# Module 1: Fluid Dynamics of Accretion

N.B.: Really, Fluid (and MHD) Dynamics of Accretion (and Planet Formation).

Approximate Plan:

[Not an astrophysics class!]

- Physics of Accretion (Basics)
- Magnetorotational Instability (MHD) and Accretion Dynamics
- From Disks → Planets
- TBD. (Galactic Disks, Spirals)

References:

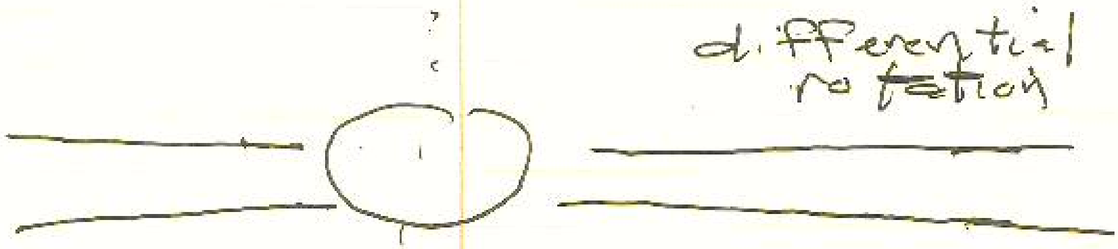
- Posted Materials → Labeled as "Module"
- P. Armitage } book review
- T. Padmanabhan, Vol 2. Theoretical Astrophysics

~ Solar System — Disk (intuitive from planarity)

S · · · ?? J S

etc.

Sun and solar system evolved from disk, in which sun formed




Some simple observations:

– where did the mass end up?

$\sim$ sun.

Obviously $M_\odot \gg$ all else.

– where is the angular momentum?

$\Rightarrow$ giant planets (distant).

$L_\odot \cong M_\odot R_\odot^2 \Omega \sim 10^{49}$ g cm$^2$/sec.

$L_J \cong 3 \times 10^{50}$ g cm$^2$/sec. Jupiter

+ Saturn, Uranus, Neptune + ....

Disk evolution:

- process of segregation of mass (in) and angular momentum (out)

- (i) Mass → concentrates in core → star, object
  accretion ← i.e. how does mass infall ?

(ii) Angular momentum → transported to periphery

obviously (ii) enables (i)

⇒ Central question of physics of angular momentum transport.

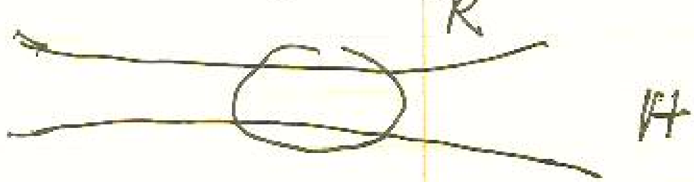
⇒ key player is turbulent viscosity, somewhat resembling notions in pipe flow.

N.B. Key early paper: Shakura and Sunyaev (11,000+ cites) prominently draws on fluid intuition and concepts.

N.B. ~ Dissipation does not lead to rigid rotation here !

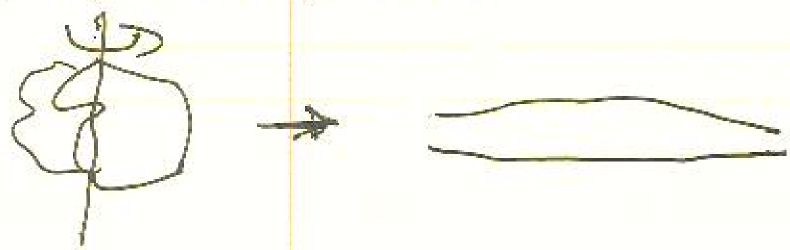
→ Basics

- Disk (thin)




$\left\{ \begin{array}{l} H \ll R. \\ \text{Flare} \end{array} \right.$

- evolved from GMC



- why disk

① infall → colliding particles mix into flow of disk.

② relaxation (I!)

