Physics 216/116

# Lecture V – Instabilities 1 (Life after spheres)

## I.) → Convection / Rayleigh–Benard

- convection: heat → ∇T → motion
  broadly relevant
- ideal physics – Schwarzschild criterion
- dissipation and Rayleigh, Prandtl number
- Rayleigh – Benard Equations.
- Boundary conditions and $Ra_{crit.}$

## II.) → Rotating Convection

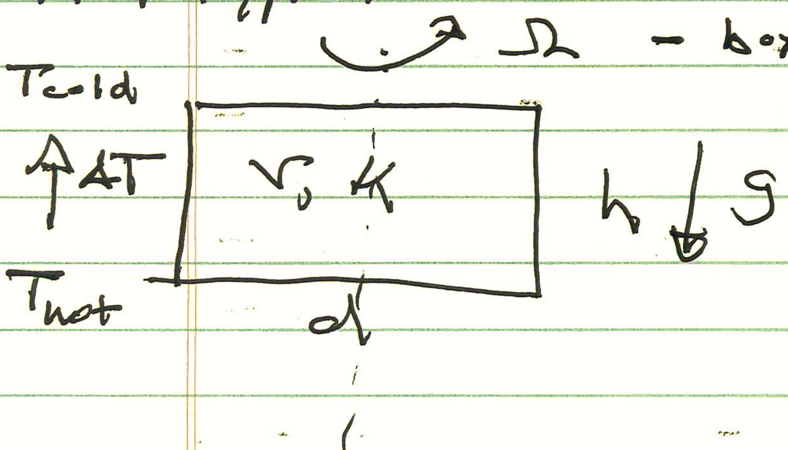
- freezing-in law with rotation
- Taylor–Proudman Theorem, implications for cells
- rotating convection
- physics of inertial waves
- relation magneto-convection, etc.

## Convection (Rayleigh–Benard)

~ thousands papers

~ central to key problems of heat transport, general circulation

Prototype:

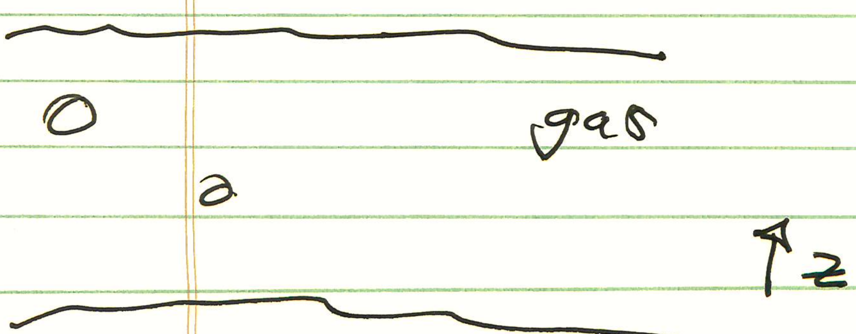



- box can rotate

~ no slip B.C.

- critical $\Delta T$ or $Ra$ for instability
- pattern structure
- effect rotation

i.) Heuristics - Ideal Fluid / Gas
i.e. stellar atmosphere
→ Schwarzchild Criterion (ideal)




$$\frac{dp}{dz} = -\rho g.$$

(eqbrm → hydrostatic)
$(g>0)$

$\partial\rho/\partial z < 0$ , $\partial P/\partial z < 0$ .

