Physics 216

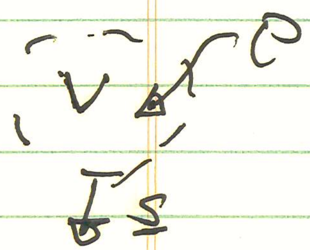
Lecture II – Ideal Fluids
(Read Landau)

- Equations
- Basic Concepts, especially $\begin{cases}\text{Kelvin's Thm} \\ \text{Potential Flow}\end{cases}$
- Induced Mass

I.) Euler Equations / Ideal Fluids

Ideal – "The Flow of Dry Water"
(Feynman)

blob



Volume V
density ρ

- argue macroscopically but really derive from Boltzmann Equation
- viscosity brings additional time scale.

① – mass conservation

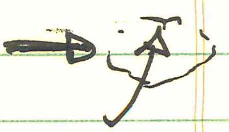
$$\frac{dM}{dt} = \frac{\partial}{\partial t}\int d^3x\, \rho(x,t) = -\int d\underline{S}\cdot(\underline{v}\rho)$$
$$= -\int d^3x\, \underline{\nabla}\cdot(\rho\underline{v})$$

so $$\boxed{\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0}$$

$\updownarrow$ — mass flux

$$\partial_t \rho + \nabla \cdot \underline{\Gamma} = 0$$

② — Momentum Conservation

$\Rightarrow$ 

blob/element

$$\underline{f} = -\underline{\nabla} p + \underline{f}_{body}$$

$\underline{f}$ → net force density on element

$\underline{\nabla} p$ → pressure gradient

$\underline{f}_{body}$ → body force

i.e. $\rho \underline{g}$

$\underline{J} \times \underline{B}/c$

⋮

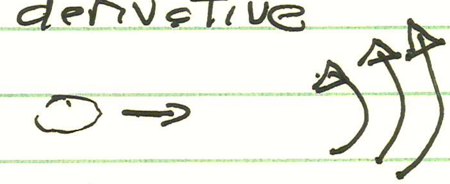
Sir Isaac $\Rightarrow$

$$\rho \underline{a} = -\underline{\nabla} p + \underline{f}$$

$\underline{a}$ → acceleration.

$$\underline{a} = \frac{d\underline{v}}{dt}$$

→ "substantive derivative"



now $$d\underline{v} = \frac{\partial \underline{v}}{\partial t} dt + d\underline{r} \cdot [redacted] \underline{\nabla} \underline{V}$$

$d\underline{v}$ → increment

$\frac{\partial \underline{v}}{\partial t}$ → local acceleration

$d\underline{r}$ → displacement

→ particle moves in inhomogeneous velocity field so

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \frac{d\underline{r}}{dt} \cdot \underline{\nabla}\,\underline{v} = \frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}$$

so

$$\boxed{\rho \frac{d\underline{v}}{dt} = \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla} p + \underline{f}}$$

Euler Eqn.

? Momentum Flux

Will show:

$$\partial_t (\rho v_i) = -\frac{\partial}{\partial x_k} \Pi_{ik}$$

so

$$\partial_t (\rho \underline{v}) = \underline{v}\,\partial_t \rho + \rho \frac{\partial \underline{v}}{\partial t}$$

$$= -\underline{v}\left(\rho(\underline{\nabla}\cdot\underline{v}) + \underline{v}\cdot\underline{\nabla}\rho\right)$$

$$+ \rho\left(-\underline{v}\cdot\underline{\nabla}\,\underline{v} - \frac{\underline{\nabla} p}{\rho}\right)$$

$$= -\left(\rho\left[\underline{v}\,(\underline{\nabla}\cdot\underline{v}) + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right] + \underline{v}(\underline{v}\cdot\underline{\nabla}\rho)\right) - \underline{\nabla} p$$

$$\Rightarrow \partial_t(\rho \underline{V}) = -\underline{\nabla}\cdot\big(\underbrace{\rho \underline{V}\,\underline{V}}_{\text{Reynolds stress tensor}} + \underbrace{\underline{\underline{I}}}_{\text{identity}} P\big)$$

Reynolds stress tensor

(analogue to Maxwell stress tensor).

so

$$\boxed{\Pi_{ik} = \rho V_i V_k + \delta_{ik} P}$$

momentum flux

$$\partial_t \int d^3x\, \rho \underline{V} = \underbrace{\frac{d}{dt}\underline{P}}_{\text{change in momentum of blob}} = -\int d\underline{S}\cdot\left(\rho \underline{V}\,\underline{V} + \underline{\underline{I}} P\right)$$

change in momentum of blob

$\Pi_{ik}\, dS_k \equiv$ momentum flux in ith direction.

→ Beyond Euler, viscous stress appears due to momentum flux from collisions, interacting with macroscopic flow gradients.

For incompressible flow ($\underline{\nabla}\cdot\underline{V} = 0$), continuity and Euler/N-S. describe flow.

→ Mass, Momentum and Energy!

In ideal fluid, no heat exchanged between fluid elements ⇒ motion adiabatic - i.e. entropy conserved along trajectories

$$\frac{ds}{dt} = 0 \qquad s = \text{entropy/mass}$$

$$\boxed{\frac{\partial s}{\partial t} + \underline{v} \cdot \underline{\nabla} s = 0}$$

→ adiabatic equation for fluid

For energy flux

$$\mathcal{E} = \frac{\rho v^2}{2} + \rho \epsilon$$

$\mathcal{E}$ ↓ total energy density of fluid

$\frac{\rho v^2}{2}$ ↓ kinetic energy density

$\rho \epsilon$ → internal energy density (i.e. thermal).

then use dynamics + thermo to derive total energy balance equation

$$\partial_t\left(\frac{\rho v^2}{2}+\rho\epsilon\right)+\nabla\cdot\left(\rho\underline{v}\left(\frac{v^2}{2}+w\right)\right)=0$$

$w = \epsilon + \frac{p}{\rho}$

enthalpy.

$$\partial_t\int d^3x\left(\frac{\rho v^2}{2}+\rho\epsilon\right) = -\int d\underline{S}\cdot\left[\rho\underline{v}\left(\frac{v^2}{2}+w\right)\right]$$

What does this mean?

$$\boxed{\underline{q} = \rho\underline{v}\left(\frac{v^2}{2}+w\right)}$$

energy flux density

→ what does it mean?