- minimum energy orbit → circle

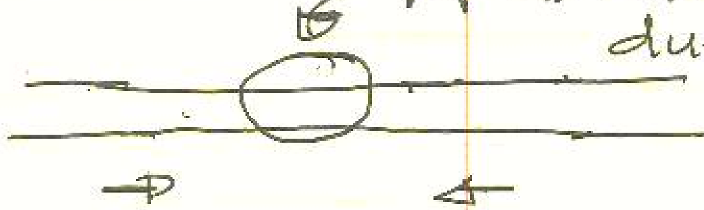
- disk as succession of circles

⇒ accretion: transfer circle-to-circle.

Disk — framework/vehicle for accretion

→ Characterizing the Disk

$\dot{M}$ → rate of mass increase due infall/accretion



$$\dot{M} \sim V_r \Sigma$$

$V_r$ → radial velocity (small, inward)

$\Sigma$ ~ column density

i.e. thin disk:

3D, but $H \ll R$

$$2D: \quad \Sigma = 2\int_0^H dz\, \rho(z)$$

Radial balance: thin justify

$$\frac{GM}{r^2}\rho = -\nabla_r P + \rho \frac{V_\theta^2}{r} - \cancel{\rho \nabla \phi}$$

mass low, neglect self gravity

$$\Rightarrow \quad \frac{GM}{r^2} = \frac{V_\theta^2}{r}$$

$\phi$ → Poisson eqn.

$$\nabla^2 \phi = 4\pi G \rho$$

$$V_\theta = (GM/r)^{1/2}$$

$$\Omega(r) = (GM/r^3)^{1/2}$$

and specific angular momentum
(i.e. angular momentum per article)

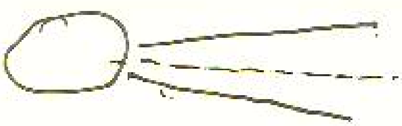
$$\boxed{r^2\Omega = (GMr)^{1/2}}$$

→ n.b. specific angular momentum <u>increases outward</u>

N.B.: $L \ll L_{\text{Eddington}}$
(weak radiation pressure

→ basic $\boxed{\text{Keplerian Disk}}$

<u>Vertical balance</u>



Pressure supports disk against gravity
<u>Hydrostatic</u>.

$$\left(\frac{GM}{R^2}\right)\frac{H}{R} \sim \frac{1}{\rho_c}\frac{\partial P_c}{\partial z} \sim \frac{P_c}{\rho_c H}$$

↑ <u>Vertical component</u>

$$\frac{P_c}{\rho_c} \sim c_s^2 \sim \frac{GM}{R^2}\frac{H^2}{R}$$

$$\sim V_\phi^2 \frac{H^2}{R^2}$$

$$\frac{GM}{r^2} = \frac{V_\phi^2}{r}$$

so $c_s \ll V_\theta$, in thin disk

$\Downarrow$ mean azimuthal flow

check: Is Keplerian profile ok?

Neglected pressure gradient radially

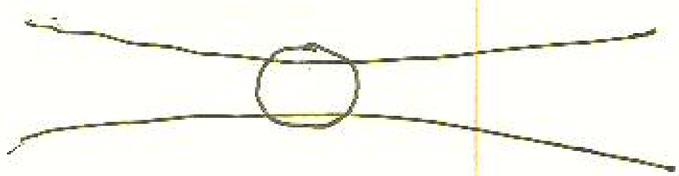
$$\frac{1}{\rho}\frac{\partial P}{\partial R} \cong \frac{P_0}{\rho_0 R} \sim \frac{c_s^2}{R} \ll \frac{V_\theta^2}{R}$$

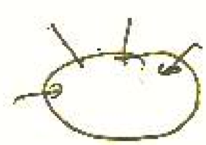
$\Downarrow$ pressure gradient

so neglect pressure is ok.