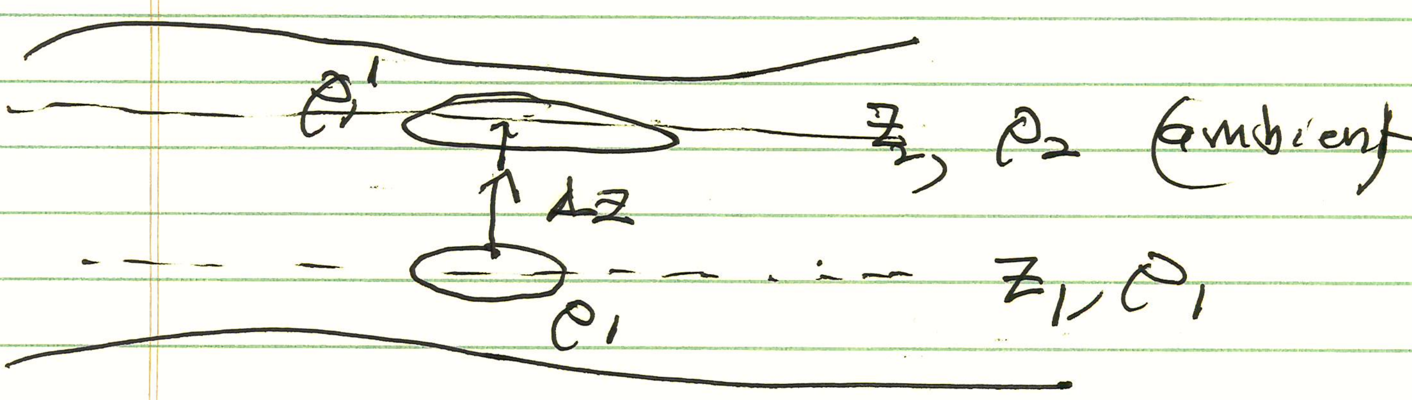
$P\rho^{-\gamma} \simeq \text{const}$ $\rightarrow$ eqn. state (ideal gas)

Basic idea:

- virtual displacement of slug/blob $\rho_1$ upward to $\rho_1'$ (after thermodynamic equilibration)

- $\rho_1' < \rho_2$ $\rightarrow$ blob buoyant, rises, unstable

  $\rho_1' > \rho_2$ $\rightarrow$ blob sinks, stable




For infinitesimal displacement:

Now, $\rho_2 = \rho_1 + \frac{\partial \rho}{\partial z} \Delta z$

What is $\rho_1'$ $\Rightarrow$ density of $\left\{ \begin{matrix} \text{perturbed} \\ \text{displaced} \end{matrix} \right\}$ blob $\rho_1$ ?

Point:

- blob $\rho_1$ equilibrates pressure with surroundings $\Rightarrow \rho_1'$

- why? $\frac{\Delta z}{c_s} \ll \tau_{rise}$

  i.e. rise time is long, slow

  $\rightarrow$ "incompressible"

then

$$P_1 \rho_1^{-\gamma} = \text{const} = P_1' \rho_1'^{-\gamma}$$

but $P_1' = P_2$ $\rightarrow$ i.e. surroundings of test blob ($\delta p = 0$)

$\hookrightarrow$ incompressible

$$P_2 = \left( P_1 + \frac{dP}{dz} \Delta z \right)$$

so

$$p_1 \rho_1^{-\gamma} = \left(p_1 + \frac{dp_1}{dz}\Delta z\right) \rho_1'^{-\gamma}$$

$$\left(\rho_1'/\rho_1\right)^{\gamma} = \left(1 + \frac{\Delta z}{p}\frac{dp_1}{dz}\right)$$

$$\rho_1' = \rho_1\left(1 + \frac{\Delta z}{p_1}\frac{dp_1}{dz}\right)^{1/\gamma}$$

$$= \left(1 + \frac{1}{\gamma}\frac{\Delta z}{p_1}\frac{dp_1}{dz}\right)\rho_1$$

$$\rho_2 = \left(1 + \frac{1}{\rho_1}\Delta z\frac{d\rho_1}{dz}\right)\rho_1$$

$$\rho_1' < \rho_2 \iff \frac{1}{\gamma}\frac{\Delta z}{p_1}\frac{dp_1}{dz} < \frac{\Delta z}{\rho_1}\frac{d\rho_1}{dz}$$

buoyand

$$\boxed{\frac{1}{\gamma}\frac{1}{p_1}\frac{dp_1}{dz} < \frac{1}{\rho_1}\frac{d\rho_1}{dz}}$$

or as both gradients negative

$$\left| \frac{1}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} \right| > \left| \frac{1}{\rho_1} \frac{d\rho_1}{dz} \right|$$

Now, $S = (\ ) \ln P\rho^{-\gamma}$

$$\frac{dS}{dz} = (\ ) \left[ \frac{1}{P} \frac{dP}{dz} - \frac{\gamma}{\rho} \frac{d\rho}{dz} \right]$$

buoyancy $\Rightarrow$ $\frac{dS}{dz} < 0$. $\Rightarrow$ free energy available.

