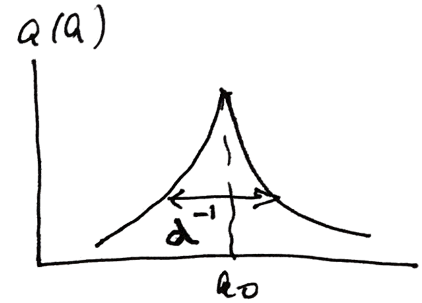
Physics 4E   Quiz 3 Solutions   Spring 2020

Problem 1



$$\Psi(x,t) = \int dk\, a(k)\, e^{i(kx - \omega(k)t)}$$

$$a(k) = A e^{-\alpha|k-k_0|}$$

(a) Electron moves at group velocity:

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0}.$$ If particle is non-relativistic, $\omega(k) = \frac{\hbar k^2}{2m}$

$$v_g = \frac{\hbar k_0}{m_e} = \frac{\hbar c\, k_0}{m_e c^2} c = \frac{1973 \times 200}{511{,}000} c = \boxed{0.77c} = \boxed{231{,}000 \frac{km}{s}}$$

This is close to the speed of light, so we should use relativity instead

$$\omega(k) = \frac{1}{\hbar}\sqrt{\hbar^2 k^2 c^2 + m_e^2 c^4}\ ; \quad v_g = \frac{\hbar k_0 c^2}{E} = \frac{pc^2}{E}$$

$$E = \sqrt{\hbar^2 k_0^2 c^2 + m_e^2 c^4} = \sqrt{1973^2 \times 200^2 + 511{,}000^2}\ eV = 645{,}624\ eV$$

$$v_g = \frac{\hbar c\, k_0}{E} c = \frac{1973 \times 200}{645{,}624} c = 0.61c = \boxed{183{,}358\ km/s}$$

(b) $\Delta x \Delta k \sim 1$. We can estimate $\Delta k \sim \frac{1}{\alpha} = \frac{1}{100} \mathring{A}^{-1}$ at $t = 0$

$$\Rightarrow \boxed{\Delta x = 100\ \mathring{A}}, \text{ or } \boxed{\Delta x \sim \alpha}$$

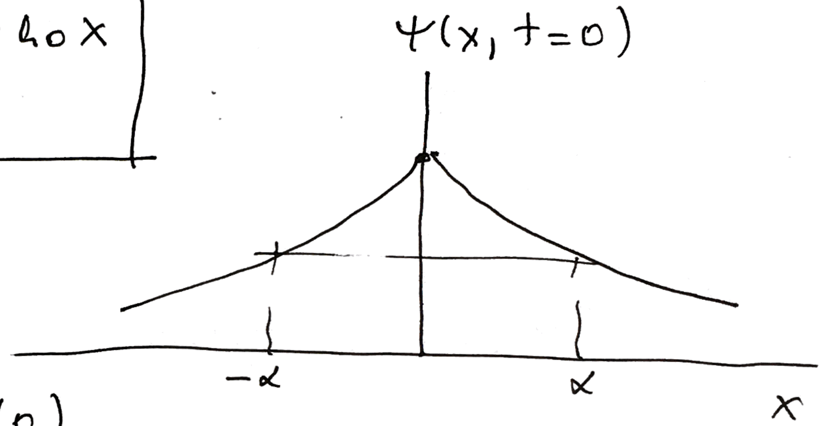
(c) $\psi(x, t=0) = A \int_{-\infty}^{\infty} dk\, e^{ikx} e^{-\alpha|k-k_0|}$

Let $k - k_0 = k'$ $\Rightarrow$

$\psi(x, t=0) = A e^{ik_0 x} \left[\int_0^{\infty} dk\, e^{k(-\alpha + ix)} + \int_{-\infty}^{0} dk\, e^{k(\alpha + ix)} \right] =$

$\Rightarrow$ using $\int_0^{\infty} dk\, e^{k(-\alpha+ix)} = \frac{1}{\alpha - ix}$, $\int_{-\infty}^{0} dk\, e^{k(\alpha+ix)} = \frac{1}{\alpha + ix}$

$\Rightarrow \boxed{\psi(x, t=0) = \frac{2A\alpha}{\alpha^2 + x^2} e^{ik_0 x}}$



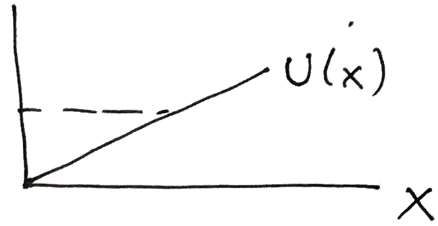
For $x = \pm \alpha$, $\psi(x) = \frac{1}{2} \psi(0)$

So ψ decays over a distance of order α

$\Rightarrow \Delta x \sim \alpha$ or $\Delta x \sim 2\alpha$

Problem 2

$U(x) = \lambda x$



(a) The lowest possible energy for a classical particle is: zero velocity, position $x = 0$, energy $= 0$.

