5

Matter Waves

5-2 The issue is: Can we use the simpler <u>classical</u> expression $p = (2mK)^{1/2}$ instead of the exact <u>relativistic</u> expression $p = \frac{K\left(1+\frac{2mc^2}{K}\right)^{1/2}}{c}$? As the relativistic expression reduces to $p = (2mK)^{1/2}$ for $K \ll 2mc^2$, we can use the classical expression whenever $K \ll 1\text{ MeV}$ because mc^2 for the electron is 0.511 MeV.

(a) Here $50\text{ eV} \ll 1\text{ MeV}$, so $p = (2mK)^{1/2}$

$$\lambda = \frac{h}{p} = \frac{h}{\left[(2)\left(\frac{0.511\text{ MeV}}{c^2}\right)(50\text{ eV})\right]^{1/2}} = \frac{hc}{[(2)(0.511\text{ MeV})(50\text{ eV})]^{1/2}}$$

$$= \frac{1\,240\text{ eV nm}}{\left[(2)\left(0.511\times10^6\right)(50)(\text{eV})^2\right]^{1/2}} = 0.173\text{ nm}$$

(b) As $50\text{ eV} \ll 1\text{ MeV}$, $p = (2mK)^{1/2}$

$$\lambda = \frac{hc}{\left[(2)\left(\frac{0.511\text{ MeV}}{c^2}\right)\left(50\times10^3\text{ eV}\right)\right]^{1/2}} = 5.49\times10^{-3}\text{ nm}.$$

As this is clearly a worse approximation than in (a) to be on the <u>safe</u> side use the relativistic expression for p: $p = K\frac{\left(1+\frac{2mc^2}{K}\right)^{1/2}}{c}$ so

$$\lambda = \frac{h}{p} = \frac{hc}{\left(K^2+2Kmc^2\right)^{1/2}} = \frac{1\,240\text{ eV nm}}{\left[\left(50\times10^3\right)^2+(2)\left(50\times10^3\right)\left(0.511\times10^6\text{ eV}\right)\right]^{1/2}}$$

$$= 5.36\times10^{-3}\text{ nm} = 0.005\,36\text{ nm}$$

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5-7 A 10 MeV proton has $K = 10 \text{ MeV} << 2mc^2 = 1\,877 \text{ MeV}$ so we can use the classical expression $p = (2mK)^{1/2}$. (See Problem 5-2)

$$\lambda = \frac{h}{p} = \frac{hc}{[(2)(938.3 \text{ MeV})(10 \text{ MeV})]^{1/2}} = \frac{1\,240 \text{ MeV fm}}{\left[(2)(938.3)(10)(\text{MeV})^2\right]^{1/2}} = 9.05 \text{ fm} = 9.05\times10^{-15} \text{ m}$$

5-8 $$\lambda = \frac{h}{p} = \frac{h}{(2mK)^{1/2}} = \frac{h}{(2meV)^{1/2}} = \left[\frac{h}{(2me)^{1/2}}\right]V^{-1/2}$$

$$\lambda = \left[\frac{6.626\times10^{-34} \text{ Js}}{\left(2\times9.105\times10^{-31} \text{ kg}\times1.602\times10^{-19} \text{ C}\right)^{1/2}}\right]V^{-1/2}$$

$$\lambda = \left[\frac{1.226\times10^{-9} \text{ kg}^{1/2}\text{m}^2}{s\text{C}^{1/2}}\right]V^{-1/2}$$

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5-10 As $\lambda = 2a_0 = 2(0.052\,9)\ \text{nm} = 0.105\,8\ \text{nm}$ the energy of the electron is nonrelativistic, so we can use

$$p = \frac{h}{\lambda}\ \text{with}\ K = \frac{p^2}{2m};$$

$$K = \frac{h^2}{2m\lambda^2} = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{2\left(9.11\times10^{-31}\ \text{kg}\right)\left(1.058\times10^{-10}\ \text{m}\right)^2} = 21.5\times10^{-18}\ \text{J} = 134\ \text{eV}$$

This is about ten times as large as the ground-state energy of hydrogen, which is 13.6 eV.

