Physics 216

Lecture II - Ideal Fluids

- Equations
- Basic Concepts - especially $\left\{\begin{array}{l}\text{Kelvin's Thm.}\\ \text{potential Flow}\end{array}\right.$
- Induced Mass, Quasi (Pseudo) Momentum

I.) Euler Equations / Ideal Fluids. "The Flow of Dry Water" - RPF

$\underline{v}, \sigma, \rho$ Volume V

$\uparrow$ $\underline{s}$ density ρ

argue macroscopically but really derive from Boltzmann Eqn.

- Mass conservation:

$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int d^3x \, \rho(\underline{x},t) = -\int d\underline{s} \, [\underline{v}\rho]$$

$$= -\int d^3x \, \underline{\nabla}\cdot(\rho \underline{v})$$

so

$$\boxed{\partial_t \rho + \underline{\nabla}\cdot(\rho\underline{v}) = 0}$$

continuity

$$\partial_t \rho + \underline{\nabla}\cdot\underline{\Gamma} = 0 \qquad \underline{\Gamma} = \rho\underline{v}$$

$\uparrow$

mass flux density

- Momentum conservation:

fluid element

$$\underbrace{\underline{f}}_{\text{net force density on element}} = -\underbrace{\underline{\nabla} p}_{\text{pressure gradient}} + \underbrace{\underline{F}}_{\text{body force}}$$

($\frac{\underline{J} \times \underline{B}}{c}$ in MHD)

or, Sir Isaac:

$$\rho \underbrace{\underline{a}}_{\text{acceleration}} = -\underline{\nabla} p + \underline{F}$$

$$\underline{a} = \frac{d\underline{v}}{dt}$$ → what does this mean (substantive derivative)

here:

$$\underbrace{d\underline{v}}_{\text{velocity increment}} = \underbrace{\frac{\partial \underline{v}}{\partial t}}_{\text{local acceleration}} dt + \underbrace{d\underline{r}}_{\text{displ.}} \cdot \underline{\nabla}\, \underline{v}$$

→ increment in $\underline{v}$

particle moves/displaced in inhomogeneous velocity field.

$$d\underline{v}/dt = \frac{\partial \underline{v}}{\partial t} + \frac{d\underline{x}}{dt} \cdot \underline{\nabla}\, \underline{v}$$

$$= \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}$$

$\Rightarrow$

$$\boxed{\rho \frac{d\underline{v}}{dt} = \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\underline{\nabla} p + \underline{F}}$$

Euler Eqn.

? Momentum Flux ?

$$\partial_t (\rho v_i) = -\frac{\partial}{\partial x_k} \Pi_{ik}$$

i.e.

$$\partial_t (\rho \underline{v}) = \underline{v}\, \partial_t \rho + \rho \frac{\partial \underline{v}}{\partial t}$$

momentum density

$$= -\underline{v}\left(\rho(\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla} \rho\right)$$
$$+ \rho\left(-\underline{v} \cdot \underline{\nabla}\, \underline{v} - \frac{\underline{\nabla} p}{\rho}\right)$$

$$= -\Big(\rho\left[\underline{v}(\underline{\nabla} \cdot \underline{v}) + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right]$$
$$+ \underline{v}\,(\underline{v} \cdot \underline{\nabla} \rho)\Big) - \underline{\nabla} p$$

$$= -\underline{\nabla} \cdot \left(\rho \underline{V}\,\underline{V} + \underline{\underline{I}}\, P \right)$$

identity tensor

Reynolds stress tensor

→ analogous to Maxwell stress tensor

so

$$\Pi_{ik} = \rho V_i V_k + \delta_{ik} P$$

and

$$\frac{\partial}{\partial t}\int d^3x\, \rho \underline{V} = \frac{d}{dt}\underline{P} = -\int d\underline{S} \cdot \left(\rho \underline{V}\,\underline{V} + \underline{\underline{I}} P \right).$$

$\Pi_{ik}\, dS_k \equiv$ momentum flux in ith direction.

$$\boxed{\Pi_{ik} = \rho V_i V_k + \delta_{ik} P.}$$

defines momentum flux

- in N-S. Eqn, viscous stress appears due momentum flux from collisions, interacting with macroscopic flow gradients

⇒ Mass, Momentum and Energy !