$w = \epsilon + p/\rho$

flux of KE and internal energy thru surface

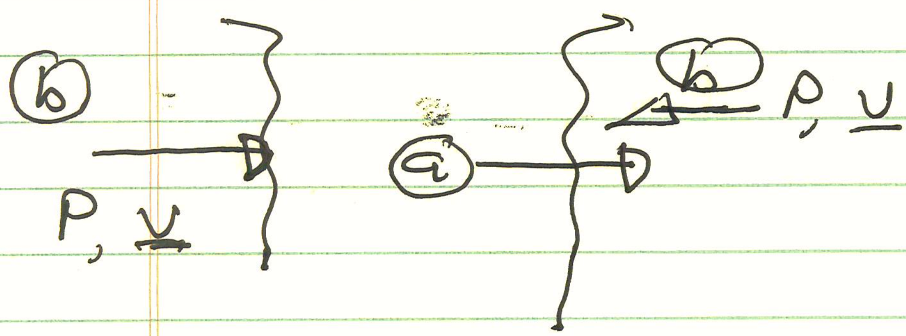
$$\int d\underline{S}\cdot\underline{q} = \int d\underline{S}\cdot\rho\underline{v}\left(\frac{v^2}{2}+\epsilon\right) \quad (a)$$

$$+\int d\underline{S}\cdot\cancel{\rho}\,\underline{v}\,\frac{p}{\cancel{\rho}} \quad (b)$$

(b) $= \int d\underline{s} \cdot \underline{v}\, P$

$= \int (\underline{v} \cdot d\underline{s})\, P$ → $P d\bar{V}$ work by pressure on fluid in blob

(a) $\equiv$ transport of energy thru the surface of the blob



Rate change of energy density

$= (a) + (b)$

To show:

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial}{\partial t}\left(\overset{①}{\frac{\rho v^2}{2}} + \overset{②}{\rho G}\right)$$

$$① = \frac{v^2}{2}\frac{\partial \rho}{\partial t} + \rho\, \underline{v} \cdot \frac{\partial \underline{v}}{\partial t}$$

$$= -\frac{V^2}{2}\underbrace{\underline{\nabla}\cdot(\rho\underline{V})}_{\text{continuity}} - \underbrace{\underline{V}\cdot\underline{\nabla}P - \rho\underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{V})}_{\text{mom. balance}}$$

but

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times\underline{\omega} + \underline{\nabla}\left(\frac{V^2}{2}\right)$$

$$\underline{\omega} = \underline{\nabla}\times\underline{V} \rightarrow \text{vorticity}$$

$$\rho\underline{V}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{V}) = \rho\underline{V}\left(-\underline{V}\times\underline{\omega} + \underline{\nabla}\frac{V^2}{2}\right)$$

$$= \rho\underline{V}\cdot\underline{\nabla}\frac{V^2}{2}$$

To deal with pressure:

$$dW = dE + d(PV)$$

Enthalpy

$$= TdS - \cancel{pdV} + VdP + \cancel{PdV}$$

$$= Tds + \frac{dP}{\rho}$$

$\Rightarrow$

$$\boxed{\underline{\nabla}P = \rho\underline{\nabla}W - \rho T\underline{\nabla}S}$$

thus:

$$① = \partial_t\left(\frac{\rho v^2}{2}\right) = -\frac{v^2}{2}\nabla\cdot(\rho \underline{v})$$
$$-\rho\, \underline{v}\cdot\nabla\left(\frac{v^2}{2}+w\right) + \rho T \underline{v}\cdot\nabla s$$

② $\partial_t(\rho\epsilon) = \ldots\ldots$

Useful to note:

$$d\epsilon = dQ - p\,dV$$
$$= T\,ds - p\,dV$$

$V = 1/\rho, \quad dV = -d\rho/\rho^2$

$$d\epsilon = T\,ds + \frac{p}{\rho^2}\,d\rho$$

so

$$d(\rho\epsilon) = \rho\, d\epsilon + \epsilon\, d\rho$$

$$d(\rho\epsilon) = \left(\frac{p}{\rho}+\epsilon\right)d\rho + \rho T\, ds$$

$$W = \epsilon + PV = \epsilon + P/\rho$$

$$d(\epsilon\rho) = W\,d\rho + \rho T\,ds$$

and

$$② = \partial_t(\rho\epsilon) = W\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}$$

$$= -W\,\underline{\nabla}\cdot(\rho\underline{V}) - \rho T\,\underline{V}\cdot\underline{\nabla}s$$

and so, combining ①, ②

$$\partial_t\left(\frac{\rho V^2}{2} + \rho\epsilon\right) = -\left(\frac{V^2}{2} + W\right)\underline{\nabla}\cdot(\rho\underline{V})$$

$$-\rho\,\underline{V}\cdot\underline{\nabla}\left(\frac{V^2}{2} + W\right)$$

$$= -\underline{\nabla}\cdot\left(\rho\underline{V}\left(\frac{V^2}{2} + W\right)\right)$$

$\Rightarrow$

$$\boxed{\partial_t\left(\frac{\rho V^2}{2} + \rho\epsilon\right) + \underline{\nabla}\cdot\left(\rho\underline{V}\left(\frac{V^2}{2} + W\right)\right) = 0}$$

→ Basic Laws and Concepts

What about vorticity $\underline{\omega} = \underline{\nabla} \times \underline{V}$?

Convenient to note:

$$dE = đQ - p\,dV$$
$$= T\,dS - p\,dV$$

$W = E + pV$ → enthalpy

then

$$dW = T\,dS + V\,dp = T\,dS + dp/\rho$$

and for isentropic flow $(dS = 0)$

$$dp/\rho = dW$$

thus can write (in isentropic case)
RHS of Euler as perfect derivative

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} W$$

Then consider circulation

$$\Gamma = \oint \underline{v}\cdot d\underline{\ell}$$

then

$$\frac{d}{dt}\oint \underline{v}\cdot d\underline{\ell} = \oint \frac{d\underline{v}}{dt}\cdot d\underline{\ell} + \oint \underline{v}\cdot \frac{d}{dt}\, d\underline{\ell}$$

$$= \oint (-\underline{\nabla} w)\cdot d\underline{\ell} + \oint \underline{v}\cdot d\underline{v}$$

$$= 0$$

so

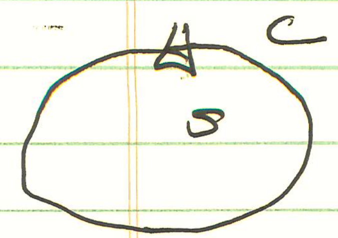
$$\boxed{\Gamma = \oint \underline{v}\cdot d\underline{\ell} = \text{const.}}$$

for ideal, isentropic fluid.