Thin $\Leftrightarrow$ Keplerian

$\Rightarrow$ Accretion



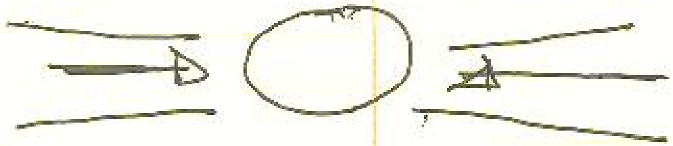


$$\dot{M} = -2\pi R \Sigma V_R \qquad (V_r < 0)$$

$\Downarrow$ gain in mass in center

$\dot{M}$ is the FOM of accretion process

→ Key Point:



- mass accretes
- $L_\theta \sim r^2 \Omega \sim r^{1/2} (GM)^{1/2}$

specific angular momentum decreases with radius

$\boxed{\text{If matter accretes, where does the angular momentum go?}}$

Answer:

- fluid must lose angular momentum

∴ transport it outward, (recall segregation)

by viscous torque.

inner radii torque outer

$$F_\nu = \nu \Sigma r \frac{d\Omega}{dr}$$

$F_\nu$ ↓ viscous force per length.

$\nu$ ↓ viscosity (what is it?)

{ inner, faster radii spin up
outer, slower radii

$$\Pi = -\nu\left(\frac{dv_\theta}{dr} - \frac{v_\theta}{r}\right)$$

$$= -\nu\left(r\frac{d\Omega}{dr} + \cancel{\Omega} - \cancel{\Omega}\right) = -\nu\left(r\frac{d\Omega}{dr}\right)$$

i.e. viscous stress should balance for solid body rotation.

N.B. [ viscous transport $\sim \Sigma \nu r \, d\Omega/dr$ (local) is a crutch $\Rightarrow$ simplest representation transport process

So torque exerted by viscous force.

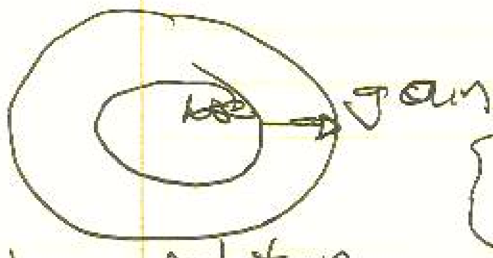
$$\overset{L}{\phantom{x}} \qquad \overset{F/L}{\phantom{x}} \quad \text{Moment}$$

$$T(r) = (2\pi R)\, F\, R$$

$$= \nu \Sigma \, 2\pi R^3 \left(\frac{d\Omega}{dr}\right), \quad \text{at } r.$$

$$\frac{d\Omega}{dr} < 0$$

$$\boxed{\begin{array}{l} \nu \Leftrightarrow \dot{M} \\ (\text{really } \nu\Sigma \Leftrightarrow \dot{M}) \end{array}}$$

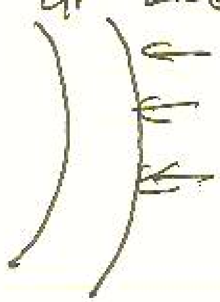



{ torque sense so outer regions gain.

<u>Key Balance relation</u>

Now consider balance of angular momentum

dr Loss in $L_s$ must balance torque.
during accretion



$$T = \frac{d}{dt} L_s \, \Delta R$$

$$\frac{d}{dt} L_s = \overset{\text{in}}{\dot{M} (R+dr)^2 \, \Omega(r+dr)} - \overset{\text{out}}{\dot{M} r^2 \Omega}$$

$$= \dot{M} \frac{d}{dr}(r^2\Omega)\, dr$$

$$\left.\frac{d L_s}{dt}\right)_{\text{lost}} = \dot{M} \frac{d}{dr}(r^2\Omega)\, dr$$

torque diffntl.

$$\frac{d}{dr}T(r) < 0$$

$$\dot M \frac{d}{dr}(r^2\Omega) = -\frac{d}{dr}\left[\nu\Sigma 2\pi R^3 \frac{d\Omega}{dr}\right]$$

$$\Omega = (GM/r^3)^{1/2}$$ for Kep. dr.

integrate.

$\Rightarrow$

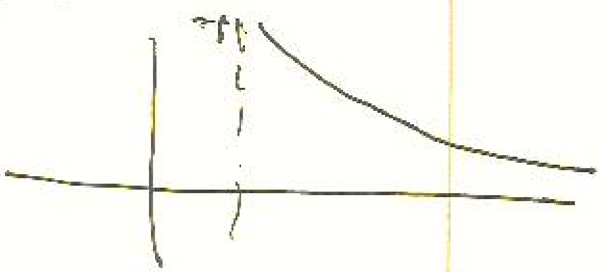
$$R^{1/2}\dot M = \nu\Sigma(3\pi)R^{1/2} + C$$

$$\boxed{\nu\Sigma R^{1/2} = \frac{\dot M}{3\pi}R^{1/2} + C}$$

($r\nu\, d\Omega/dr$)

shear vanishes at some inner radius $R_*$

(boundary specific)



$$\boxed{\nu\Sigma = \frac{\dot M}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]}$$

- tie viscosity to $\dot M$ (aggregation)
- viscosity role in accretion.