(b) For a quantum particle of mass m:

if $\Delta x \sim x$ is the uncertainty in the position, $\Delta p \sim \frac{\hbar}{x}$

The energy is kinetic + potential energy $= \frac{p^2}{2m} + U(x)$

$$E = \frac{(\Delta p)^2}{2m} + U(x) = \frac{\hbar^2}{2mx^2} + \lambda x$$

Minimize with respect to x:

$$\frac{dE}{dx} = -\frac{\hbar^2}{mx^3} + \lambda = 0 \Rightarrow x^3 = \frac{\hbar^2}{m\lambda} \Rightarrow x = \left(\frac{\hbar^2}{m\lambda}\right)^{1/3}$$

electron: $x_e = \left(\frac{\hbar^2}{m_e \lambda}\right)^{1/3} = \left(\frac{7.62\,\text{eV}\,\text{Å}^2}{1\,\text{eV}/\text{Å}}\right)^{1/3} = \boxed{1.97\,\text{Å}}$ (b)

For a proton: $\frac{m_p}{m_e} = 1836 \Rightarrow \boxed{x_p = \frac{1.97\,\text{Å}}{1836^{1/3}} = 0.16\,\text{Å}}$ (b)

(c) Kinetic energy:

$$K = \frac{\hbar^2}{2mx^2} \Rightarrow \boxed{K_e = \frac{7.62\,\text{eV}}{2 \times 1.97^2} = 0.98\,\text{eV}} \quad \boxed{K_p = \frac{K_e}{m_p/m_e} = 0.00053\,\text{eV}}$$

(d) $K = \frac{\hbar^2}{2m}\left(\frac{\hbar^2}{m\lambda}\right)^{-2/3} = \left(\frac{\hbar^2}{m}\right)^{1/3} \cdot \frac{1}{2} \cdot \lambda^{2/3}$

$U = \lambda\left(\frac{\hbar^2}{m\lambda}\right)^{1/3} = \left(\frac{\hbar^2}{m}\right)^{1/3} \cdot \lambda^{2/3}$

$\Rightarrow \boxed{\frac{K}{U} = \frac{1}{2}}$ smaller than 1.

Problem 3

(a) $E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 = E_1 n^2$

$E_1 = 4\,eV \Rightarrow \boxed{E_2 = 16\,eV}$

(b) $E_1 = \frac{\hbar^2\pi^2}{2mL^2} \Rightarrow L^2 = \frac{\hbar^2\pi^2}{2mE_1} = \frac{7.62}{2}\frac{\pi^2}{4}\,Å^2 \Rightarrow$

$\Rightarrow \boxed{L = 3.07\,Å}$

(c) We have $E_1 = U_0/2$. For a finite well,

$E = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{k_0^2}{k^2} = \frac{U_0}{E}$. For $E_1 = \frac{U_0}{2}$, $\boxed{\frac{k_0^2}{k^2} = 2}$

$U_0 = \frac{\hbar^2 k_0^2}{2m}$ $\qquad \boxed{\alpha = \sqrt{\frac{2m(U_0-E)^2}{\hbar^2}} = \sqrt{k_0^2 - k^2}}$

The condition for a solution is $\tan(ka) = \frac{\alpha}{k}$, or

$\tan(ka) = \sqrt{\frac{k_0^2}{k^2} - 1} = \sqrt{2-1} = 1$

$\Rightarrow ka = \frac{\pi}{4} \Rightarrow \boxed{k = \frac{\pi}{2L}}$ with $L = 2a$ the width of the well.

So the energy is $E_1 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}\cdot\frac{1}{4} = E_1\cdot\frac{1}{4} \Rightarrow$

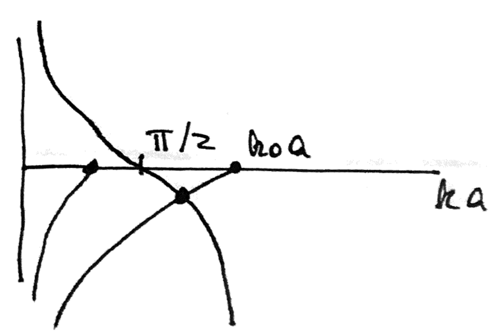
$\boxed{E_1 = 1\,eV,\ U_0 = 2\,eV}$

(d) In the infinite well, $E_2 = 4E_1$, that is larger than U_0, suggests there is no first excited state.

To do it right, look at the condition for the first excited state:

$\cot(ka) = -\frac{\alpha}{k}$. To get a solution,

we need $k_0 a > \frac{\pi}{2}$. ~~[illegible]~~



$\boxed{k_0 a = \frac{\sqrt{2}\pi}{4} < \pi/2} \Rightarrow$ no first excited state