT)^{3/2}} = \dfrac{\left(2.69 \times 10^{22}\ \mathrm{m}^{-3}\right)\left(1.174 \times 10^{-102}\ (\mathrm{Js})^3\right)}{(8)\left(4.14 \times 10^{-21}\ \mathrm{J} \times 6.64 \times 10^{-27}\ \mathrm{kg}\right)^{3/2}} = 2.74 \times 10^{-11} \ll 1$. Thus the

MB distribution is a valid approximation for He gas at STP.

(b)   At 4K, $\dfrac{N}{V} = \dfrac{0.145}{4\ \left(\mathrm{moles/cm}^3\right) \times 6.02 \times 10^{23}\ \mathrm{atoms/mole}} = 2.18 \times 10^{28}\ \mathrm{atom/m}^3$ and

$k_{\mathrm{B}}T = 5.52 \times 10^{-23}\ \mathrm{J}$ so

$\dfrac{(N/V)\hbar^3}{8(k_{\mathrm{B}}mT)^{3/2}} = \dfrac{\left(2.18 \times 10^{28}\ \mathrm{m}^{-3}\right)\left(1.174 \times 10^{-102}\ (\mathrm{Js})^3\right)}{(8)\left(5.52 \times 10^{-23}\ \mathrm{J} \times 6.64 \times 10^{-27}\ \mathrm{kg}\right)^{3/2}} = 1.44 \times 10^{-2}$. This is not $\ll 1$ so

we should use the BE distribution to describe liquid He.

10-13   (a) $C = (3R)\left(\dfrac{\hbar\omega}{k_B T_E}\right)^2 \dfrac{e^{\hbar\omega/k_B T_E}}{\left(e^{\hbar\omega/k_B T_E} - 1\right)^2}$. For $T = T_E$, $k_B T_E = \hbar\omega$, so

$$C = (3R)\left(\frac{\hbar\omega}{\hbar\omega}\right)^2 \frac{e^{\hbar\omega/\hbar\omega}}{\left(e^{\hbar\omega/\hbar\omega} - 1\right)^2} = (3R)\frac{e}{(e-1)^2} = (3R)(0.920\,7) = 2.76R .$$

        Using $R = 1.986$ cal/mol K $\Rightarrow C = 5.48$ cal/mol K .

     (b)   From Figure 10.9, T_E lead ≈ 100 K , T_E aluminum ≈ 300 K , T_E silicon $\simeq 500$ K .
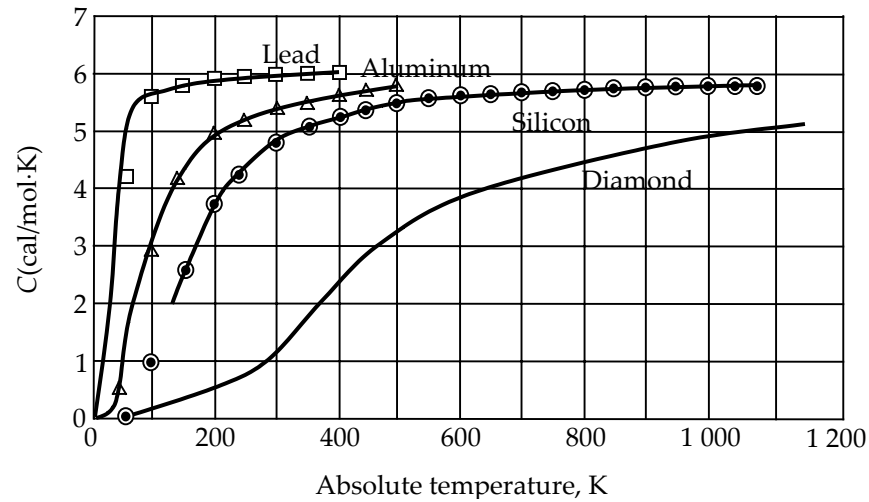
     (c)   Using $C = (3R)\left(\dfrac{T_E}{T}\right)^2 \dfrac{e^{T_E/T}}{\left(e^{T_E/T} - 1\right)^2} = (5.97$ cal/mol K$)\left(\dfrac{T_E/T}{e^{T_E/T} - 1}\right)^2 e^{T_E/T}$ heat capacities

        for lead, aluminum, and silicon were obtained. These results can be summarized in the following tables.