In ideal fluid, no heat exchanged between fluid elements ⇒ motion adiabatic - i.e. entropy conserved along trajectories

$$\frac{ds}{dt} = 0$$

$s \equiv$ entropy per mass.

$$\Rightarrow \boxed{\frac{\partial s}{\partial t} + \underline{v} \cdot \underline{\nabla} s = 0}$$

adiabatic equation for fluid.

For energy flux:

$$\mathcal{E} = \rho \frac{v^2}{2} + \rho \epsilon$$

$\mathcal{E}$ ↓ total energy density of fluid element

$\rho \frac{v^2}{2}$ ↓ kinetic energy density

$\rho\epsilon$ → internal energy density (i.e. thermal).

Now as with momentum, consider $\partial \mathcal{E}/\partial t$;

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t}\Big(\underbrace{\rho \frac{v^2}{2}}_{1} + \underbrace{\rho \epsilon}_{2}\Big)$$

$$\frac{\partial}{\partial t}\left(\frac{\rho \underline{V}^2}{2}\right) = \frac{\underline{V}^2}{2}\frac{\partial \rho}{\partial t} + \rho \underline{V}\cdot\frac{\partial \underline{V}}{\partial t}$$

$$= -\frac{\underline{V}^2}{2}\underbrace{\underline{\nabla}\cdot(\rho \underline{V})}_{\text{cont.}} \underbrace{- \underline{V}\cdot\underline{\nabla} P - \rho \underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\underline{V})}_{\text{momentum balance}}$$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times\underline{\omega} + \frac{1}{2}\nabla(V^2)$$

$$\underline{\omega} = \underline{\nabla}\times\underline{V} \rightarrow \text{vorticity}$$

so

$$\rho \underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\underline{V}) = \rho \underline{V}\cdot\left(-\underline{V}\times\underline{\omega} + \underline{\nabla}\left(\frac{V^2}{2}\right)\right)$$

$$= \rho \underline{V}\cdot\underline{\nabla}\frac{V^2}{2}$$

and

$$dW = dE + d(PV)$$

enthalpy

$$= T\,dS + V\,dP$$

$$= T\,dS + \frac{dP}{\rho}$$

so

$$\underline{\nabla} P = \rho \nabla W - \rho T \underline{\nabla} S.$$

thus, ①

$$\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right) = -\frac{V^2}{2}\underline{\nabla}\cdot(\rho\underline{V}) - \rho\underline{V}\cdot\underline{\nabla}\left(\frac{V^2}{2}+w\right)$$

$$+\rho T\underline{V}\cdot\underline{\nabla}s$$

For ②:

$$\frac{\partial}{\partial t}(\rho\epsilon) =$$

useful to transform using thermodynamic identity:

$$d\epsilon = dQ - p\,dV$$
$$= T\,dS - p\,dV$$

but $V = 1/\rho$

$$dV = -d\rho/\rho^2$$

$$d\epsilon = T\,dS + \frac{p}{\rho^2}d\rho$$

so $d(\rho\epsilon) = \rho\,d\epsilon + \epsilon\,d\rho$

So

$$d(\rho\epsilon) = \left(\frac{p}{\rho} + \epsilon\right) d\rho + \rho T ds$$

$$\epsilon + \frac{p}{\rho} = \epsilon + pV = w$$

enthalpy

So

$$d(\rho\epsilon) = w\, d\rho + \rho T ds$$

$$② \quad \frac{\partial}{\partial t}(\rho\epsilon) = w \frac{\partial \rho}{\partial t} + \rho T \frac{\partial s}{\partial t}$$

$$= -w\, \underline{\nabla}\cdot(\rho \underline{V}) - \rho T\, \underline{V}\cdot\underline{\nabla} s$$

So, combining ①, ②:

$$\frac{\partial}{\partial t}\left(\rho \frac{V^2}{2} + \rho\epsilon\right) = -\frac{V^2}{2}\underline{\nabla}\cdot(\rho\underline{V}) - \rho\underline{V}\cdot\underline{\nabla}\left(\frac{V^2}{2} + w\right) - w\,\underline{\nabla}\cdot(\rho\underline{V}) \cancel{- \rho T\, \underline{V}\cdot\underline{\nabla} s} \cancel{+ \rho T\, \underline{V}\cdot\underline{\nabla} s}$$

$$= -\left(\frac{V^2}{2} + w\right)\underline{\nabla}\cdot(\rho\underline{V}) - \rho\underline{V}\cdot\underline{\nabla}\left(\frac{V^2}{2} + w\right)$$

$$= -\underline{\nabla}\cdot\left(\rho\underline{V}\left(\frac{V^2}{2} + w\right)\right)$$