Kelvin's Thm.

Circulation conserved

N.B.: - broken by viscosity
- $\underline{\nabla}\cdot\underline{v} \underset{\neq}{=} 0$ irrelevant.
- Analogy in mechanics is Poincaré - Cartan invariant



$$I = \oint \underline{p}\cdot d\underline{q}$$

$dI/dt = 0$ for Hamiltonian system.

and elementary vector calculus,

normal to plane enclosed area.

$$\Gamma = \oint_c \underline{v} \cdot d\underline{\ell} = \int_A \underline{\omega} \cdot d\underline{S}$$

$$\nabla \times \underline{v} = \underline{\omega}$$

What is vorticity:

- describes rotation of fluid element
- $\underline{\omega}$ is 2 × effective local angular velocity of the fluid

$$\delta \underline{v} = (\underline{\omega} \times \underline{r})/2$$

* Vorticity is the non-trivial element in fluid dynamics! Vorticity is central to all interesting topics.

How evolve vorticity?

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} w$$

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{v} \times (\underline{\nabla} \times \underline{v}) + \underline{\nabla} \frac{v^2}{2}$$

$$= -\underline{v} \times \underline{\omega} + \underline{\nabla} \frac{v^2}{2}$$

so

$$\frac{\partial \underline{v}}{\partial t} - \underbrace{\underline{v} \times \underline{\omega}}_{\text{Magnus Force}} = -\underline{\nabla}\left(w + \frac{v^2}{2}\right)$$

then $\underline{\nabla} \times$

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})} \rightarrow \text{induction equation}$$

$$= -\underline{v} \cdot \underline{\nabla}\, \underline{\omega} + \underline{\omega} \cdot \underline{\nabla}\, \underline{v} - \underline{\omega}\, \underline{\nabla} \cdot \underline{v}$$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v} - \underline{\omega}\, (\underline{\nabla} \cdot \underline{v})$$

and with continuity:

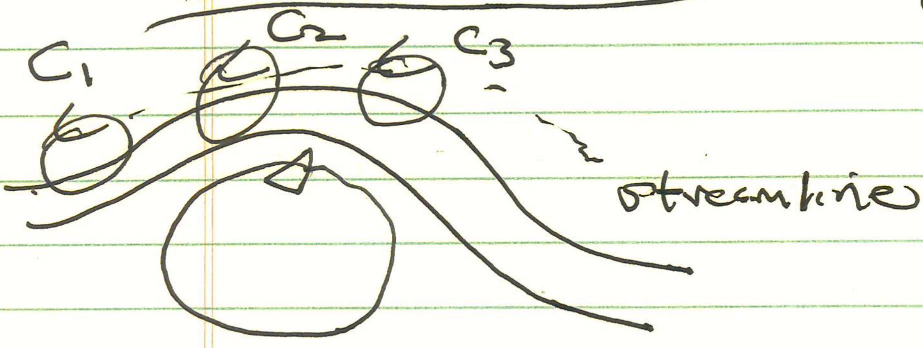
$$\frac{d}{dt}(\underline{\omega}/\rho) = \frac{\underline{\omega}}{\rho} \cdot \underline{\nabla}\, v \rightarrow \frac{\underline{\omega}}{\rho} \text{ "Frozen-in".}$$

<u>TBS</u>

Can derive Kelvin's Thm from induction eqn.

No page 15!

→ **Potential Flow** (copious analogies with electrostatics)

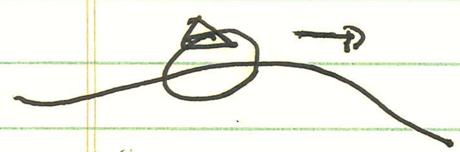


– Consider streamlines

Fluid flows along these, so

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}$$

If $\underline{\omega} = 0$ at any point on streamline, Kelvin's thm ⇒ $\underline{\omega} = 0$ everywhere on line.



tiny loop, then pull along line, and invoke Kelvin's theorem.

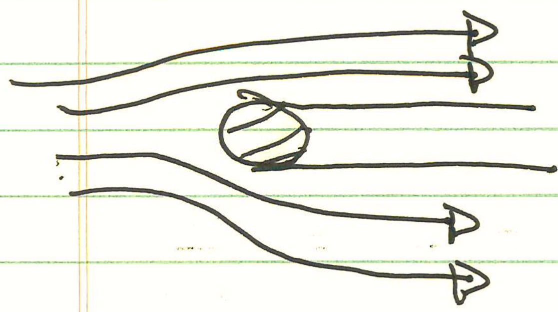
$$\oint_{C_1} \underline{v}\cdot d\ell = \int_{A_1} \underline{\omega}\cdot ds = 0 \qquad \underline{\underline{\text{so}}}$$

$$\oint_{C_n} \underline{v}\cdot d\ell = \int_{A_n} \underline{\omega}\cdot ds = 0, \qquad \text{all } C_n \text{ along line}$$

- flow with $\underline{\omega} = 0$ everywhere is <u>potential</u> or <u>irrotational</u> flow.

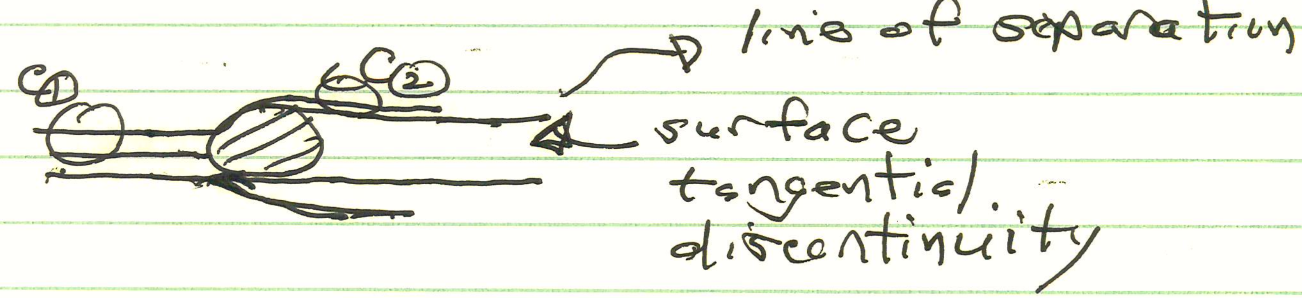
<u>Important</u>: Fails for separation

<u>i.e.</u> consider flow around sphere



- streamlines separate from the body
- surface of tangential discontinuity appears in velocity component

<u>i.e.</u>



- cannot infer $\oint_{C_2} \underline{v} \cdot d\underline{\ell}$ from $\oint_{C_1} \underline{v} \cdot d\underline{\ell}$, due to separation – induced tangential discontinuity

- viscosity important in boundary layer. (No slip B.C.)