→ What else happens?

$\nu \rightarrow$ viscous dissipation $\rightarrow$ heating

$\sim \nu\rho(\nabla v)^2$

$$\underset{\text{dissip}}{D} \sim \nu\Sigma\left(R\frac{d\Omega}{dr}\right)^2$$

$$\simeq \frac{3GM\dot{M}}{4\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{1/2}\right]$$

from above.

so for $R \gg R_*$

energy (released / dissipated) between $R+dr$, $R$:

$$2\pi R\,dr\,\frac{3GM\dot{M}}{4\pi R^3} \sim \boxed{\frac{3GM\dot{M}}{2R^2}}$$

$$\sim 3\,\frac{GM\dot{M}}{2R^2}$$

$$\Downarrow$$

$$\sim 3\,\dot{KE}$$

$$\Downarrow$$

excess from energy released at smaller radii and transported to $R$ by $\nu$.

$L \equiv$ Luminosity

dissipated energy radiated

$$L = \int_{R_*}^{\infty} D(R)\, 2\pi R\, dR$$

$$= \frac{GM\dot{M}}{2R_*} \sim \tfrac{1}{2}\, \Delta \text{ Grav Pot En. in accretion}$$

i.e. links luminosity to energy dissipated in accretion ~ other half in $V_\phi$ at $R_*$

Treating thin disk as black body,

$$2\sigma T_s^4 \sim D(R)$$

radiates top, bottom

$$\Rightarrow T_s \sim (\dot{M})^{1/4}$$

temp. $\sim r^{-3/4}$

→ What of the viscosity ??

→ we have treated viscosity as a coefficient

collisional

$\Rightarrow \nu \sim c_s\, l_{mfp}$

→ Disks are hot → Plasma

→ $l_{mfp} \sim 1/n\sigma$

$\sigma \sim \pi b^2$

crudely

impact parameter

$\frac{Ze^2}{b} \simeq \frac{3}{2} k_B T$

Rutherford/ Coulomb scattering. (no worry re: $\ln\Lambda$).

$\nu_{coll}$ feeble

→ $Re \sim 2\times10^{9}\left(\frac{M}{M_{\odot}}\right)^{1/2}\frac{R}{10^{10}\,\mathrm{cm}}\, n T^{-5/2}$

$\gg 1$ huge.

Collisional viscosity irrelevant!

→ Points toward a turbulent viscosity as the agent for accretion.

Recall pipe flow!

→ What is the mechanism for producing the viscosity ?

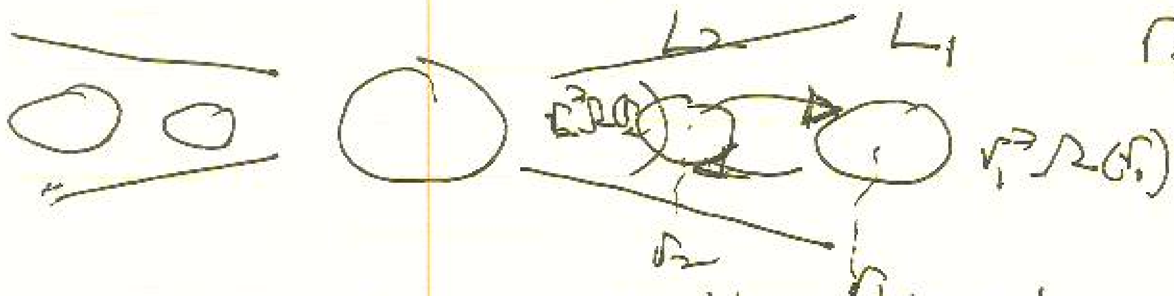
i.e. Pipe flow → shear flow instability
→ shear flow turbulence.