5-11 (a) In this problem, the electron must be treated relativistically because we must use relativity when $pc \approx mc^2$. (See problem 5-5). the momentum of the electron is

$$p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{10^{-14}\ \text{m}} = 6.626\times10^{-20}\ \text{kg}\cdot\text{m/s}$$

and $pc = 124\ \text{MeV} >> mc^2 = 0.511\ \text{MeV}$. The energy of the electron is

$$\begin{aligned} E &= \left(p^2c^2 + m^2c^4\right)^{1/2} \\ &= \left[\left(6.626\times10^{-20}\ \text{kg}\cdot\text{m/s}\right)^2\left(3\times10^8\ \text{m/s}\right)^2 + \left(0.511\times10^6\ \text{eV}\right)^2\left(1.602\times10^{-19}\ \text{J/eV}\right)^2\right]^{1/2} \\ &= 1.99\times10^{-11}\ \text{J} = 1.24\times10^8\ \text{eV} \end{aligned}$$

so that $K = E - mc^2 \approx 124\ \text{MeV}$.

(b) The kinetic energy is too large to expect that the electron could be confined to a region the size of the nucleus.

5-12 Using $p = \frac{h}{\lambda} = mv$, we find that $v = \frac{h}{m\lambda} = \frac{6.626\times10^{-34}\ \text{J}\cdot\text{s}}{\left(9.11\times10^{-31}\ \text{kg}\right)\left(1\times10^{-10}\ \text{m}\right)} = 7.27\times10^6\ \text{m/s}$. From the principle of conservation of energy, we get

$$eV = \frac{mv^2}{2} = \frac{\left(9.11\times10^{-31}\ \text{kg}\right)\left(7.27\times10^6\ \text{m/s}\right)^2}{2} = 2.41\times10^{-17}\ \text{J} = 151\ \text{eV}.$$

Therefore $V = 151\ \text{V}$.

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5-15 For a free, non-relativistic electron $E = \frac{m_e v_0^2}{2} = \frac{p^2}{2m_e}$. As the wavenumber and angular frequency of the electron's de Broglie wave are given by $p = \hbar k$ and $E = \hbar\omega$, substituting these results gives the dispersion relation $\omega = \frac{\hbar k^2}{2m_e}$. So $v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m_e} = \frac{p}{m_e} = v_0$.

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5-17 $E^2 = p^2c^2 + (m_ec^2)^2$

$E = \left[p^2c^2 + (m_ec^2)^2\right]^{1/2}$. As $E = \hbar\omega$ and $p = \hbar k$

$\hbar\omega = \left[\hbar^2k^2c^2 + (m_ec^2)^2\right]^{1/2}$ or

$$\omega(k) = \left[k^2c^2 + \frac{(m_ec^2)^2}{\hbar^2}\right]^{1/2}$$

$$v_p = \frac{\omega}{k} = \frac{\left[k^2c^2 + (m_ec^2/\hbar)^2\right]^{1/2}}{k} = \left[c^2 + \left(\frac{m_ec^2}{\hbar k}\right)^2\right]^{1/2}$$

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{1}{2}\left[k^2c^2 + \left(\frac{m_ec^2}{\hbar}\right)^2\right]^{-1/2} 2kc^2 = \frac{kc^2}{\left[k^2c^2 + (m_ec^2/\hbar)^2\right]^{1/2}}$$

$$v_pv_g = \left\{\frac{\left[k^2c^2 + (m_ec^2/\hbar)^2\right]^{1/2}}{k}\right\}\left\{\left[k^2c^2 + (m_ec^2/\hbar)^2\right]^{1/2}\right\} = c^2$$

Therefore, $v_g < c$ if $v_p > c$.

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5-23 (a) $\Delta p \Delta x = m \Delta v \Delta x \geq \frac{\hbar}{2}$

$$\Delta v \geq \frac{h}{4\pi m \Delta x} = \frac{2\pi \text{ J} \cdot \text{s}}{4\pi (2 \text{ kg})(1 \text{ m})} = 0.25 \text{ m/s}$$

(b) The duck might move by $(0.25 \text{ m/s})(5 \text{ s}) = 1.25 \text{ m}$. With original position uncertainty of 1m, we can think of Δx growing to $1 \text{ m} + 1.25 \text{ m} = 2.25 \text{ m}$.