Thus, have:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho \epsilon\right) + \underline{\nabla}\cdot\left(\rho \underline{v}\left(\frac{v^2}{2} + w\right)\right) = 0$$

so

$$\frac{\partial}{\partial t}\int^{V} d^3x \left(\frac{\rho v^2}{2} + \rho\epsilon\right) = -\int d\underline{s}\cdot\left[\rho \underline{v}\left(\frac{v^2}{2} + w\right)\right]$$

change in energy in volume V

energy flux density thru surface.

$$\boxed{\begin{array}{l}\text{energy density flux} \\ \underline{Q} = \rho \underline{v}\left(\frac{v^2}{2} + w\right)\end{array}}$$

accompanies Π_{ik}, $\underline{\Gamma}$.

→ Meaning:

$$w = \epsilon + \frac{P}{\rho}$$

so

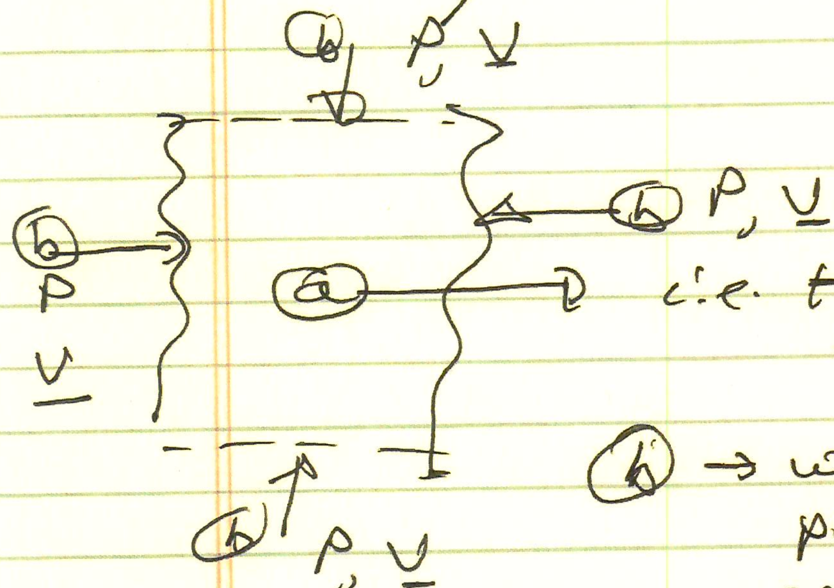
$$\int d\underline{s}\cdot\underline{Q} = \underbrace{\int d\underline{s}\cdot\rho\underline{v}\left(\frac{v^2}{2} + \epsilon\right)}_{(a)} + \underbrace{\int d\underline{s}\,\rho\underline{v}\,\frac{P}{\rho}}_{(b)}$$

(a) flux of KE and internal energy thru surface

$$(b) = \int d\underline{S} \cdot \underline{V}\, P$$

$\rightarrow P\, d\mathcal{V}$ work done by pressure forces on fluid within surface.

$$= \int \underbrace{(\underline{V} \cdot d\underline{S})}_{d\mathcal{V}/dt}\, P$$



(a) $\rightarrow$ i.e. transport thru surface

(b) $\rightarrow$ work done by pressure forces on fluid within S.

II.) Basic Concepts

Now, convenient to note:

$$dE = đQ - p\,dV$$
$$= T\,dS - p\,dV$$

$$W = E + pV$$

→ enthalpy as Legendre transform of entropy

then

$$dW = T\,dS + V\,dp$$

$$= T\,dS + \frac{dp}{\rho}$$

so, for isentropic motions ($dS = 0$),

$dp/\rho = dW$ or $\dfrac{\underline{\nabla} p}{\rho} = \underline{\nabla} W$

→ has advantage of RHS of Euler Eqn. as perfect derivative,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = \frac{d\underline{v}}{dt} = -\underline{\nabla} W$$

Then, can immediately note:

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = 0$$

→ Circulation conserved for inviscid, isentropic fluid

↓

Kelvin's thm.

i.e.