Classic — Rayleigh
→ stability of differential rotating fluid

→ Disk stability

$L_1 > L_2$

$r_2 < r_1$




Consider incompressible interchange of two rings ($k_\theta = 0$ for perturbation),
— conserve angular momentum of each ring. ($k_\theta = 0$)
— energy ↑ or ↓

$$\Delta E = E_{after} - E_{before}$$

$$= \left(\frac{L_2^2}{2r_1^2} + \frac{L_1^2}{2r_2^2}\right) - \left(\frac{L_1^2}{2r_1^2} + \frac{L_2^2}{2r_2^2}\right)$$

$$= \frac{L_2^2}{2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + L_1^2\left(\frac{1}{2r_2^2} - \frac{1}{2r_1^2}\right)$$

$$= \underbrace{\frac{1}{2}\left(L_1^2 - L_2^2\right)}_{>0}\underbrace{\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)}_{>0}$$

$$\boxed{\Delta E > 0}$$

→ interchange <u>increases</u> the energy

→ <u>stable</u> radial stratification → $L^2$ incr. with radius

## Equations

$$\rho \frac{d\widetilde{v_r}}{dt} = -\nabla_r \widehat{P} + \rho \widetilde{\frac{v_\theta^2}{r}}$$

$$\partial_t + v_\theta \nabla_\theta + \cdots$$

$$\rho \frac{d}{dt} \widetilde{v_z} = -\nabla_z \widehat{P}$$

$$\frac{d}{dt} L_\theta = 0$$

To calculate;

$$\rho \frac{\partial_t}{\partial t} \tilde{V}_r = -\partial_r \tilde{P} + 2\Omega \tilde{V}_\theta$$

$$\rho\, \partial_t \tilde{V}_z = -\partial_z \tilde{P}$$

and conservation of angular momentum,

$$\frac{d}{dt} r \tilde{V}_\theta = -\tilde{V}_r\, \partial_r (r^2 \Omega)$$

much akin to R-B, with

$T \Leftrightarrow L_\theta$

$\nabla \langle T \rangle \Leftrightarrow \partial_r (r^2 \Omega)$

$$\Rightarrow \quad \omega^2 = \frac{k_z^2}{k_r^2 + k_z^2} \left( \frac{2\Omega}{r} \partial_r (r^2 \Omega) \right)$$

$$\boxed{\omega^2 = \frac{k_z^2}{k^2}\, \Phi}$$

$\Phi \to$ Rayleigh discriminant

$- \kappa^2 \to$ epicyclic frequency

$\to$ stable buoyancy wave for

$\partial_r (r^2 \Omega) > 0$

($L^2$ incr. with radius)

(uniform rotation:
$\Phi = 4\Omega^2 \to$
inertial wave)

- axisymmetric interchange stable
- no inflection point (HW)

⇒ <u>no obvious hydrodynamic linear instability.</u>

Contrast → pipe flow.

<u>Also</u>

~ Nonlinear instability → <u>hard</u> <u>work</u>.

see H. Ji.

~ convection:

~ $\omega^2 = (k_z^2 \Phi + k_r^2 N^2)/k^2$

$N^2 < 0$, but axisymmetric rolls ⇒ <u>inward</u> transport ?

~ difficult to excite ($N^2 < 0$)

~ non-axisymmetric problematic

<u>Plasma!</u>

⇒ B-fields & M. R. I.

(next)

In the meantime:

$$\nu \sim v_T \, l_T$$

$$\sim \alpha \, c_s H \quad \begin{cases} \text{Shakura-Sunyaev} \\ \text{prescription} \end{cases}$$

$$\Downarrow$$

$$\underline{\alpha < 1}$$

What is $\alpha$?

N.B. Is this scaling sensible?

$\alpha$ viscosity is classic parametrization for disk.

→ A Closer Look!?

→ Picture / Model relies on viscous transport angular momentum

$\sim \Pi \sim -\Sigma \nu \, r \frac{d\Omega}{dr}$

→ would expect:

→ dissipation leads to rigid rotation

→ minimum energy state is uniform rotation

how understand?

how reconcile?

Of course → segregation / (N.B. Accretion happens).

