Physics 4E Final Exam Solutions Spring 2020

Problem 1

For the Bohr atom, $\frac{m_e v^2}{r} = \frac{ke^2 Z}{r^2} \Rightarrow K = \frac{1}{2} m_e v^2 = \frac{ke^2 Z}{2r}$

We are told the orbit has radius $r = 0.529\,\text{Å} = a_0$ and $K = 340\,eV$, so

$$K = \frac{ke^2 Z}{2a_0} = 340\,eV \Rightarrow 13.6\,eV \cdot Z = 340\,eV \Rightarrow \boxed{Z = 25}$$

Then, since $r = r_n = \frac{a_0}{Z} n^2 = a_0 \Rightarrow n^2 = Z \Rightarrow \boxed{n = 5}$

Since $L = n\hbar$ in the Bohr model, $\boxed{L = 5\hbar}$ (a)

(b) The energies are $E_n = -\frac{E_0 Z^2}{n^2}$, $E_0 = 13.6\,eV$

So $E_n - E_1 = E_0 Z^2 \left(1 - \frac{1}{n^2}\right) = E_0 Z^2 \cdot \frac{24}{25}$

$$\Rightarrow E_5 - E_1 = 8160\,eV \Rightarrow \boxed{\lambda = \frac{hc}{E_5 - E_1} = 1.52\,\text{Å}}$$ is the (b)

wavelength of emitted photon.

(c) de Broglie wavelength:

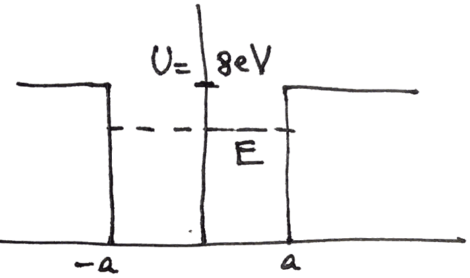
$$\lambda_{dB} = \frac{h}{p}, \quad p = mv, \quad L = mvr = pr = n\hbar$$

$$\Rightarrow \lambda_{dB} = \frac{h \cdot r}{p \cdot r} = \frac{h \cdot r}{L} = \frac{h \cdot a_0}{n\hbar} = \frac{2\pi a_0}{n}$$

So $\boxed{\lambda_{dB} = \frac{2\pi a_0}{5} = 0.665\,\text{Å}}$ (c)

Problem 2

$E = 7\,eV$



For the lowest state, we have the condition

$$\tan(ka) = \frac{\alpha}{k} \quad \text{with} \quad \alpha = \sqrt{\frac{2m_e(U-E)}{\hbar^2}} \quad \text{and} \quad E = \frac{\hbar^2 k^2}{2m_e}$$

Using $\frac{\hbar^2}{m_e} = 7.62\,eV\,\mathring{A}^2$,

$$\alpha = \sqrt{\frac{2}{7.62}(8-7)}\ \mathring{A}^{-1} \Rightarrow \boxed{\alpha = 0.512\,\mathring{A}^{-1}}$$

and $$k = \sqrt{\frac{2m_e E}{\hbar^2}} = \sqrt{\frac{2}{7.62}\times 7}\ \mathring{A}^{-1} \Rightarrow \boxed{k = 1.355\,\mathring{A}^{-1}}$$

$$\Rightarrow \frac{\alpha}{k} = 0.378 \Rightarrow ka = \tan^{-1} 0.378 = 0.361 \Rightarrow \boxed{a = 0.267\,\mathring{A}} \ \text{(a)}$$

(b) For an infinite well of this depth,

$$E = \frac{\hbar^2\pi^2}{2m_e(2a)^2} = \frac{7.62}{2}\frac{\pi^2}{(2a)^2}\,eV\mathring{A}^2 \Rightarrow \boxed{E = 132\,eV \gg 7\,eV}$$

The reason the energy here is much lower is because the wavefunction of the electron extends well beyond the region $-a$ to a, into the forbidden region, because E is close to U, so Δx is much larger.