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d\underline{\ell}}{dt}$$

$$= \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d}{dt} d\underline{\ell}$$

$$= \oint (-\underline{\nabla} w)\cdot d\underline{\ell} + \oint \underline{v}\cdot d\underline{v}$$

$$= 0 + 0 .$$

i.e.

$$\boxed{\oint_{C} \underline{v}\cdot d\underline{\ell} = \text{const}}$$

for closed contour in ideal, isentropic fluid

{Note: no use $\underline{\nabla} \cdot \underline{V} = 0$

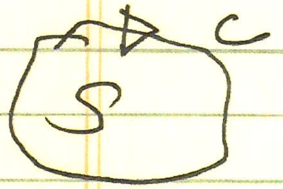
often:

$$\Gamma = \oint \underline{v} \cdot d\underline{\ell} \quad \Rightarrow \quad \text{conserved.}$$

(absolute)

N.B.: Obvious analogy in mechanics is Poincare – Cartan invariant.

$$I_{PC} = \oint \underline{p} \cdot d\underline{q} \qquad \frac{d}{dt} I_{PC} = 0$$



for Hamiltonian system.

Now, elementary vector calc. $\Rightarrow$

normal to enclosed area.

$$\Gamma = \oint_C \underline{v} \cdot d\underline{\ell} = \int_A \underline{\omega} \cdot d\underline{S} \quad \rightarrow \text{cons.}$$

$$\underline{\omega} = \nabla \times \underline{V}$$

Vorticity

what is vorticity?

→ describes rotation of fluid element.

→ $\underline{\omega}$ is 2× effective local angular velocity of fluid.

i.e. $\delta \underline{v} = (\underline{\omega} \times \underline{r})/2$.

→ Vorticity is the non-trivial element in fluid dynamics, beyond Bernoulli's Law and potential flow. Vorticity is central to all interesting topics.

Now,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\underline{\nabla} w$$

$$\underline{v}\cdot\underline{\nabla}\,\underline{v} = -\underline{v}\times(\underline{\nabla}\times\underline{v}) + \underline{\nabla}\frac{v^2}{2}$$

$$= -\underline{v}\times\underline{\omega} + \underline{\nabla}\frac{v^2}{2}$$

so

Magnus Force

$$\frac{\partial v}{\partial t} - \underline{v}\times\underline{\omega} = -\underline{\nabla}\left(w + \frac{v^2}{2}\right)$$

then $\underline{\nabla}\times$

⇒ Ideal vorticity (Induction) equation:

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})}$$

$$= - \underline{v} \cdot \underline{\nabla} \underline{\omega} + \underline{\omega} \cdot \underline{\nabla}\, \underline{v} - \underline{\omega}\, \underline{\nabla} \cdot \underline{v}$$

$$\boxed{\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v} - \underline{\omega}\, \underline{\nabla} \cdot \underline{v}}$$

or, with continuity:

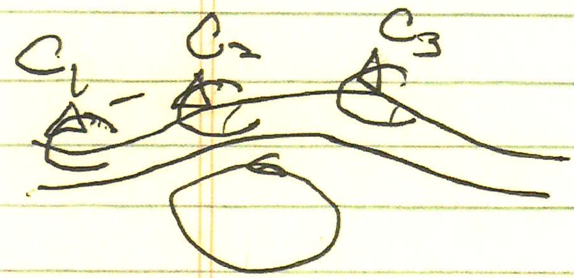
$$\boxed{\frac{d}{dt} \underline{\omega}/\rho = \underline{\omega}/\rho \cdot \underline{\nabla}\, \underline{v}}$$

→ "$\underline{\omega}/\rho$ frozen-in"

→ can derive Kelvin's Thm from induction equation

→ viscosity breaks circulation conservation.

III.) Potential Flow



excludes case of separation!

– Consider fluid streamlines

i.e. stream lines are lines along which fluid flows, i.e.

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

then if $\underline{\omega} = 0$ at any point on streamline, Kelvin's thm $\Rightarrow \underline{\omega} = 0$ everywhere on line

i.e. Easily seen by considering circulation around infinitesimal loop "pulled" along line. Thus if:

$$\oint_{C_1} \underline{v} \cdot d\underline{\ell} = \int_{A_1} \underline{\omega} \cdot d\underline{s} = 0 \quad , \text{ then}$$

$$\oint_{C_n} \underline{v} \cdot d\underline{\ell} = \int_{A_n} \underline{\omega} \cdot d\underline{s} = 0 \quad , \text{ all } C_n$$

- Flow with $\underline{\omega} = 0 = \underline{\nabla} \times \underline{v}$ in all space

  $\Rightarrow$ potential, irrotational flow.

  $\rightarrow \underline{\omega} \neq 0 \rightarrow$ vortical rotational