Theorem: For given density distribution and total angular momentum, the motion of least energy is uniform rotation.

L-B+P

$E_{internal}$, $E_{grav}$ fixed

$$T = \frac{1}{2}\int \rho v^2 d^3x = \frac{1}{2}\int v^2 dm$$

$$\int R v_\phi \, dm = \underset{\text{fixed}}{L}$$

→ angular momentum.

For moment of inertia:

$$I = \int R^2 \rho \, d^3x = \int R^2 \, dm$$

then:

$$\int v_\phi^2 \, dm \, I = \int v_\phi^2 \, dm \int R^2 \, dm$$
$$\geq \left( \int R v_\phi \, dm \right)^2$$
$$= L^2$$

Schwarz inequality !

so

$$\boxed{\int v_\phi^2 \, dm \, I \geq L^2}$$

$$\int v_\phi^2 \, dm \geq \frac{L^2}{I}$$

equality if $\boxed{v_\phi \sim \underbrace{\Omega}_{\text{const}} R}$ → uniform rotation

ie $\Omega^2 \int R^2 \, dm = \frac{L^2}{I}$

$$\Omega^2 = L^2 / I^2 \checkmark$$

Solid body rotation is minimum energy state.

→ Accretion ? ?

What of disk?

- Keplerian balance
- small viscosity
- accretion

⇒ Answer: (will show).

Minimum energy configuration is:

→ almost all mass accretes to center

→ 1 particle carries angular momentum in circular orbit at ∞ !

how show?

But: i) Expect for viscous stress transport $L_\theta$, that minimum energy state is uniform rotation

$$\Pi \sim -\nu\, r\, \partial\Omega/\partial r$$

→ ii) How reconcile with accretion ??

Now → re-visit 2 particle argument, but seek

→ lower energy
→ conserve total angular momentum
→ conserve total mass

→ contrast Rayleigh — there angular momentum of each particle conserved. Here, sum

$L_\theta/M$ → specific angular momentum
$E/M$ → " energy.

For specific $L_\theta = h$,

$$\mathcal{E} = \frac{1}{2}\left(u_r^2 + u_z^2\right) + \frac{h^2}{2R^2} - \psi(R, z)$$

grav., etc

↑ maximal on $z = 0$.

So minimal $\epsilon$, given $h$:

$u_r = u_z = 0$, $z = 0$; is where

$h^2/2R^2 - \Psi(R, 0)$ minimal.

i.e. $\epsilon(h) = \min \epsilon = \min_R \left[ \frac{h^2}{2R^2} - \Psi(R, 0) \right]$

$\Rightarrow R_h$ s/t $\frac{\partial}{\partial R}\left\{ \frac{1}{2} h^2/R^2 - \Psi(R, 0) \right\} = 0$

radius for min. energy, given $h$ → radius of minimum energy circular orbit

so

$$\boxed{\begin{aligned} \epsilon(h) &= \frac{1}{2} h^2/R_h^{\,2} - \Psi(R_h, 0) \\ &= \frac{V^2}{2} - \Psi \end{aligned}}$$

min energy.

For $\boxed{\frac{d\epsilon}{dh} = \epsilon'(h) = \frac{\partial \epsilon}{\partial h} = \frac{h}{R_h^{\,2}} = \Omega}$

$\epsilon(h)$ already stationary w/R variations $R$.

i.e. $\frac{d\epsilon}{dh} = \frac{\partial \epsilon}{\partial h} + \cancel{\frac{\partial \epsilon}{\partial R} \frac{dR}{dh}}$

① Now, minimize energy 2 particles, keeping total angular momentum constant

- minimize energy each, L const
  → 2 circles
- lower → minimize via exchange?
  what happens?