5-24 (a) $\Delta x \Delta p = \hbar$ so if $\Delta x = r$, $\Delta p \approx \frac{\hbar}{r}$

(b) $$K = \frac{p^2}{2m_e} \approx \frac{(\Delta p)^2}{2m_e} = \frac{\hbar^2}{2m_e r^2}$$

$$U = -\frac{ke^2}{r}$$

$$E = \frac{\hbar^2}{2m_e r^2} - \frac{ke^2}{r}$$

(c) To minimize E take $\frac{dE}{dr} = -\frac{\hbar^2}{m_e r^3} + \frac{ke^2}{r^2} = 0 \Rightarrow r = \frac{\hbar^2}{m_e ke^2} = \text{Bohr radius} = a_0$. Then

$$E = \left(\frac{\hbar}{2m_e}\right)\left(\frac{m_e ke^2}{\hbar^2}\right)^2 - ke^2\left(\frac{m_e ke^2}{\hbar^2}\right) = \frac{m_e k^2 e^4}{2\hbar^2} = -13.6 \text{ eV}.$$

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5-26 The full width at half-maximum (FWHM) is 110 MeV. So $\Delta E = 55$ MeV and using $\Delta E_{\min}\Delta t_{\min} = \frac{\hbar}{2}$,

$$\Delta t_{\min} = \frac{\hbar}{2\Delta E} = \frac{6.58\times10^{-16}\ \text{eV}\cdot\text{s}}{2\left(55\times10^{6}\ \text{eV}\right)} \cong 6.0\times10^{-24}\ \text{s}$$

$$\tau = \text{lifetime} \sim 2\Delta t_{\min} = 1.2\times10^{-23}\ \text{s}$$

5-27 For a single slit with width a, minima are given by $\sin\theta = \frac{n\lambda}{a}$ where $n = 1, 2, 3, \ldots$ and $\sin\theta \approx \tan\theta = \frac{x}{L}$, $\frac{x_1}{L} = \frac{\lambda}{a}$ and $\frac{x_2}{L} = \frac{2\lambda}{a} \Rightarrow \frac{x_2 - x_1}{L} = \frac{\lambda}{a}$ or

$$\lambda = \frac{a\Delta x}{L} = \frac{5\ \text{Å}\times 2.1\ \text{cm}}{20\ \text{cm}} = 0.525\ \text{Å}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{\left(1.24\times10^4\ \text{eV}\cdot\text{Å}\right)^2}{2\left(5.11\times10^5\ \text{eV}\right)\left(0.525\ \text{Å}\right)^2} = 546\ \text{eV}$$

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5-32 (a) $f=\dfrac{E}{h}=\dfrac{(1.8)\left(1.6\times10^{-19}\text{ J}\right)}{6.63\times10^{-34}\text{ J}\cdot\text{s}}=4.34\times10^{14}\text{ Hz}$

(b) $\lambda=\dfrac{c}{f}=691\text{ nm}$

(c) $\Delta E\geq\dfrac{\hbar}{\Delta t}=\dfrac{6.63\times10^{-34}\text{ J}\cdot\text{s}}{2\pi\left(2\times10^{-6}\text{ s}\right)}$

$\Delta E\geq5.276\times10^{-29}\text{ J}=3.30\times10^{-10}\text{ eV}$

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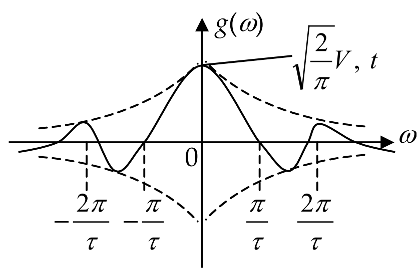
5-34 (a) $g(\omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}V(t)\left(\cos\omega t-i\sin\omega t\right)dt$, $V(t)\sin\omega t$ is an odd function so this integral vanishes leaving $g(\omega)=2(2\pi)^{-1/2}\int_{0}^{\tau}V_0\cos\omega t\,dt=\left(\dfrac{2}{\pi}\right)^{1/2}V_0\dfrac{\sin\omega\tau}{\omega}$. A sketch of $g(\omega)$ is given below.