Now, for isentropic fluids:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} w$$

potential flow

if $\underline{\omega} = 0$, $\underline{v} = \underline{\nabla} \phi$

stream function

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{v} \times \underline{\omega} + \underline{\nabla}(v^2/2)$$

$$= \underline{\nabla}(v^2/2)$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{\nabla}(v^2/2) = -\nabla w$$

$$\underline{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} + w\right) = 0$$

So

wave dynamical equation for potential flow:

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + w = f(t)}$$

defined for each stream line

- $\frac{\partial \phi}{\partial t} = 0$, recover $(ds = 0)$

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{const.}$$ (Bernoulli Law)

- potential not uniquely defined, as $\underline{v} = \underline{\nabla} \phi$.

Consider incompressible potential flow:

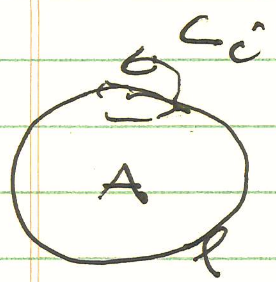
- $\underline{v} = \underline{\nabla} \phi$, $\underline{\nabla} \cdot \underline{v} = 0$

$$\boxed{\begin{gathered} \nabla^2 \phi = 0 \\ \frac{\partial \phi}{\partial t} + \frac{(\underline{\nabla} \phi)^2}{2} + \frac{p}{\rho} = f(t) \end{gathered}}$$

For static flow, with gravity:

$$\boxed{\frac{v^2}{2} + \frac{P}{\rho} + gz = \text{const}}$$

N.B.: In potential flow, streamlines must be <u>open</u>



$$\oint_{C_i} \underline{v} \cdot d\underline{\ell} = \int_{A_i} \underline{\omega} \cdot d\underline{S}_i = 0$$

$\underline{\omega} = 0$ along line

but then,

$$\int_{\underline{A}} \underline{\omega} \cdot dS = 0$$

but

$$= \oint_{SL} \underline{v} \cdot d\underline{\ell}$$

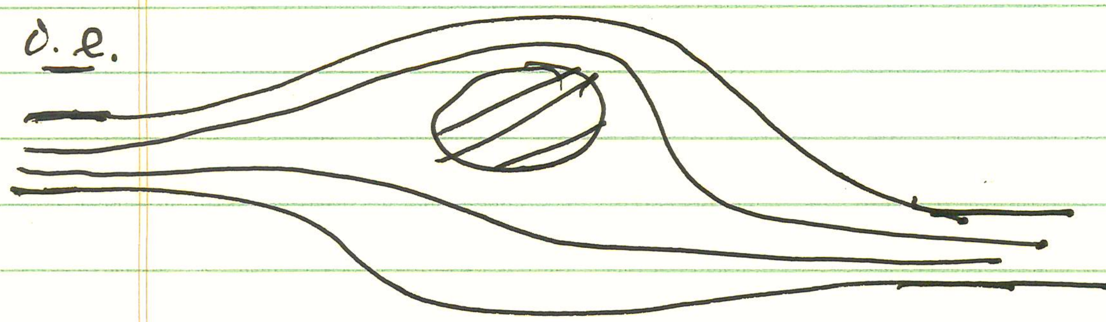
but $\oint \underline{v} \cdot d\underline{\ell} \neq 0$ → fluid flow.

⇒ contradiction. ⇒ $\boxed{\text{streamlines must be } \underline{\text{open}}.}$

Also, streamlines (for potential flow) should not intersect boundaries.

Generally, potential flow problems apply to infinite media, some distance from [redacted] surfaces, boundaries.

i.e.



sphere in $\underline{V} = V_0 \hat{z}$ flow, for locations away from sphere, is typical potential flow problem.

Aside: What does "incompressibility" mean? When is $\underline{\nabla} \cdot \underline{V} = 0$ a good approximation?

$\rightarrow |V| \ll c_s$ $\qquad c_s^2 = dP/d\rho$

$\left(\frac{l}{T}\right)^2 \ll c_s^2$

↳ length, time scale ratio

i.e. compare terms in continuity equation:

$$\frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V}$$

$$\frac{\Delta\rho}{\tilde{\tau}} \qquad \rho\frac{\tilde{V}}{\ell}$$

For $\tilde{V}$:

$$\frac{\partial v}{\partial t} = -\frac{\nabla P}{\rho}$$

$$\frac{\tilde{V}}{\tilde{\tau}} \sim \frac{c_s^2}{\ell\rho}\Delta\rho$$

so

$$\frac{\Delta\rho}{\tau} \text{ vs } \frac{\cancel{\rho}}{\ell}\frac{\tau c_s^2}{\ell\cancel{\rho}}\Delta\rho$$

Now, $\left|\underline{\nabla}\cdot\underline{V}\right| \gg \left|\frac{1}{\rho}\frac{d\rho}{dt}\right|$

means $\underline{\nabla}\cdot\underline{V} \cong 0$, to good approximation.

so, incompressible if:

$$\frac{\tau\, c_s^2\, \Delta\rho}{\ell^2} \gg \frac{\Delta\rho}{\tau}$$

$$\Rightarrow \boxed{c_s^2 \gg \frac{\ell^2}{\tau^2}}$$

→ criteria in terms length time scales of flow.

$$\Leftrightarrow \quad c_s^2 \gg \frac{\omega^2}{k^2}$$

Note: Long time favors incompressible

so $\nabla \cdot \underline{V} \cong 0$ if

- flow speeds subsonic
- times slow compared to time to traverse 1 spatial scale at acoustic speed.

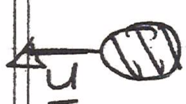
$$\underline{\omega} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z} = \hat{z}\left(-\nabla^2\psi\right)$$

$$\frac{d\underline{\omega}}{dt} = 0 \Rightarrow \begin{cases} +\dfrac{\partial}{\partial t}\nabla^2\psi + \underline{\nabla}\psi \times \hat{z}\cdot\nabla\nabla^2\psi = 0 \\ \text{2D incompressible fluid eqn.} \end{cases}$$

iv.) Problems in Potential Flow

a.) Incompressible Potential Flow Around Sphere

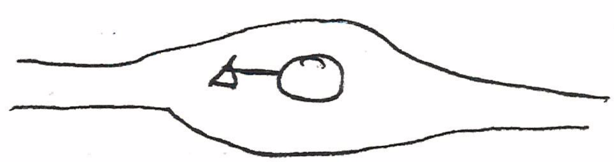
Consider rigid sphere in motion at $\underline{u}$ in infinite fluid



Flow Pattern ?