$\frac{dS}{dz}$ $<$ / $=$ / $>$ $0$ — superadiabatically / adiabatically / subadiabatically stratified

$$\frac{dS}{dz} < 0 \Rightarrow \frac{1}{P} \frac{dP}{dz} < \frac{\gamma}{\rho} \frac{d\rho}{dz}$$

$$P = k_B \rho T$$

— $\gamma$ captures essential thermal properties

$$\boxed{\frac{1}{T} \frac{dT}{dz} < \frac{(\gamma - 1)}{\rho} \frac{d\rho}{dz}}$$

— $\gamma - 1$ specifies how steep $\nabla T$ must be relative to density.

(iv) Scales

— "Incompressibility"

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\underset{①}{\partial_t \tilde{\rho}} + \underset{②}{v_z \, d\rho / dz} + \underset{③}{\rho_0 \underline{\nabla} \cdot \underline{\tilde{v}}} = 0$$

$\frac{1}{\rho_0} \tilde{\rho} / T$ , $\frac{\tilde{v}_z}{L_\rho}$ , $k_z \tilde{v}_z$

$T \gg \underbrace{(k \, c_s)^{-1}}_{=\text{, sound}}$ → time long relative to sound transit time

(i.e. $\frac{1}{T k c_s}$) $\frac{\tilde{\rho}}{\rho_0} \sim \frac{\tilde{v}}{c_s}$

[incompressible — no density perturbation]

drop ①

$L_\rho \sim L_s > k_z^{-1}$ → drop ②

$\underline{\nabla} \cdot \underline{v} \cong 0$ for $\left\{ \begin{array}{l} \text{wavelengths} < \text{scale height} \\ |\tilde{v}| \ll c_s \\ \tilde{T} \gg \lambda / c_s \end{array} \right.$

→ simplest subsonic extension,

$\nabla \cdot (\rho \underline{v}) = 0$ → incompressible mass flow. (anelastic).

$\underline{\nabla} \cdot \underline{v} + \frac{\tilde{v}_z}{\rho} \frac{d\rho}{dz} = 0$ → decouples sound wave

→ retains finite scale height

→ modifies freezing-in law.

To identify scales, convenient to work with $s$, in full analysis (as in ideal fluid):

$$\partial_t s + \underline{v} \cdot \underline{\nabla} s = 0$$

$$s' = \underbrace{\langle s \rangle}_{\text{mean}} + \tilde{s}$$

$\tilde{s}$ ↳ fluctuation

$$\frac{\partial \tilde{s}}{\partial t} + \tilde{v} \frac{\partial \langle s \rangle}{\partial z} = 0$$

$$s' \sim \ln (P \rho^{-\gamma}) \sim \ln (T \rho^{-(\gamma-1)})$$

so

$$\frac{\tilde{S}}{S_0} \cong \left[\frac{\tilde{T}}{T_0} - (\gamma-1)\frac{\tilde{\rho}}{\rho_0}\right]$$

Now, $\underline{\nabla}\cdot\underline{v} \simeq 0$, so $\delta p = 0$.

$$\left[\frac{\partial \underline{v}}{\partial t} = -\frac{\underline{\nabla} p}{\rho} + \underline{g}\right.$$

$$\partial_t \underline{\nabla}\cdot\underline{v} = -\frac{\nabla^2 \tilde{p}}{\rho_0} + \underline{\nabla}\cdot\underline{g}$$

(scales)

$$\underline{\nabla}\cdot\underline{v} \cong 0 \Rightarrow \quad \left.\begin{aligned} &\nabla^2\tilde{p} \cong 0 \\ &k^2 \tilde{p}_k = 0 \\ &\frac{\tilde{\rho}}{\rho_0} = -\frac{\tilde{T}_k}{T_0} \end{aligned}\right]$$

$$\frac{\tilde{\rho}}{\rho_0} = -\frac{\tilde{T}}{T_0}$$

$$\boxed{\frac{\tilde{S}}{S_0} = \gamma\frac{\tilde{T}}{T_0}}$$

→ entropy perturbation tied to temperature perturbation, alone.

