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Tunneling Phenomena

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10\text{ MeV}$ and $m = \frac{938.28\text{ MeV}}{c^2}$. Using $\hbar = 197.3\text{ MeV fm}/c\left(1\text{ fm} = 10^{-15}\text{ m}\right)$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3\text{ MeV fm}/c}{\left[(2)\left(938.28\text{ MeV}/c^2\right)(10\text{ MeV})\right]^{1/2}} = 1.44\text{ fm}.$$

7-11 (a) The matter wave reflected from the trailing edge of the well $(x = L)$ must travel the extra distance *2L* before combining with the wave reflected from the leading edge $(x = 0)$. For $\lambda_2 = 2L$, these two waves interfere destructively since the latter suffers a phase shift of 180° upon reflection, as discussed in Example 7.3.

(b) The wave functions in all three regions are free particle plane waves. In regions 1 and 3 where $U(x) = U$ we have

$$\Psi(x,\ t) = Ae^{i(k'x - \omega t)} + Be^{i(-k'x - \omega t)} \qquad x < 0$$
$$\Psi(x,\ t) = Fe^{i(k'x - \omega t)} + Ge^{i(-k'x - \omega t)} \qquad x < 0$$

with $k' = \dfrac{[2m(E-U)]^{1/2}}{\hbar}$. In this case $G = 0$ since the particle is incident from the left. In region 2 where $U(x) = 0$ we have

$$\Psi(x,\ t) = Ce^{i(-kx-\omega t)} + De^{i(kx-\omega t)} \qquad 0 < x < L$$

with $k = \frac{(2mE)^{1/2}}{\hbar} = \frac{2\pi}{\lambda_2} = \frac{\pi}{L}$ for the case of interest. The wave function and its slope are continuous everywhere, and in particular at the well edges $x = 0$ and $x = L$. Thus, we must require

$$A + B = C + D \qquad [\text{continuity of } \Psi \text{ at } x = 0]$$

$$k'A - k'B = kD - kC \qquad \left[\text{continuity of } \frac{\partial \Psi}{\partial x} \text{ at } x = 0\right]$$

$$Ce^{-ikL} + De^{ikL} = Fe^{ik'L} \qquad [\text{continuity of } \Psi \text{ at } x = L]$$

$$kDe^{ikL} - kCe^{-ikL} = k'Fe^{ik'L} \qquad \left[\text{continuity of } \frac{\partial \Psi}{\partial x} \text{ at } x = L\right]$$

For $kL = \pi$, $e^{\pm ikL} = -1$ and the last two requirements can be combined to give $kD - kC = k'C + k'D$. Substituting this into the second requirement implies $A - B = C + D$, which is consistent with the first requirement only if $B = 0$, i.e., no reflected wave in region 1.

7-14 Suppose the marble has mass 20 g. Suppose the wall of the box is 12 cm high and 2 mm thick. While it is inside the wall, $U = mgy = (0.02\ \text{kg})(9.8\ \text{m/s}^2)(0.12\ \text{m}) = 0.023\,5\ \text{J}$ and $E = K = \frac{1}{2}mv^2 = \frac{1}{2}(0.02\ \text{kg})(0.8\ \text{m/s})^2 = 0.006\,4\ \text{J}$. Then

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar} = \frac{\sqrt{2(0.02\ \text{kg})(0.017\,1\ \text{J})}}{1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 2.5\times10^{32}\ \text{m}^{-1}$$

and the transmission coefficient is

$$e^{-2\alpha L} = e^{-2\left(2.5\times10^{32}\right)\left(2\times10^{-3}\right)} = e^{-10\times10^{29}} = e^{-2.30\left(4.3\times10^{29}\right)} = 10^{-4.3\times10^{29}} \sim 10^{-10^{30}}\ .$$

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$, the first approximation to the decay length δ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(30\ \text{MeV})\right]^{1/2}} = 0.415\,6\ \text{fm}\,.$$

This gives an effective width for the (infinite) well of $R+\delta = 9.415\,6\ \text{fm}$, and a ground state energy $E_1 = \dfrac{\pi^2(197.3\ \text{MeV fm}/c)^2}{2\left(3\,755.8\ \text{MeV}/c^2\right)(9.415\,6\ \text{fm})^2} = 0.577\ \text{MeV}$. From this E we calculate $U - E = 29.42\ \text{MeV}$ and a new decay length

$$\delta = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(29.42\ \text{MeV})\right]^{1/2}} = 0.419\,7\ \text{fm}\,.$$

This, in turn, increases the effective well width to 9.419 7 fm and lowers the ground state energy to $E_1 = 0.576\text{ MeV}$. Since our estimate for E has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of E_1, the alpha particle in the ground state has speed $v_1 = \left(\frac{2E_1}{m}\right)^{1/2} = \left[\frac{2(0.576\text{ MeV})}{(3\,755.8\text{ MeV}/c^2)}\right]^{1/2} = 0.017\,5c$. In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \frac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E - U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9\text{ fm}$, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2\text{ eV})\left(\frac{5.29\times10^4\text{ fm}}{9\text{ fm}}\right) = 28.14\text{ MeV}.$$

From this we conclude that $U = -1.86\text{ MeV}$ to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E = 4.05\text{ MeV}$ has kinetic energy $4.05 + 1.86 = 5.91\text{ MeV}$ inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\text{ MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m}\right)^{1/2} = \left[\frac{2(5.91)}{3\,755.8\text{ MeV}/c^2}\right]^{1/2} = 0.056c.$$

Thus, we find for the collision frequency $f = \frac{v}{2R} = \frac{0.056c}{2(9\text{ fm})} = 9.35\times10^{20}\text{ Hz}$.