each particle: $\epsilon(h)$

so

$$E = m\,\epsilon(h)$$

∴

$$E = m_1 \epsilon(h_1) + m_2 \epsilon(h_2)$$

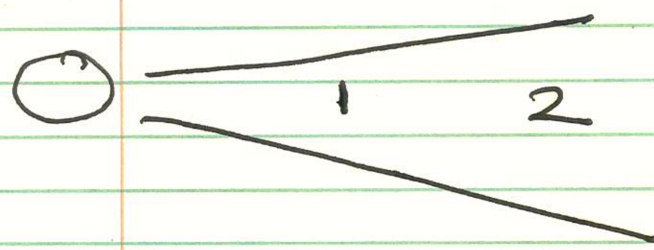
$$H = m_1 h_1 + m_2 h_2$$

so $\quad dH = 0 \Rightarrow m_1 dh_1 + m_2 dh_2 = 0$

$$dE = m_1 dh_1 \epsilon'(h_1) + m_2 dh_2 \epsilon'(h_2)$$

so $\quad dE = m_1 dh_1 \left[ \epsilon'(h_1) - \epsilon'(h_2) \right]$

$\Rightarrow$ $$dE = m\, dh_1 (\Omega_1 - \Omega_2)$$




$\Omega_1 > \Omega_2$

$$dE < 0 \Rightarrow \boxed{\begin{array}{l} dh_1 < 0 \\ dh_2 > 0 \end{array}}$$

$\Rightarrow$ orbit 2, of lower angular velocity, gains angular momentum while orbit 1, of higher angular velocity, lowers / loses angular momentum.

$\Rightarrow$ (minimization) (relaxation) of energy by exchange of angular momentum to orbit of lower $\Omega$ $\rightarrow$ i.e. outward

$\Rightarrow$ energy lowered if angular momentum (flows) outward.

③ Now consider only that total mass fixed $\Rightarrow$ orbits exchange mass too;

so $\boxed{dE} = d\left[M_1 G(h_1) + M_2 G(h_2)\right]$

$$\left[dM = 0\ ,\ dM_1 = -dM_2\right]$$

$$\left[\begin{array}{l} dH = 0 = dH_1 + dH_2 \\ d(M_1 h_1) + d(M_2 h_2) = 0 \end{array}\right.$$

now

$$dE = dM_1\, G(h_1) + M_1 G'(h_1)\, dh_1 + dM_2\, G(h_2) + M_2 G'(h_2)\, dh_2$$

$$= dM_1\left[G(h_1) - h_1 G'(h_1)\right] + d(M_1 h_1) G'(h_1) + dM_2\left[G(h_2) - h_2 G'(h_2)\right] + d(M_2 h_2) G'(h_2)$$

$\Rightarrow$

$$dE = dM_1 \left\{\left[G(h_1) - h_1 \Omega_1\right] - \left[G(h_2) - h_2 \Omega_2\right]\right\} + dH_1 (\Omega_1 - \Omega_2)$$

$\therefore$ $<0$, for $dH_1<0$

$$dE = dH_1 (\Omega_1 - \Omega_2)$$

sm. $\ominus$ lg

$$+ dM_1 \left\{ \left[ \mathcal{G}(h_1) - h_1 \Omega_1 \right] - \left[ \mathcal{G}(h_2) - h_2 \Omega_2 \right] \right\}$$

Now $\dfrac{d}{dR}(\mathcal{G} - h\Omega) = \dfrac{d}{dR}\left(-\dfrac{1}{2}V^2 - \psi\right)$

$$= -V\frac{dV}{dR} + \frac{V^2}{R}$$

cent bal.

$$= -V\left(\frac{dV}{dR} - \frac{V}{R}\right)$$

$$= -RV\frac{d}{dR}\left(\frac{V}{R}\right) > 0$$

$\therefore$

$$\frac{d}{dR}(\mathcal{G} - h\Omega) > 0$$

$<0$ for $dH_1 < 0$

$\therefore$ $dE = dH_1 (\Omega_1 - \Omega_2)$

$$+ dM_1 \left\{ \Delta(\mathcal{G} - h\Omega) \right\}$$

$<0$ for $dM_1 > 0$

so

$dE < 0$ for

→ ② gains $h$

i.e. [ angular momentum coupled outward

→ ① gains $M$

i.e. mass accretes.


i.e. final, minimum energy state:

⇒ { all mass accretes
but
1 particle at ∞ to carry angular momentum.

~ Contrast to solid body rotation!