For estimation:

$$\frac{\partial \tilde{V}_z}{\partial t} = -\frac{\partial_z \tilde{p}}{\rho_0} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$\delta p \to 0$

$$\cong g\frac{\tilde{T}}{T_0}\hat{z}$$

$$\gamma\frac{\partial \tilde{T}/T_0}{\partial t} = -\tilde{V}_z\frac{dS_0}{dz}$$

so

$$\frac{\tilde{T}}{T_0} \sim \frac{1}{\gamma}\tau_b \tilde{V}_z \frac{dS_0}{dz}$$

so

$$\boxed{\frac{1}{\tau_b^2} \sim \frac{g}{\gamma}\frac{dS_0}{dz}}$$ → buoyancy time scale.

Now, consider dissipation:

viscosity:

$$\partial_t \to \partial_t - \nu\nabla^2, \quad 1/\tau_\nu \sim \frac{\nu}{\ell^2}$$

thermal:

$$\partial_t \to \partial_t - \kappa\nabla^2, \quad 1/\tau_\kappa \sim \kappa/\ell^2$$

→ Diffusion effects will smear out heat parcel if:

$$1/\tau_r\tau_\kappa \gtrsim 1/\tau_b^2$$

i.e. parcel needs free energy sufficient to overcome dissipation

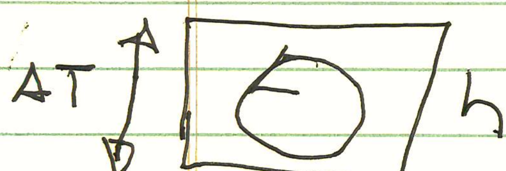
$$\frac{\tau_r\tau_\kappa}{\tau_b^2} \sim \frac{g}{\gamma s}\frac{\partial\langle s\rangle}{\partial z}\,\ell^4/\nu\kappa$$

$$\boxed{Ra = \frac{g}{\gamma s_0}\frac{\partial\langle s\rangle}{\partial z}\,\ell^4/\nu\kappa}$$

Rayleigh #

key dimensionless number in fluid mechanics.

For fluid in box:




$\rho_0 \equiv$ const

$$\delta\rho = -\alpha\,\delta T$$

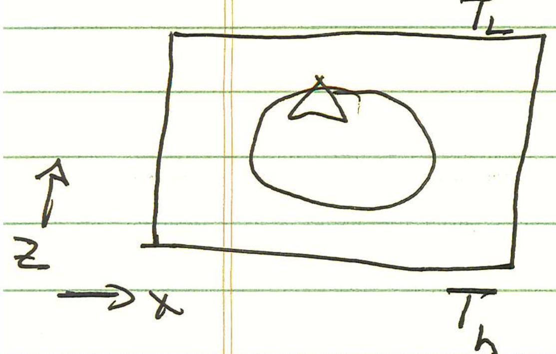
$\alpha$: coeff of thermal expansion

$$\boxed{Ra \equiv \frac{g\,\Delta T\,\alpha\,h^3}{\nu\kappa}}$$

clearly need $Ra > Ra_{crit}$ for convective instability.

- free energy
- sufficient to overcome damping

iii) Calculation




- consider vorticity $\perp$ to $x, z$

$\Rightarrow \omega_y$

- $\underline{V} = \nabla\phi \times \hat{y}$

$$\omega_y = -\partial_x^2\phi - \partial_y^2\phi$$

eqbrm: hydro
$\rho_0$ const
$\nabla T = g, +$ b.c.

Now, $\rho = \rho_0 + \tilde{\rho}$

where $\tilde{\rho} = -\rho_0 \alpha \tilde{T}$

$$= \frac{-1}{\rho}\frac{\partial\rho}{\partial T}\Big)_p \equiv \text{coeff thermal expansion}$$

$$P_0 = -\rho_0 g z \quad \text{(hydrostatics)}$$

$$= -\rho_0 g z$$

N.B. incompressibility:

$$\Delta p \sim \rho_0 g h$$

$\hookrightarrow$ ht

$$p = \rho c_s^2$$

$$\frac{\Delta p}{p} \sim \frac{\Delta \rho}{\rho_0} \sim \frac{gh}{c_s^2}$$

need $\frac{\Delta \rho}{\rho_0} \ll 1$ for validity of $\left\{ \begin{array}{l} \rho_0 = \text{const} \\ \text{hydrostatics} \end{array} \right.$

so $\frac{\Delta \rho}{\rho_0} \ll 1 \rightarrow \frac{gh}{c_s^2} \ll 1$ and $\frac{gh}{c_s^2} \ll \alpha \Delta T$

(thermally induced strat.)