(c) $$\boxed{\frac{P(x=0)}{P(x=a)} = \frac{1}{\cos^2(ka)} = 1.14}$$ since wavefunction is $\psi(x) = A\cos(kx)$

(d) $\psi_1(x) = A\cos(kx)$ for $x < a$

$\psi_2(x) = Be^{-\alpha x}$ for $x > a$

$$\frac{P(x=0)}{P(x=2a)} = \frac{A^2}{B^2 e^{-2\alpha\cdot 2a}}$$

By continuity, $A\cos(ka) = Be^{-\alpha a}$

$$\Rightarrow \frac{A^2}{B^2} = \frac{e^{-2\alpha a}}{\cos^2 ka} \Rightarrow \frac{P(x=0)}{P(x=2a)} = \frac{e^{-2\alpha a}}{\cos^2 ka}\cdot\frac{1}{e^{-4\alpha a}} = \frac{e^{2\alpha a}}{\cos^2 ka} = 1.50$$

so $$\boxed{\frac{P(x=0)}{P(x=2a)} = 1.50}$$

Problem 3

$\psi_1(x) = A x e^{-\frac{\lambda}{2}x^2}$ for $x<0$

Potential is harmonic oscillator potential. $E_n = \hbar\omega(n+\frac{1}{2})$

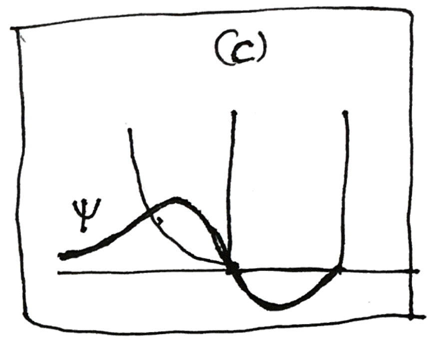
This is the first excited state in a harmonic oscillator potential.

$\Rightarrow E = \frac{3}{2}\hbar\omega$. Since $\hbar\omega = 3eV \Rightarrow \boxed{E = 4.5eV}$ (a)

For the harmonic oscillator wavefunction, exponential is

$e^{-\frac{m\omega}{2\hbar}x^2} \Rightarrow \lambda = \frac{m\omega}{\hbar} = \frac{m\hbar\omega}{\hbar^2} = \frac{3eV}{7.62eV\text{Å}^2} = 0.39\text{Å}^{-2}$

so $\boxed{\lambda = 0.39\text{Å}^{-2}}$ (a)



(b) For $x \geq 0$, $U(x) = 0 \Rightarrow$

$\psi_2(x) = B\cos(kx) + C\sin(kx)$

and the energy is $E = \frac{\hbar^2 k^2}{2m}$

Continuity of ψ: $\psi_1(x=0) = 0 = \psi_2(x=0) \Rightarrow B = 0$

Continuity of ψ': $\psi_1'(x=0) = A = kC\cos(0) = kC \Rightarrow \boxed{A = kC}$

So: $\boxed{\psi_2(x) = C\sin(kx)}$. Since $\psi_2(x=L) = 0 \Rightarrow$

$kL = n\pi \Rightarrow \boxed{k = \frac{n\pi}{L}}$. So $\boxed{E = \frac{\hbar^2\pi^2}{2mL^2}n^2 = 4.5eV}$

smallest L is for $n=1$: $L^2 = \frac{\hbar^2\pi^2}{2m \times 4.5eV} = \frac{\pi^2 \cdot 7.62}{2 \cdot 4.5}\text{Å}^2 = 8.36\text{Å}^2 \Rightarrow$

$\Rightarrow \boxed{L = 2.89\text{Å}}$

(d) $\int_0^\infty dx\, x^2 e^{-\lambda x^2} = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} \equiv N_1$; $\int_0^L dx \sin^2\frac{\pi}{L}x = \frac{L}{2} \equiv N_2$

So: $P_L \equiv \int_{-\infty}^0 dx\, |\psi_1(x)|^2 = A^2 \cdot N_1$; $P_R \equiv \int_0^\infty dx\, |\psi_2(x)|^2 = C^2 N_2$

so $\frac{P_\ell}{P_r} = \frac{A^2}{C^2}\frac{N_1}{N_2} = k^2 \frac{N_1}{N_2}$

we have: $N_1 = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} = 1.82$; $N_2 = \frac{L}{2} = 1.45$

$k = \frac{\pi}{L} = 1.09 \Rightarrow \frac{P_\ell}{P_r} = 1.09^2 \cdot \frac{1.82}{1.45} = 1.48$

$\boxed{\text{So electron is 1.48 times more likely to be found at } x < 0 \text{ than at } x > 0.}$