Now :

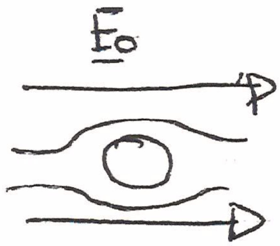
- intuitively, expect :



i.e. equivalent to $\begin{cases} \text{sphere at rest} \\ \underline{V}\big|_{\infty}^{\text{fluid}} = -\underline{u} \end{cases}$

Electrostatic analogy: Conducting sphere in uniform electric field

i.e.



$\phi = -\underline{E}_0 \cdot \underline{r} + \phi_{sphere}$

ϕ_{sphere} is dipole field.

Dipole moment determined by b.c.
i.e. $\phi = \text{const} = 0$ on sphere surface

Now, for potential flow (incompressible):

$\nabla^2 \phi = 0 \qquad \underline{V} = \underline{\nabla} \phi$

$$V_n = \underline{V} \cdot \hat{n} = \underline{u} \cdot \hat{n} \Big|_{surface}$$

(i.e. normal velocity = sphere velocity on surface)

By analogy with electrostatics, can solve via:
- multipole expansion
- B.C.'s determine effective "charge" distribution

Recall e.s. $\Rightarrow$ $\nabla^2 \phi = -4\pi\rho$

$$\phi = \int d^3x' \, \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

For $\underline{x}$ outside region ρ :

$$\phi(\underline{x}) = \int d^3x' \, \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= \int d^3x' \, \frac{\rho(\underline{x}')}{|\underline{x}-\cancel{\underline{x}'}|} - \int d^3x' \, \underline{x}'\rho(\underline{x}') \cdot \underline{\nabla}\left(\frac{1}{|\underline{x}|}\right) + \cdots$$

$$= \frac{Q}{|\underline{x}|} - \underline{d}\cdot\underline{\nabla}\left(1/|\underline{x}|\right) + \cdots$$

monopole $\qquad$ dipole $\qquad$ quadrupole

Thus, can write down general solution for potential flow streamlines around body as multipole expansion.

– $Q = 0$ (no sources, sinks)

$\therefore$ in general dipole dominates

– in 2D, same story with $\ln|x-x'| \Rightarrow 1/|x-x'|$

Here: $\underline{u} = u\hat{z}$ (flow velocity) (spherical symmetry) (body velocity)

$$V_n\Big|_R = V_r\Big|_R = u\hat{z}\cdot\hat{n} = u\cos\theta$$ boundary condition

$u \to \hat{z}$

Now, $$\phi(\underline{x}) = \underline{A}\cdot\underline{\nabla}\left(1/|\underline{x}|\right)$$

$$\underline{A} = A\hat{z}$$ (dipole moment in $\hat{z}$ direction)

$$\phi = -A\frac{\cos\theta}{r^2}$$

$$V_r = 2A\cos\theta/r^3$$

$V_r = u\cos\theta$ on surf

$$\Rightarrow \frac{2A\cos\theta}{R^3} = u\cos\theta$$

$$\Rightarrow A = \frac{R^3}{2}u$$

$$\boxed{\begin{aligned}\phi &= -uR^3\cos\theta/2r^2 \\ \underline{V} &= \nabla\phi\end{aligned}}$$

determines general flow field

Note:

– can recover from $\phi = \sum_{\ell} \left(a_{\ell} r^{\ell} + b_{\ell}/r^{\ell+1} \right) P_{\ell}(\cos\theta)$ expansion and b.c.'s.

regularity at ∞

– if sphere in uniform field:

$$\phi = U_0 r\cos\theta + \phi_{sphere}$$

determine from $V_n = 0$

– to determine pressure distribution on sphere, recall:

$$\rho \frac{\partial \phi}{\partial t} + \frac{\rho V^2}{2} + p = p_0$$

incompressible Bernoulli Eqn.

p_0: ambient pressure at ∞

Thus, can immediately write:

$$p(\underline{x}) = p_0 - \frac{\rho}{2} \nabla\phi \cdot \nabla\phi - \rho \frac{\partial \phi}{\partial t}$$

– $\phi(\underline{x}) \equiv$ determined a la' above via $\nabla^2\phi = 0$ and b.c.'s.

As sphere in motion (but uniform):

$$\frac{\partial \phi}{\partial t} = -\underline{u} \cdot \nabla \phi + \cancel{\frac{\partial \phi}{\partial \underline{y}} \cdot \dot{\underline{y}}} \quad \dot{u} = 0$$

so

$$\boxed{P(\underline{x}) = P_0 - \frac{\rho}{2} \nabla \phi \cdot \nabla \phi - \underline{u} \cdot \nabla \phi}$$

Generally, leads to concept of stagnation point

i.e. for Bernoulli Eqn. for incompressible fluid:

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{const.} = P_0$$

Now, consider fixed body in fluid with $\begin{cases} V_\infty = u_0 \\ P_\infty = P_0 \end{cases}$

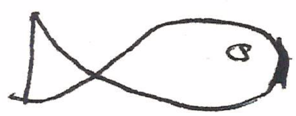
As $\underline{V} = 0$ on surface body:

$$P_{max} = P\Big|_{bdy} = P_0 + \frac{1}{2} \rho u^2$$

- stagnation point ($\underline{V} = 0$) on body is point of maximal pressure

- maximal pressure determined by $\begin{cases} P_0 \\ \text{speed} \end{cases}$

a.



→ fish skeleton strongest on front face, weakest elsewhere

⟺ front face is point of maximal pressure (head)

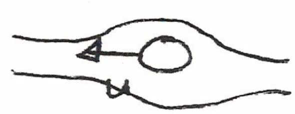
⟺ eye lens adjusts to allow for speed-induced pressure changes.

b.) Drag Force and Induced Mass

bubble

→ what force on [illegible] ball u'

→ Heuristics: Consider rigid body in water.



Slow body motion ⇒ potential flow around sphere
⇒ energy in fluid motion, too!