(b) As the major contribution to this pulse comes from ω's between $-\frac{\pi}{\tau}$ and $\frac{\pi}{\tau}$, let $\Delta\omega \approx \frac{\pi}{\tau}$ and since $\Delta t = \tau$.

$$\Delta\omega\Delta t = \left(\frac{\pi}{\tau}\right)\tau = \pi$$

(c) Substituting $\Delta t = 0.5\ \mu\text{s}$ in $\Delta\omega = \frac{\pi}{\Delta t}$ we find $\frac{\Delta 1}{2\Delta t} = \frac{1}{2\left(0.5\times10^{-6}\ \text{s}\right)} = 1\times10^{6}\ \text{Hz}$. As the range is $2\Delta f$, the range is $2\times10^{6}\ \text{Hz}$. For $\Delta t = 0.5\ \text{ns}$, the range is $2\Delta f = 2\times10^{9}\ \text{Hz}$.

5-35 (a) $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} a(k)e^{ikx}dk = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha^2(k-k_0)^2}e^{ikx}dk = \frac{A}{\sqrt{2\pi}}e^{-\alpha^2k_0^2}\int_{-\infty}^{+\infty} e^{-\alpha^2\left(k^2-\left(2k_0+ix/\alpha^2\right)k\right)}dk$.

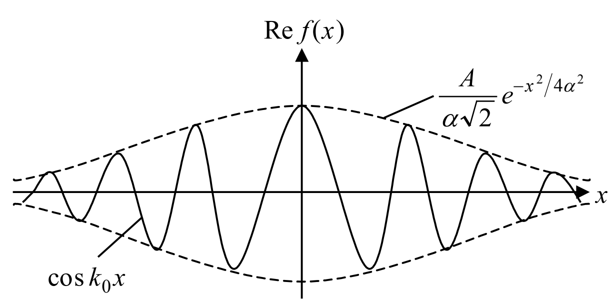
Now complete the square in order to get the integral into the standard form $\int_{-\infty}^{+\infty} e^{-az^2}dz$:

$$e^{-\alpha^2\left(k^2-\left(2k_0+ix/\alpha^2\right)k\right)} = e^{+\alpha^2\left(k_0+ix/2\alpha^2\right)^2}e^{-\alpha^2\left(k-\left(k_0+ix/2\alpha^2\right)\right)^2}$$

$$f(x) = \frac{A}{\sqrt{2\pi}}e^{-\alpha^2k_0^2}e^{\alpha^2\left(k_0+ix/2\alpha^2\right)^2}\int_{k=-\infty}^{+\infty} e^{-\alpha^2\left(k-\left(k_0+ix/2\alpha^2\right)\right)^2}dk$$

$$= \frac{A}{\sqrt{2\pi}}e^{-x^2/4\alpha^2}e^{ik_0x}\int_{z=-\infty}^{+\infty} e^{-\alpha^2z^2}dz$$

where $z = k-\left(k_0+\frac{ix}{2\alpha^2}\right)$. Since $\int_{z=-\infty}^{+\infty} e^{-\alpha^2z^2}dz = \frac{\pi^{1/2}}{\alpha}$, $f(x) = \frac{A}{\alpha\sqrt{2}}e^{-x^2/4\alpha^2}e^{ik_0x}$. The real part of $f(x)$, $\text{Re}\,f(x)$ is $\text{Re}\,f(x) = \frac{A}{\alpha\sqrt{2}}e^{-x^2 4\alpha^2}\cos k_0x$ and is a gaussian envelope multiplying a harmonic wave with wave number k_0. A plot of $\text{Re}\,f(x)$ is shown below:



Comparing $\frac{A}{\alpha\sqrt{2}}e^{-x^2 4\alpha^2}$ to $Ae^{-(x/2\Delta x)^2}$ implies $\Delta x = \alpha$.

(c) By same reasoning because $\alpha^2 = \dfrac{1}{4\Delta k^2}$, $\Delta k = \dfrac{1}{2\alpha}$. Finally $\Delta x \Delta k = \alpha\left(\dfrac{1}{2\alpha}\right) = \dfrac{1}{2}$.

5-36 $E = K = \dfrac{1}{2}mu^2 = hf$ and $\lambda = \dfrac{h}{mu}$. $v_{\text{phase}} = f\lambda = \dfrac{mu^2}{2h}\dfrac{h}{mu} = \dfrac{u}{2} = v_{\text{phase}}$. This is different from the speed u at which the particle transports mass, energy, and momentum.

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