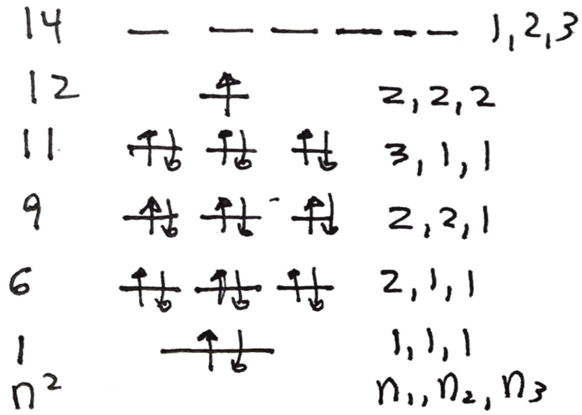
Problem 4

3D box, $L = 6\,\text{Å}$

$$E_{n_1,n_2,n_3} = \frac{\hbar^2\pi^2}{2m_eL^2}\left(n_1^2+n_2^2+n_3^2\right)$$

$$\equiv E_0\, n^2;\quad n^2 = n_1^2+n_2^2+n_3^2$$

$$E_0 = \frac{\hbar^2\pi^2}{2m_eL^2} = 1.045\,\text{eV}$$



there are 21 electrons as shown in the picture.

(a) the longest wavelength photon is for the transition

(3, 1, 1) to (2, 2, 2) $\Rightarrow$ $\Delta E = E_0(12-11) = E_0$

$$\Rightarrow \boxed{\lambda = \frac{hc}{\Delta E} = 11{,}866\,\text{Å}}$$

(b) The electron in the highest energy state has $n^2 = 12$. Its degeneracy is 1 (not including spin). The next highest energy state has $n^2 = 14$, and degeneracy 6. So the electron in $n^2 = 12$ will be equally likely to be in $n^2 = 14$ at temperature that satisfies

$$1 \cdot e^{-E(n^2=12)/k_BT} = 6 \cdot e^{-E(n^2=14)/k_BT} \Rightarrow$$

$$\frac{\Delta E}{k_BT} = \ln 6 \Rightarrow k_BT = \frac{\Delta E}{\ln 6} \quad \text{with } \Delta E = E_0(14-12) = 2.09\,\text{eV}$$

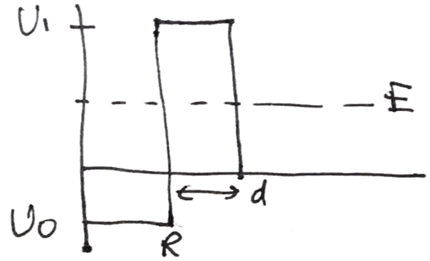
$$\Rightarrow \quad k_BT = 1.166\ \text{eV} \Rightarrow \boxed{T = 13{,}531\ \text{K}}$$

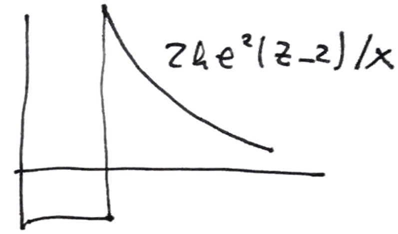
(c) The Fermi energy is the energy of the level occupied by the highest energy electron $n^2 = 12$

$$\boxed{E_F = E_0 \cdot 12 = 12.54\,\text{eV}}$$

Problem 5

(a) $R = 3\times10^{-5}\,\text{Å}$





$$E = \frac{\hbar^2\pi^2}{2m_\alpha R^2} = \frac{7.62}{2\times7295}\times\frac{\pi^2}{9\times10^{-10}}\,\text{eV} = 5.73\ \text{MeV}$$

$\boxed{E = 5.73\ \text{MeV}}$ (a).

(b) $E = \dfrac{2ke^2(Z-2)}{R+d} \Rightarrow R+d = \dfrac{2ke^2(Z-2)}{E} = \dfrac{2\cdot14.4\times46}{5.73\times10^6}\,\text{Å} = 23\times10^{-5}\,\text{Å}$

$\Rightarrow \boxed{d = 2\times10^{-4}\,\text{Å}}$

(c) $T = e^{-2\alpha d}$. $\alpha = \sqrt{\dfrac{2m_\alpha}{\hbar^2}(U_1-E)} = \sqrt{\dfrac{2\cdot7295\,m_e}{\hbar^2}(U_1-E)}$