Thus, for F_{ext} to move body in fluid, need work against
- inertia of body (obvious)
- inertia of fluid, excited into potential flow

Thus, for body in water, need interpret Newton's 2nd Law as:

$$\underline{F}_{ext} = M_{eff} \frac{d\underline{u}}{dt}$$

$$M_{eff} = \underset{\text{mass of body}}{M} + m_{induced}$$

→ induced mass of fluid in potential flow around body (mass of fluid flow which 'dresses' the body)

[illegible] [illegible]

To calculate induced mass:

① – calculate energy in potential flow around rigid body in uniform motion in fluid

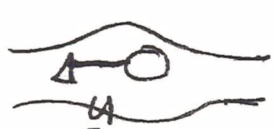
② – use $\boxed{dE = d\underline{P} \cdot \underline{u}}$ to determine momentum in fluid

as $\underline{P} = \underline{P}(\underline{u})$ ⇒ $\boxed{P_i = m_{ik} u_k}$

∴ m_{ik} is induced mass tensor!

→ Calculation: Consider rigid body moving in fluid

i.e.



Now, for flow field outside body, multipole expansion solution to $\nabla^2 \phi = 0$ yields

$$\phi = \cancel{\frac{c}{r}} + \underline{A}\cdot\underline{\nabla}\left(1/r\right) + \cdots\cdots$$

monopole (vanishes → no sources)

dipole (dominant multipole at large radius)

→ dipole moment : $A = C\,R_s^3\,\underline{u}$
($C = 1/2$, sphere)

$\therefore$

$$\phi = \underline{A}\cdot\underline{\nabla}\left(1/r\right)$$

$$= -\underline{A}\cdot\underline{r}/r^3 = -\underline{A}\cdot\hat{n}/r^2 .$$

$$\Phi = -A\frac{\cos\theta}{r^2}$$

$$\underline{V} = \underline{\nabla}\phi = \underline{A}\cdot\nabla\,\nabla(1/r)$$

$$= (\underline{A}\cdot\underline{\nabla})\left(-\underline{r}/r^3\right)$$

$$\underline{V} = \left(3(\underline{A}\cdot\hat{n})\,\hat{n} - \underline{A}\right)/r^3$$

$$u_r = \frac{2A\cos\theta}{r^3}$$

$$u\cos\theta = \frac{2A\cos\theta}{R^3}$$

$$A = \frac{u}{2}R^3$$

Now, for energy, seek calculate fluid energy in Volume V enclosed within radius R around body. Take $R^3 \gg V_0 \equiv$ volume of body.

~~Thus~~:

$$E = \frac{1}{2}\rho\int dV\,|\underline{\hat{V}}|^2$$

$$= \frac{1}{2}\rho\int d^3x\left(\underline{u}^2 + |\hat{V}|^2 - \underline{u}^2\right)$$

out $\quad |\nabla\phi|^2 - u^2 = (\underline{V}+\underline{u})\cdot(\underline{V}-\underline{u})$

$$= \nabla(\phi+\underline{u}\cdot\underline{r})\cdot(\underline{V}-\underline{u})$$

$$= \underline{\nabla}\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]$$

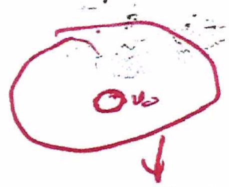
as $\quad \underline{V} = \underline{\nabla}\phi \qquad \underline{\nabla}\cdot\underline{V} = 0$

$\quad\;\; \underline{u} = \text{const.} \qquad \underline{\nabla}\cdot\underline{u} = 0$

$$\therefore \quad E = \frac{1}{2}\rho\int d^3x\left[u^2 + \nabla\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]\right]$$

$$= \frac{1}{2}\rho u^2(\underbrace{V}_{\text{volume space}} - \underbrace{V_0}_{\text{Volume object/body}}) + \frac{1}{2}\rho\int d\underline{s}\cdot\left[(\phi+\underline{u}\cdot\underline{r})(\underline{V}-\underline{u})\right]$$

$V = \frac{4\pi}{3}R^3$



$(\underline{V}-\underline{u})\cdot d\underline{s} = 0$ on R_0 surface

Now, $\quad d\underline{s} = \hat{n}\, R^2 d\Omega$, on outer surface

$$E = \frac{1}{2}\rho u^2(V-V_0)$$

$$+ \frac{1}{2}\rho\int R^2 d\Omega\left[(\hat{n}\cdot\underline{V} - \hat{n}\cdot\underline{u})(\phi+\underline{u}\cdot\underline{R})\right]$$

$$E = \frac{1}{2}\rho u^2 (V - V_0)$$
$$+ \frac{1}{2}\rho \int R^2 d\Omega \left[\left(2\frac{\underline{A}\cdot\hat{n}}{R^3} - \underline{u}\cdot\hat{n} \right) \left(\frac{-\underline{A}\cdot\hat{n}}{R^2} + R\,\underline{u}\cdot\hat{n} \right) \right]$$

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int R^2 d\Omega \left[\cancel{-2\frac{(\underline{A}\cdot\hat{n})^2}{R^5}} \right.$$ vanishes for large R

$$\left. + \frac{(\underline{u}\cdot\hat{n})(\underline{A}\cdot\hat{n})}{R^2} + \frac{2(\underline{A}\cdot\hat{n})(\underline{u}\cdot\hat{n})}{R^2} - R\,(\underline{u}\cdot\hat{n})^2 \right]$$

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int \cancel{R}^3 d\Omega \left[\frac{3(\underline{A}\cdot\hat{n})(\underline{u}\cdot\hat{n})}{\cancel{R}^2} - R^3(\underline{u}\cdot\hat{n})^2 \right]$$

Thus finally,

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int d\Omega \left[3(\underline{A}\cdot\hat{n})(\underline{u}\cdot\hat{n}) - R^3(\underline{u}\cdot\hat{n})^2 \right]$$

$$d\Omega = d\theta \sin\theta \, d\phi$$

if $\int d\Omega \, (\;) = \langle (\;) \rangle$

$$\Rightarrow \langle (\underline{A}\cdot\hat{n})(\underline{B}\cdot\hat{n}) \rangle = \frac{1}{2}\delta_{ij} A_i B_j = \frac{1}{3}\underline{A}\cdot\underline{B}$$

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \left[4\pi \underline{A}\cdot\underline{u} - \frac{4\pi}{3} R^3 u^2 \right]$$

$$= \frac{1}{2}\rho \left[4\pi \underline{A}\cdot\underline{u} - u^2 V_0 \right]$$

Thus finally, $\boxed{E = \frac{\rho}{2}\left[4\pi \underline{A}\cdot\underline{u} - u^2 V_0 \right]}$ energy in potential flow and body

