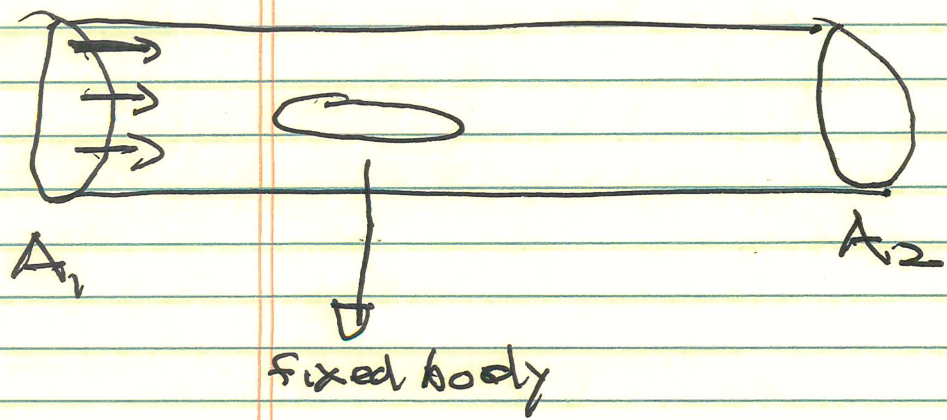
Lecture VII Wakes: Stability and Structure

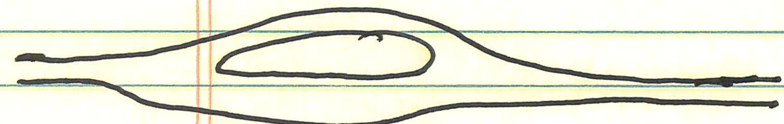
→ Recall D'Alembert's Paradox



Drag Force = Difference in Momentum Flux thru ends

$$F_d = \int_{A_1} da_1 \, (P_1 + \rho V_1^2) - \int_{A_2} da_2 (P_2 + \rho V_2^2)$$

– For ideal fluid, upstream/downstream symmetry of flow



i.e. $P_1 = P_2$

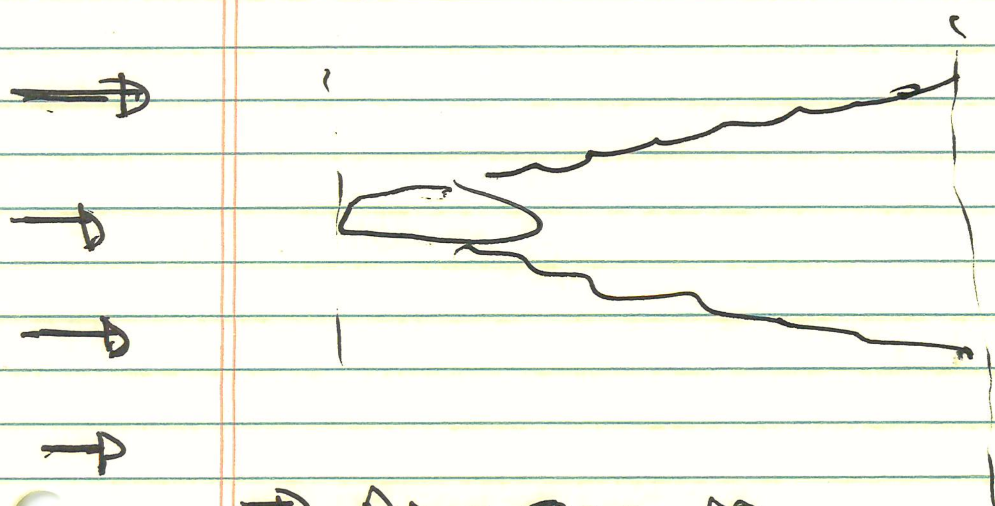
$V_1 = V_2$

∴ $F_d = 0$

i.e. effect of flow is enhanced inertia (induced mass).

- for viscous fluid (i.e. no slip boundary condition) → upstream/downstream asymmetry

→ wake

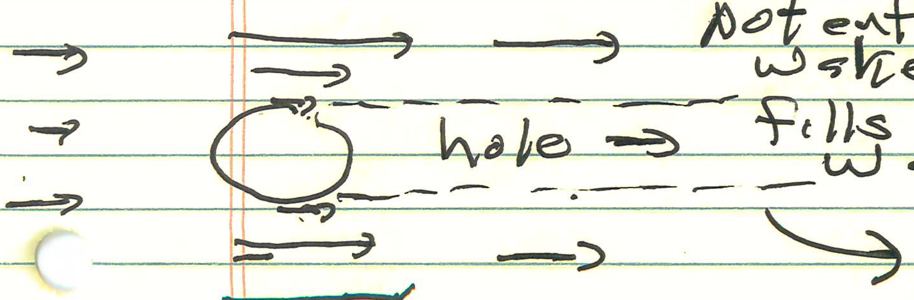


⇒ Drag occurs

Aim is to understand structure and dynamics of wake.

→ Recall: No-slip B.C.'s

⇒ Separation occurs



potential flow outside wake

fills in to form wake

line of separation

→ wake is rotational / vortical for $Re > Re_{crit}$

— how does hole fill in?