$\Rightarrow \alpha = \sqrt{\dfrac{2\cdot7295}{7.62}\times(15-5.73)\cdot10^6}\ \text{Å}^{-1} = 1.33\times10^5\ \text{Å}^{-1}$

$\Rightarrow 2\alpha d = 53.2 \Rightarrow \boxed{T = e^{-53.2} = 7.86\times10^{-24}}$ (c)

(d) $f = \dfrac{v}{2R}$ is attempt frequency. Kinetic energy is $K = E - U_0 = E$ here

$K = \frac{1}{2}m_\alpha v^2 \Rightarrow v = \sqrt{\dfrac{2K}{m_\alpha}} \Rightarrow \dfrac{v}{c} = \sqrt{\dfrac{2K}{m_\alpha c^2}} = \sqrt{\dfrac{2K}{7295\,m_e c^2}}$

$\Rightarrow \dfrac{v}{c} = \sqrt{\dfrac{2\cdot5.73\ \text{MeV}}{7295\times0.511\ \text{MeV}}} = 0.0554$. $c = 3\times10^8\,\dfrac{\text{m}}{\text{s}} = 3\times10^{18}\,\dfrac{\text{Å}}{\text{s}}$

$\Rightarrow v = 0.166\times10^{18}\ \text{Å/s} \Rightarrow f = \dfrac{0.166\times10^{18}}{6\times10^{-5}}\ \text{s}^{-1}$

$\Rightarrow \boxed{f = 2.77\times10^{21}\ \text{s}^{-1}}$ (d)

(e) decay rate $= fT = 7.86\times10^{-24}\times2.77\times10^{21}\,\text{s}^{-1} = 0.0218\,\text{s}^{-1}$

$\Rightarrow \boxed{\text{lifetime} = \dfrac{1}{\text{decay rate}} = 46\ \text{seconds}}$

Problem 6

Assume now $U_0 = -2\,\text{MeV}$

$E - U_0 = 5.73\,\text{MeV} \Rightarrow \boxed{E = 3.73\,\text{MeV}}$ (a)

(b) $R + d = \dfrac{2ke^2(Z-2)}{E} = 35.5 \times 10^{-5}\,\text{Å}$

$\Rightarrow \boxed{d = 32.5 \times 10^{-5}\,\text{Å}}$

(c) $\alpha = \sqrt{\dfrac{2m_\alpha}{\hbar^2}(U_1 - E)} = \alpha_{old}\sqrt{\dfrac{15-3.73}{15-5.73}} = \alpha_{old} \times 1.10$

$\Rightarrow \boxed{\alpha = 1.47 \times 10^5\,\text{Å}^{-1}}$

$\Rightarrow 2\alpha d = 95.3 \Rightarrow \boxed{T = e^{-95.3} = 4.01 \times 10^{-42}}$ (c)

(d) $K = E - U_0 = 3.73\,\text{MeV} - (-2\,\text{MeV}) = 5.73\,\text{MeV}$ same as before

$\Rightarrow$ attempt frequency is same, $\boxed{f = 2.77 \times 10^{21}\,s^{-1}}$ (d)

(e) decay rate :

$$fT = \cancel{2}\ 4.01 \times 10^{-42} \times 2.77 \times 10^{21}\,s^{-1} = 1.11 \times 10^{-20}\,s^{-1}$$

$\Rightarrow$ lifetime $= \dfrac{1}{\text{decay rate}} = 0.90 \times 10^{20}\,s$

$\Rightarrow \boxed{\text{lifetime} = 2.85 \times 10^{12}\ \text{years}}$

Problem 7

$n = 3, \quad l = 0, 1, 2, \quad m_l = \pm l, \ldots 0, \quad m_s = \pm \frac{1}{2} \Rightarrow$

$l = 0, \quad m_l = 0, \quad m_s = \pm 1/2$

$l = 1, \quad m_l = -1 \quad$ "

$m_l = 0 \quad$ "

$m_l = 1 \quad$ "

$l = 2 \quad m_l = 2 \quad$ "

$m_l = 1 \quad$ "

$m_l = 0 \quad$ "

$m_l = -1 \quad$ "

$m_l = -2 \quad$ "

$\Rightarrow \boxed{18 \text{ states}}$

In a magnetic field, $E(B) = E + \mu_B B (m_l + 2m_s)$

highest energy state: $m_l = 2, \; m_s = 1/2$

lowest " " $m_l = -2, \; m_s = -1/2$

$$\Delta E = \mu_B B \cdot 6 = 5.79 \times 10^{-5} \cdot 15.6 \, eV = 5.2 \times 10^{-3} eV$$