Now, $\underline{A} = \underline{A}(u) \Rightarrow E = \frac{1}{2} m_{ik} u_i u_k$ defines induced mass tensor

$$dE = \underline{u}\cdot d\underline{P}$$

$$\Rightarrow \boxed{\underline{P} = \rho_{H_2O}\left[4\pi \underline{A} - V_0 \underline{u} \right]}$$ momentum in potential flow

Now, consider external force acting system, where system = body + fluid (in pot. flow)

i.e. $\underline{f}_{ext} = \dfrac{d\underline{P}_{fluid}}{dt} + M_{body}\dfrac{d\underline{u}}{dt}$

$\Rightarrow \quad f_i = \left(M\delta_{ik} + m_{ik}\right)\dfrac{du_k}{dt}$

$\therefore \rightarrow$ effective mass of "system" is sum of
- body mass
- induced mass of fluid in potential flow around body

$\rightarrow$ note induced mass is determined purely by body shape (i.e. via volume and dipole moment)

i.e. for sphere $\quad \underline{A} = \dfrac{R_0^3}{2}\underline{u}$

$$\underline{P} = \rho\left[4\pi\frac{R_0^3}{2}\underline{u} - \frac{4\pi}{3}R_0^3\underline{u}\right]$$

$$= \rho\,\frac{2}{3}\pi R_0^3\,\underline{u}$$

$$m_{induced} = \rho\,\frac{2}{3}\pi R_0^3$$

In general $M_{induced} \sim \# \rho R_0^3$

$$\sim \# \underbrace{\rho V_0}_{\text{displaced mass fluid}}$$

$\#$ → numerical factor, shape dependent

→ Example of "renormalization" in classical physics "dressing field" in continuum i.e. {renorm., polaron, debye shield, etc.}

i.e. in quantum electrodynamics → electron polarizes vacuum

$$\rightarrow m_e = m_e^{bare} + m_e^{V.P.}$$

$$(E = mc^2)$$

in classical potential flow → moving a sphere in H_2O requires that some energy go into surrounding media (the water!)

(skip)

→ Enhanced inertia due induced mass may, alternatively, be viewed as drag force on body (careful of phase!)

mom. transmitted to fluid

i.e. $$\underline{f}_{ext} = \frac{d\underline{P}_{fluid}}{dt} + M\frac{d\underline{v}}{dt}$$

drag!

$$\therefore \quad M \frac{d\underline{u}}{dt} = \underline{f}_{ext} - \frac{d\underline{P}_{fluid}}{dt}$$

$$= \underline{f}_{ext} + \underline{f}_{drag,\,lift} \qquad \underline{f}_{drag} \sim \dot{u}$$

$\underline{f}_{drag} = -\frac{d\underline{P}_{fluid}}{dt}$, along direction motion.

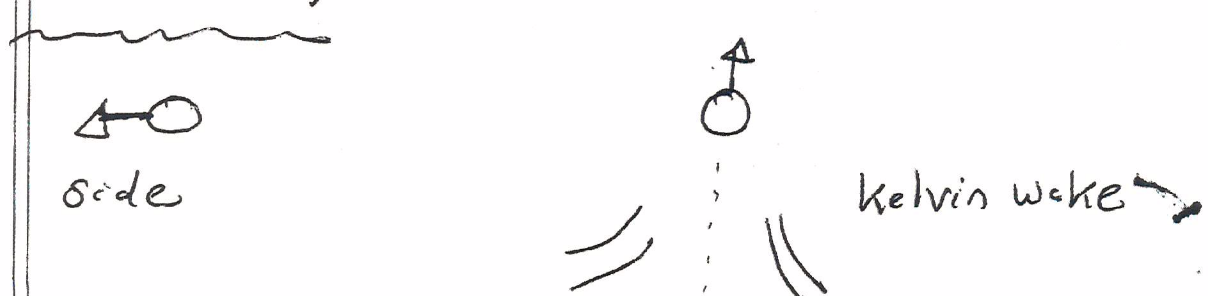
$\underline{f}_{lift} = -\frac{d\underline{P}_{fluid}}{dt}$, $\perp$ direction of motion.

Note: → if body in uniform motion in ideal (fantasy) fluid $\underline{f}_{drag} = \underline{f}_{lift} = 0$ } D'Alembert's Paradox

→ need no external force to maintain uniform motion

as – no dissipation (ideal fluid)
– no loss of energy to ∞ $(v \sim 1/R^3)$

→ but if body near surface



body will radiate surface waves to ∞ (wake) ⇒ wave drag induced energy loss!

example: Obtain:

a) - eqn. of motion for oscillating sphere in fluid

b) - sphere in oscillating fluid

a) for sphere $\underline{A} = \frac{1}{2} R^3 \underline{u}$

for oscillating sphere

$$f_{ext} = m a_{sphere} + (m \dot{\underline{v}})_{induced}$$

acceleration of dressing

$$m \dot{\underline{v}} = M_{ind} \dot{\underline{u}}$$

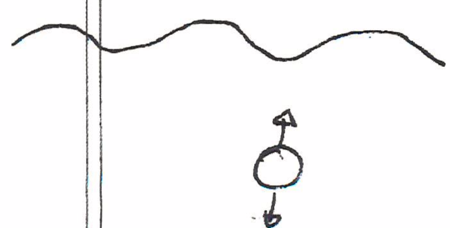
$$M_{ind} = \frac{2}{3} \pi R^3 \rho_{H_2O}$$

virtual mass

$$\therefore \quad f_{ext} = \frac{4\pi}{3} R^3 \left(\rho_{sph} + \frac{\rho_{H_2O}}{2} \right) \frac{du}{dt}$$

→ Related Problem:

- consider body in fluid, which is set in motion by external agent.



Relate $\underline{u}$ body to $\underline{v}$ fluid !?