Lead	$T_E = 100$ K		
T(K)	C(cal/(mol K))	T(K)	C(cal/(mol K))
50	4.32	250	5.92
100	5.49	300	5.94
150	5.74	350	5.96
200	5.83	400	6.09
Aluminum	$T_E = 300$ K		
T(K)	C(cal/(mol K))	T(K)	C(cal/(mol K))
50	0.535	250	5.30
100	2.96	300	5.509
150	4.32	350	5.62

| 200 | 4.97 | 400 | 5.70 |
| Silicon | $T_E = 500$ K | | |
T (K)	C (cal/(mol K))	T (K)	C (cal/(mol K))
50	0.027	600	5.64
100	1.02	650	5.67
150	2.55	700	5.74
200	3.64	750	5.75
250	4.97	800	5.78
300	4.76	850	5.81
350	5.05	900	5.84
400	5.25	950	5.85
450	5.41	1 000	5.83
500	5.50	1 050	5.85
550	5.59	1 100	5.95

These values are now plotted on Figure 10.9 as shown.

10-14 (a) $E_F = 7.05$ eV at 0 K for copper.

$$E_{av} = \frac{3}{5} E_F = \frac{3}{5}(7.05 \text{ eV}) = 4.23 \text{ eV}$$

(b) E_{av} (per molecule) $= \frac{3}{2} k_B T = 4.23$ eV

$$T = \frac{(4.23 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})}{(1.5)(1.38 \times 10^{-23} \text{ J/K})} = 3.27 \times 10^4 \text{ K} = 32\,700 \text{ K}$$

10-15   Al: $E_F = 11.63$ eV

(a) $E_F = \frac{h^2}{2m_e}\left(\frac{3n}{8\pi}\right)^{2/3}$ or $n = \frac{8\pi}{3}\left(\frac{2m_e E_F}{h^2}\right)^{3/2}$ so

$$n = \frac{8\pi}{3}\left[\frac{(2)(9.11\times10^{-31}\ \text{kg})(11.63\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})}{(6.625\times10^{-34}\ \text{Js})^2}\right]^{3/2} = 1.80\times10^{29}\ \text{free electrons/m}^3.$$

(b) $n' = \dfrac{\rho N_A}{M} = \dfrac{(2.7\ \text{g/cm}^3)(6.02\times10^{23}\ \text{atoms/mole})}{27\ \text{g/mole}}$

$n' = 6.02\times10^{22}\ \text{atoms/cm}^3 = 6.02\times10^{28}\ \text{atoms/m}^3$

Valence $= \dfrac{n}{n'} = \dfrac{18\times10^{28}}{6\times10^{28}} = 3$

10-16 $E_{\text{av}} = \dfrac{1}{n}\displaystyle\int_0^\infty EN(E)dE$ where $n = \dfrac{N}{V}$. At $T = 0$, $E = 0$ for $E > E_F$. We can take $N(E) = CE^{1/2}$

since $f_{\text{FD}}(E) = 1$ for $E < E_F$ and $f_{\text{FD}}(E) = 0$ for $E > E_F$.

$$E_{\text{av}} = \frac{1}{n}\int_0^{E_F} CE^{3/2}dE = \frac{C}{n}\int_0^{E_F} E^{3/2}dE = \frac{2C}{5n}E_F^{5/2}.$$

Using the condition $n = \displaystyle\int_0^\infty N(E)dE$ we find $\dfrac{C}{n} = \dfrac{3}{2}E_F^{-3/2}$, so that $E_{\text{av}} = \left(\dfrac{2}{5}\right)\left(\dfrac{3}{2}\right)E_F^{-3/2}E_F^{5/2} = \dfrac{3}{5}E_F$.

10-24   (a)   The density of states at the energy E is $g(E) = CE^{1/2}$. Hence, the ratio of the number
of allowed levels at 8.5 eV to the number of allowed levels at 7.0 eV is

$$\frac{g(8.5)}{g(7.0)} = \frac{C(8.5)^{1/2}}{C(7.0)^{1/2}} = \left(\frac{8.5}{7.0}\right)^{1/2} = 1.1 \, .$$

(b)   The number of electrons occupying an energy level E is $N(E) \propto \dfrac{E^{1/2}}{e^{(E-E_F)/k_BT} + 1}$, hence
the ratio required is

$$\frac{N(8.5)}{N(7.0)} = \left(\frac{8.5}{7.0}\right)^{1/2} \frac{e^0 + 1}{e^{(8.5-7.0)/k_BT} + 1} = (1.1)\frac{2}{e^{1.5/0.0259} + 1} = 1.55 \times 10^{-25} \, .$$

Comparing this with (a), we see that very few states with $E > E_F$ are occupied.