$\boxed{\Delta E = 5.2 \times 10^{-3} eV}$ (b)

(c) $\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2}$; by symmetry, $\langle L_x \rangle = 0 \Rightarrow$

$\Delta L_x = \sqrt{\langle L_x^2 \rangle}$. Now $L^2 = L_x^2 + L_y^2 + L_z^2$

since $\langle L_x^2 \rangle = \langle L_y^2 \rangle$ by symmetry, $\langle L_x^2 \rangle = \frac{1}{2}(\langle L^2 \rangle - \langle L_z \rangle^2)$

and $\langle L^2 \rangle = L^2 = \hbar^2 l(l+1)$, $\langle L_z^2 \rangle = L_z^2 = \hbar^2 m_l^2 \Rightarrow$

$\langle L_x^2 \rangle = \frac{\hbar^2}{2}\left(l(l+1) - m_l^2\right) \Rightarrow \boxed{\Delta L_x = \frac{\hbar}{\sqrt{2}} \sqrt{(l(l+1)) - m_l^2)}}$

Largest ΔL_x is for $l = 2, m_l = 0$: $\boxed{\Delta L_x = \frac{\hbar}{\sqrt{2}} \sqrt{6} = \sqrt{3}\,\hbar = 1.73\hbar}$

Smallest ΔL_x is for $l = 1, m_l = 1$; $\boxed{\Delta L_x = \frac{\hbar}{\sqrt{2}} \sqrt{2 - 1} = \frac{\hbar}{\sqrt{2}} = 0.707\hbar}$

Problem 8

For a harmonic oscillator, $\mathcal{E}_n = \hbar\omega(n + \frac{1}{2})$.

The average energy per oscillator at temperature T is

$$\langle \mathcal{E} \rangle = \frac{\hbar\omega}{2} + \hbar\omega \langle n \rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

At $T = 0$, $\mathcal{E}_0 = \frac{\hbar\omega}{2}$. So,

$$\frac{\langle \mathcal{E} \rangle}{\mathcal{E}_0} = \frac{U}{U_0} = 1.0001 = \frac{\frac{1}{2}\hbar\omega\left(1 + \frac{2}{e^{\hbar\omega/kT} - 1}\right)}{\frac{1}{2}\hbar\omega} = 1 + \frac{2}{e^{\hbar\omega/kT} - 1} \Rightarrow$$

$$\Rightarrow \frac{2}{e^{\hbar\omega/kT} - 1} = 0.0001 \Rightarrow e^{\hbar\omega/kT} = 20{,}001 \Rightarrow \frac{\hbar\omega}{kT} = 9.90$$

$$\Rightarrow \hbar\omega = 9.90\, kT = 9.90 \times \frac{1}{11{,}600\ \mathrm{K}}\, \mathrm{eV} \cdot 12\ \mathrm{K} = 0.01\ \mathrm{eV}$$

$$\Rightarrow \hbar\omega = 0.01\ \mathrm{eV} \Rightarrow \boxed{\mathcal{E}_0 = \frac{\hbar\omega}{2} = 0.005\ \mathrm{eV}} \text{ (a)}$$

(b) $T_E = \frac{\hbar\omega}{k} = 9.90 \times 12\ \mathrm{K} = 118.8\ \mathrm{K}$ $\boxed{T_E = 118.8\ \mathrm{K}}$ (b)

(c) At $T = 4000\ \mathrm{K}$, since $T \gg T_E$, the heat capacity is the classical value: $\boxed{U = N_A k_B = R = 8.314\ \mathrm{J/K}}$

this follows from $\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \xrightarrow{\text{high } T} \frac{\hbar\omega}{1 + \frac{\hbar\omega}{kT} - 1} = kT$

(d) Heat capacity is: $C = \frac{dU}{dT} = N_A k_B \left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2}$

$$\Rightarrow C = R \cdot \left(\frac{T_E}{T}\right)^2 \frac{e^{T_E/T}}{(e^{T_E/T} - 1)^2}$$

for $T = 40\ \mathrm{K}$, $\frac{T_E}{T} = 2.97 \Rightarrow$

$$\Rightarrow C = 0.50R \Rightarrow \boxed{\frac{C(40\mathrm{K})}{C(4000\mathrm{K})} = 0.5} \text{ (d)}$$