- Now $\underline{v} \equiv$ velocity of unperturbed flow

$$\frac{\|\underline{\nabla}\,\underline{v}\|\, R_0}{\|\underline{v}\|} \ll 1 \Rightarrow \underline{v} \sim \text{const over scale of body}$$

(potential flow valid)

so if body fully carried along by fluid ($\underline{v}=\underline{u}$), then force on it would equal force on volume of displaced fluid

i.e. $$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{v}}{dt}$$

but body moves relative to fluid, so that fluid aquires momentum

→ drag duo relative motion

i.e. $$d\underline{P}_{fluid}/dt = -\underline{\underline{m}}\cdot\frac{d}{dt}[\underline{u}-\underline{v}]$$

∴ so really,

$$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}(\underline{u} - \underline{v})$$

$$\frac{d}{dt}(M u_i) = \rho V_0 \frac{dv_i}{dt} - m_{ik} \frac{d}{dt}(u_k - v_k)$$

$\Rightarrow$

$$M u_i = \rho V_0 v_i - m_{ik}(u_k - v_k)$$

∴

$$(M\delta_{ik} + m_{ik}) u_k = (\rho V_0 \delta_{ik} + m_{ik}) v_k$$

$$\boxed{u_k = \left(\frac{\rho V_0 \delta_{ik} + m_{ik}}{M\delta_{ik} + m_{ik}} \right) v_k}$$

Note: $\rho V_0 < M$ (body heavier than displaced fluid) $\rightarrow$ body lags

$\rho V_0 > M$ $\rightarrow$ body leads

$\rho V_0 = M$ $\quad u_k = v_k$.

Thus

$$M \frac{d\underline{u}}{dt} = \rho_f V \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}\left[\underline{u} - \underline{v}\right]$$

$$(M \delta_{ij} + m_{ij}) \frac{du_j}{dt} = (M_f \delta_{ij} + m_{ij}) \frac{dV_j}{dt}$$

$$\therefore \quad u_j = \left[(M_f \delta_{ij} + m_{ij}) / (M \delta_{ij} + m_{ij}) \right] V_j$$

$$M_f = \rho_f V_0$$
$$M = \rho V_0$$

$\Rightarrow \quad \underline{u} = \underline{v}$ if $\rho_f = \rho$

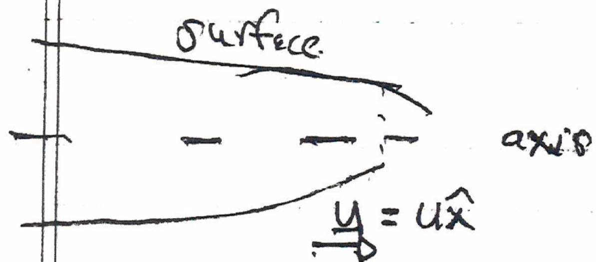
$u < V$ if $\rho_f < \rho$ $\rightarrow$ heavy object lags

$\rho_f \equiv$ fluid density
$\rho \equiv$ body density

$u > V$ if $\rho_f > \rho$ $\rightarrow$ light object leads.

c.) Potential Flow - General Slender Body

- Till now, have considered simple body potential flows, i.e. sphere, cylinder,
Here consider general body from surface of revolution



i.e. – general, axially symmetric slender body

– slender ⇔ $W/L \ll 1$

Now, observe analogy with electrostatics again,

i.e. e.s. ⇒ $\phi(x) = \int d^3x' \, \rho(x') / |\underline{x} - \underline{x}'|$

potential flow $(A \sim uV)$

$$\phi(x) = \frac{1}{4\pi}\int d^3x' \, (\dot{\rho}(x')/\rho_0)/|\underline{x} - \underline{x}'|$$

$\frac{\dot{\rho}(x')}{\rho_0}$ ≡ normalized density of fluid flowing accross cross-section of body

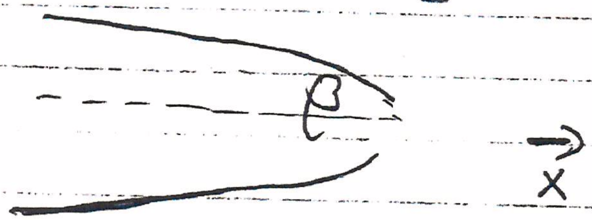
→ yields $A \sim V_0 u$ etc.

∴ $\phi(x) = \frac{1}{4\pi |x|^2} \int d^3x' \, \frac{\dot{\rho}(x')}{\rho_0} \, x'$ + h.o.t.

↓

dipole term dominates

Now, — body slender $\rightarrow \frac{W}{L} < 1 \Rightarrow \beta \ll 1$



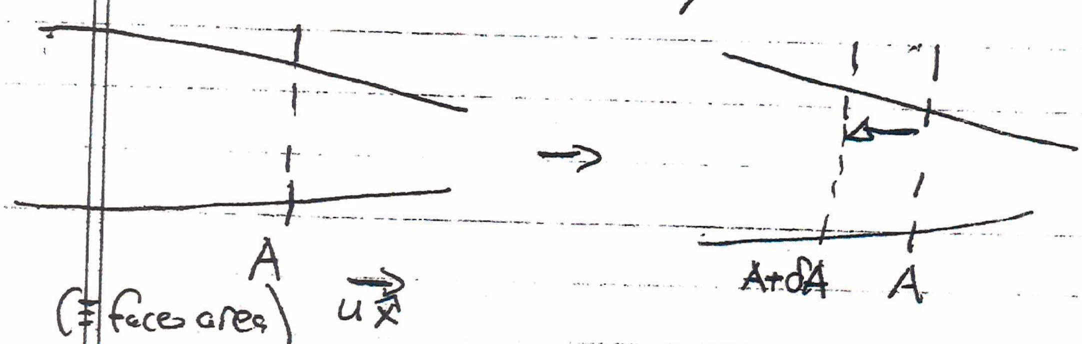
— $\nabla \cdot \underline{V} = 0$ and axial symmetry $\Rightarrow$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r V_r\right) = 0$$

$$\therefore \quad \frac{V_r}{V_x} \sim \frac{\Delta r}{\Delta x} \sim \beta \sim \frac{W}{L} \ll 1$$

$\Rightarrow$ need only consider $\hat{x}$ fluid motion

$\therefore$ to compute dipole moment, need $\rho(x')/\rho_0$ for fluid flow across body



Net $$\frac{\dot{\rho}}{\rho_0} = u\left[A + \delta A - A\right] = u\frac{\partial A}{\partial x}dx$$

$\Rightarrow \dot{\rho}(x')/\rho_0 = u \frac{\partial A}{\partial x}$

$$\therefore \quad \phi(x) = \frac{1}{4\pi |x|^2} \int dx' x' u \frac{\partial A(x')}{\partial x'}$$

$$= \frac{-u}{4\pi |x|^2} \int dx' \, A(x')$$

$$= \frac{-u}{4\pi |x|^2} V$$

$V \equiv$ volume of body $= \int dx' \, A(x')$

$\Rightarrow$ yields intuitive result:

$$\phi(x) = \underbrace{-u \, V_{body}}_{\text{effective dipole moment for slender body.}} / 4\pi r^2$$