Important contrast:

2-particle arguments:

- Rayleigh: exchange particles keeping angular momentum of <u>each</u> constant

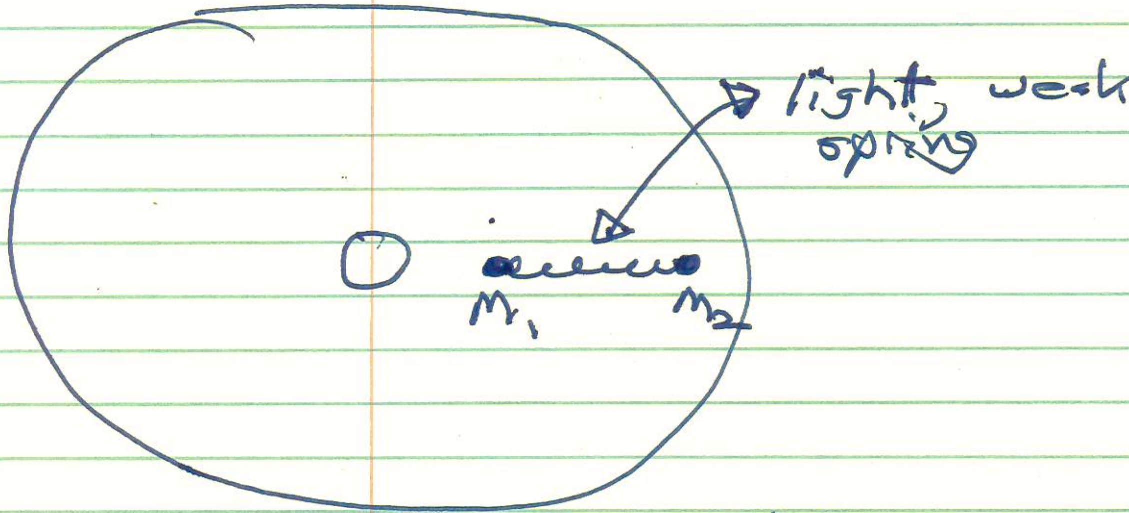
- Lynden-Bell: keep sum of angular momenta constant, allowing exchange between each

[ L-B allows inter-particle angular momentum transfer.

How? — waves
— magnetic fields ⇒ MRI
⋮

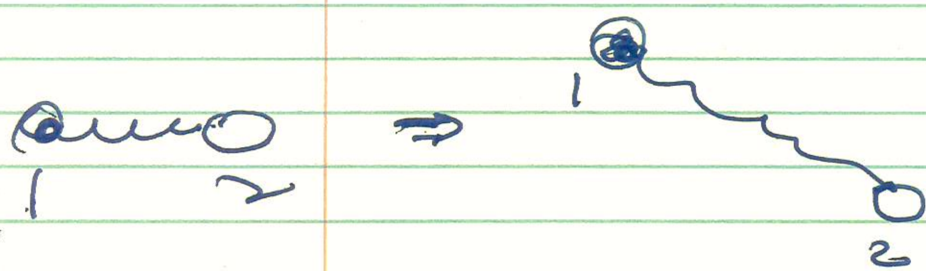
## → MRT

Considerer:




$$\frac{d\Omega}{dr} < 0$$

- $d\Omega/dr = 0$ → no spring extension
- $d\Omega/dr < 0$




① fast → pulls ahead

② slow → falls behind

∴ spring extended.

but now:

— spring:

- pulls back on 1

$\Rightarrow$ 1 loses angular momentum

- pulls ahead on 2

$\Rightarrow$ 2 gains angular momentum

but: $h^2/2R^2$ vs $\frac{GM}{r}$ ;

particles

— 1 loses angular momentum $\rightarrow$

must drop to lower radius

$R_1 \rightarrow R_1 - \Delta R_1$

2 gains angular momentum $\rightarrow$

must move to large radius

$R_2 \Rightarrow R_2 + \Delta R_2$

So

⇒ Spring extended further!

etc

⇒ $\boxed{\text{instability.}}$ → MRI
Magneto-rotational instability.

Upshot of instability is

→ $M_2$ gaining angular momentum
$M_1$ losing angular momentum

⇒

① infall

② move out

⇒ $\boxed{\text{transfer of angular momentum to larger radii via elastic coupling.}}$

- driven by $d\Omega/dr$
- spring ($\underline{B}$ field) coupling allows angular momentum transfer.

⇒ need develop MHD for

MRI.