Now,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\frac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{v} - g \hat{z}$$

$$p = p_0 + \tilde{p}$$

$$\rho = \rho_0 + \tilde{\rho}$$

$$\frac{\underline{\nabla} p}{\rho} = \frac{\underline{\nabla} p_0}{\rho_0} + \frac{\underline{\nabla} \tilde{p}}{\rho_0} - \frac{\underline{\nabla} p_0}{\rho_0^2} \tilde{\rho}$$

$$= -g \hat{z} + \frac{\underline{\nabla} \tilde{p}}{\rho_0} - g \hat{z} \alpha \tilde{T}$$

$$\frac{\nabla p}{\rho} = \underbrace{-g\hat{z}}_{\text{eqlbrm}} + \underbrace{\nabla\left(\frac{\tilde{p}}{\rho_0}\right)}_{\nabla\cdot\tilde{\underline{v}}=0} \underbrace{- g\alpha\tilde{T}\hat{z}}_{\text{buoyancy}}$$

so $$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\frac{\nabla p}{\rho} - g\hat{z}$$

eqlbrm

$$0 = -\frac{\nabla p_0}{\rho_0} - g\hat{z} = g\hat{z} - g\hat{z} = 0$$

pert.

$$\boxed{\frac{\partial \tilde{v}}{\partial t} = -\nabla\left(\frac{\tilde{p}}{\rho_0}\right) + g\alpha\tilde{T}\hat{z}}$$

$\hat{y}\cdot\nabla\times$ and $\underline{v} = \nabla\phi\times\hat{y}$

$$\boxed{\begin{aligned} -\frac{\partial}{\partial t}\nabla^2\phi &= g\alpha\frac{\partial}{\partial x}\left(\frac{\tilde{T}}{T_0}\right) - \nu\nabla^2\left(-\nabla^2\phi\right) \\ \partial_t\frac{\tilde{T}}{T_0} &= -\tilde{v}_z\frac{dT_0}{dz} + \kappa\nabla^2 T \end{aligned}}$$

$\tilde{v}_z = \partial_x\tilde{\phi}$

Now universal notation:

$\hat{v}_z \rightarrow w$

$\frac{\hat{T}}{T_0} \rightarrow \Theta$

$\omega_z \rightarrow \omega$

$\beta = -\frac{dT}{dz} = \frac{\Delta T}{h}$

$\delta\rho \equiv -\alpha\delta T$

see Chandrasekhar

and $\hat{z} \cdot \left( \underline{\nabla} \times \underline{\nabla} \times NSE \right) \Rightarrow$

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \nabla^2 W &= g\alpha \left( \partial_x^2 \Theta \right) + \nu \nabla^2 \nabla^2 W \\ \frac{\partial \Theta}{\partial t} &= \beta W + \kappa \nabla^2 \Theta \end{aligned}}$$

standard form.

For local theory: dispersion relation.

$$(-i\omega + \nu k^2)(-i\omega + \kappa k^2)$$

$$= g \alpha \beta \frac{k_x^2}{k^2}$$

$$k^2 = k_x^2 + k_z^2$$

N.B.: $\begin{cases} \nabla_h^2 \to \partial_x^2 + \cancel{\partial_y^2} \\ \nabla^2 = \partial_x^2 + \partial_z^2 \end{cases}$

and: $\begin{cases} T_0 = T_b - \beta z \\ \beta = \Delta T / h \end{cases}$

can de-dimensionalize:

length $\to h$
time $\to h^2/\kappa$

$\underline{V} \to h/(h^2/\kappa) = \kappa/h$

$\kappa^2/h^2 \leftarrow P/\rho_0$

$T \to \kappa\nu/\alpha g h^3$

$\Rightarrow$ de-dim

$$\boxed{\begin{aligned} \partial_t \nabla^2 W &= P \nabla^4 W + \partial_x^2 \Theta \\ \partial_t \Theta &= \nabla^2 \Theta + Ra\, W \end{aligned}}$$

2 param: $Ra = g\alpha\beta h^3/\nu\kappa$ → Rayleigh #

$P = \nu/\kappa$ → Prandtl #
relative strength dissipation

Game now becomes:

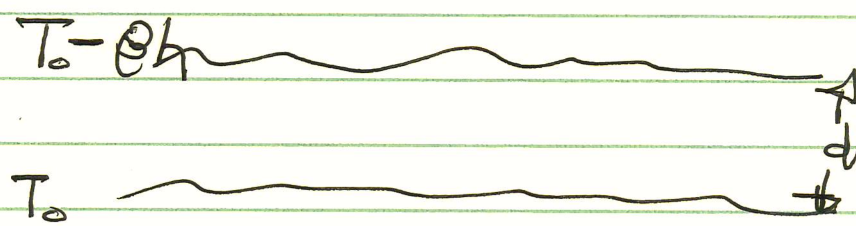
- take $Pr \sim 1$.
- scan $k_x$ o/+
- $Ra_{crit}$ for onset instability.

⇒ <u>Laborious;</u>

What are boundary conditions?

⇒ B.C. set effect of dissp'n and thus $Ra$ crit.

Assume wide tank (don't concern lateral)




$\tilde{T}$ fixed : $\tilde{\Theta} = 0$ at $z = 0, h$

walls : $\tilde{W} = \tilde{V}_z = 0$ at $z = 0, h$

but $6^{th}$ order system (4 for $w$, 2 for $\theta$)

→ need 2 more.

Can envision two scenarios for two more b.c.'s:

① no-slip

② stress free (Rayleigh 1916, 2 free boundaries)

① No-Slip (rigid)

$$\Rightarrow \tilde{V}_h \Big|_{0,h} = 0$$

horizontal/tangential velocity vanishes at 0, h

but, work with w ?!

$$\underline{\nabla} \cdot \underline{v} = 0$$

$$\nabla_h \tilde{V}_h + \partial_z \tilde{V}_z = 0$$

as all $\nabla_h$ of $\tilde{V}_h$ vanish as $\tilde{V}_h$ vanishes

$$\Leftrightarrow k \times \tilde{V}_h(\underline{z}) \Big|_{0,h} = 0 \quad \text{as} \quad V_h(z) \Big|_{0,h} = 0$$

$$\Rightarrow \nabla_h \tilde{V}_h = 0$$

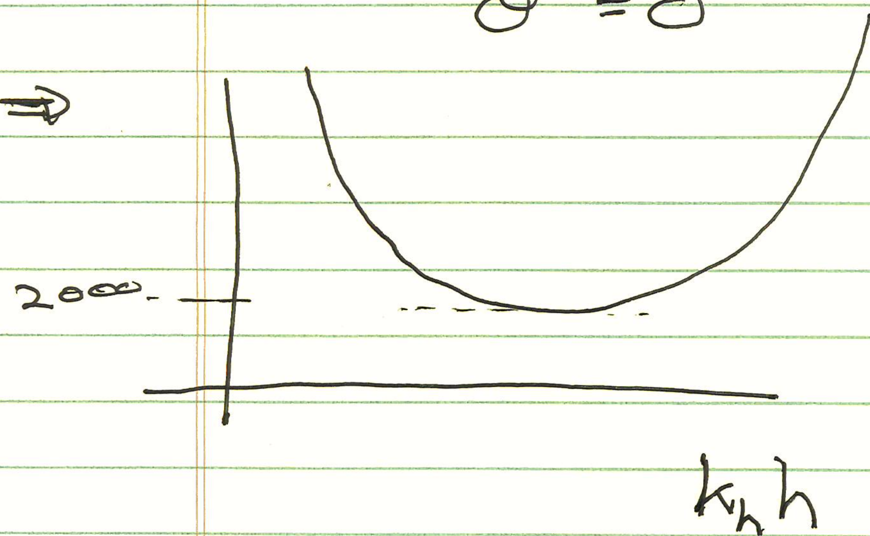
so $\boxed{\partial_z W = 0}$.

B.C.'s : $\partial_z W \big|_{0,h} = 0$

$W = 0 \big|_{0,h}$

$\Theta = 0$

$\Rightarrow$




Chandra. Fig. 39

$Ra_{crit} \sim 2000$

- Where does the shape come from?

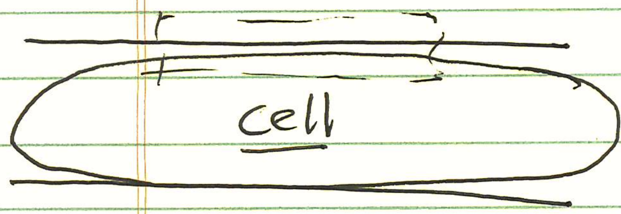
- high $k$? - $\nu k^2 = \nu(k_h^2 + k_v^2)$

$\kappa k^2 = \kappa(k_h^2 + k_v^2)$

rising $k_h$ $\rightarrow$ rising dissipation due diffusion

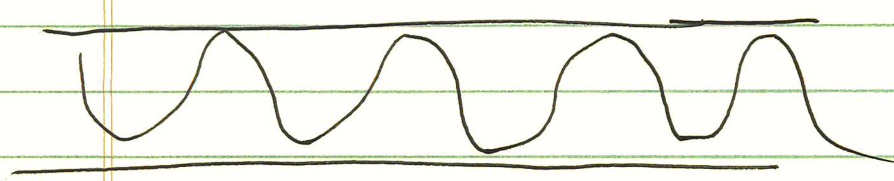
$\rightarrow$ rising $Ra_{crit}$.

– low k ? → top, bottom boundary layer due no slip.



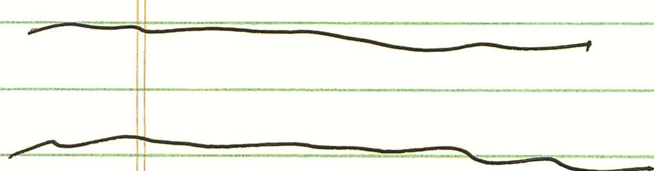

→ dissipation due viscous effects + $\tilde{V}_h = 0$ condition at 0, h.

i.e. compare:



→ greater curvature but less effect $\tilde{V}_h = 0$.

② stress free:



Free surface at top, bottom → no stress.

$$\underline{\underline{\tau}} = -\eta \, \partial_z V_h$$

↓

shear stress delivered to surface.

so need,

$$\partial_z \tilde{V}_h \Big|_{0,h} = 0 .$$

Have $\quad \partial_h V_h = -\partial_z V_z$

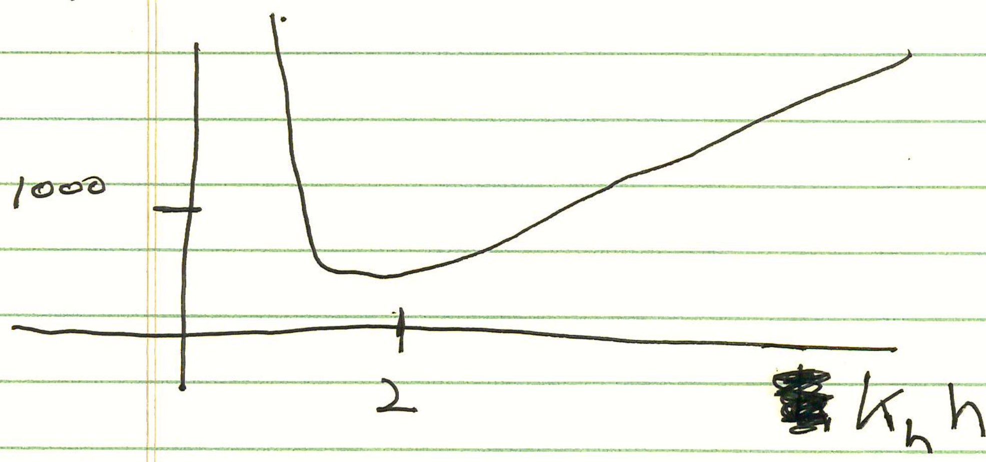
$$\partial_z \partial_h V_h = -\partial_z^2 V_z$$

$$\partial_h \partial_z V_h = 0 = -\partial_z^2 V_z$$

so

$$\boxed{\left.\partial_z^2 W\right|_{0,\,h} = 0}$$

and have:




$$Ra_{crit} \sim \frac{2\pi^4}{4}$$

for $\quad k_h h_{crit} \sim \pi/\sqrt{2}$

→ substantially lower $Ra_{crit}$ due stress free b.c. → no longer fighting no-slip condition.

- high k → increased dissipation

  low k → effects layers at boundary.