(1)

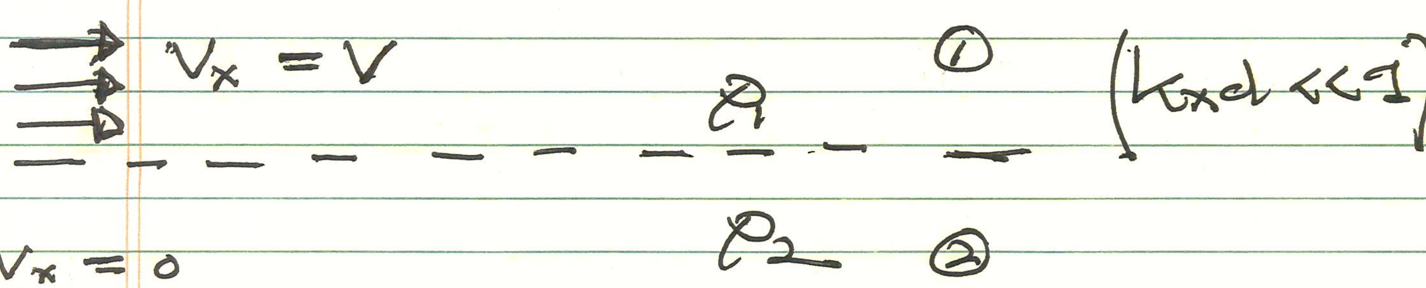
⇒ Separation is unstable

⇒ Kelvin-Helmholtz Instability

KH

→ free energy → ∇V

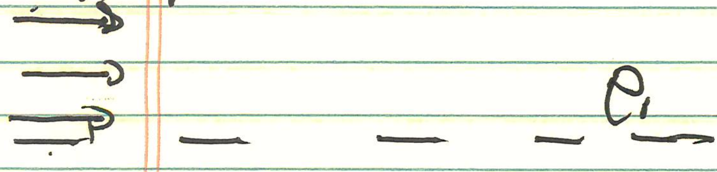
- simplification: interface



- $\nabla V = 0$, except interface
- $\omega = \partial V_x / \partial z = 0$, except interface
- can treat as potential flow in regions (1), (2) and match at interface
- interface ripples → dynamic b.c.

Physical ideas:

① context


ρ_1

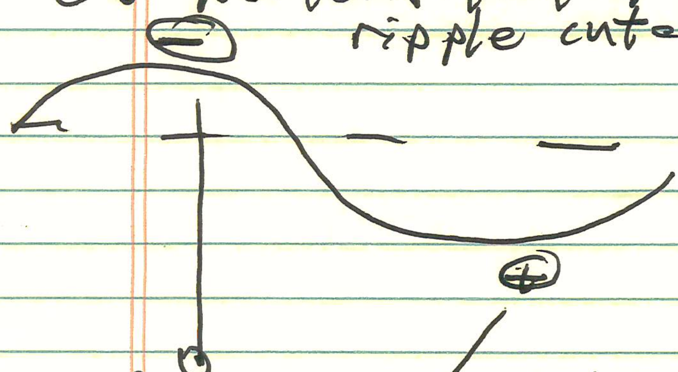
ρ_2

① high v, low p

② zero v, high p

$$p + \frac{\rho v^2}{2} = \text{const.}$$

② δv perturbation ⇒ ripple interface

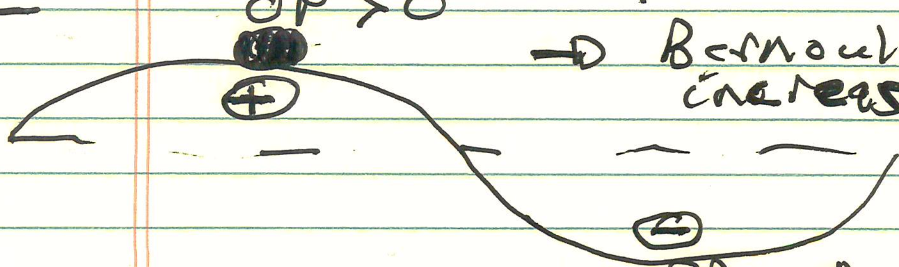


$\delta v_1 < 0$ ⇒ - ripple brings in slow fluid
- flow in region ① drops

$\delta v_2 > 0$ - ripple brings in fast fluid
- flow in ② increases

but

$\delta p > 0$ ⇒ Bernoulli ⇒ pressure increases



δp ⊖ → pressure decreases

so

$$\left.\begin{array}{lll} \delta p > 0 & \Rightarrow & \delta V < 0 \text{ further} \\ \delta p < 0 & \Rightarrow & \delta V > 0 \text{ further} \end{array}\right\} \Rightarrow$$

reinforcing initial perturbation!

N.B. K-H instability drives viscous mixing via turbulence, mixing, billows etc.

To calculate:

$p + \rho_1 \frac{V^2}{2} = \text{const}$

assures equilibrium

so

$\underline{\nabla} \cdot \underline{V} = 0$

$\underline{V} = \underline{\nabla}\phi$ $\qquad\qquad$ $\underline{\omega} = 0$, except interface

$\nabla^2\phi = 0$ $\qquad\qquad$ $(\partial_z^2 - k^2)\phi = 0$

wave along interface (symmetry)

$$\Rightarrow \quad \phi = \sum_k \phi_k \, e^{ikx} \, e^{-k|z|} \, e^{-i\omega_k t}$$

decays away from interface

matching conditions

→ pressure continuity

$$\tilde{P}_1(0_+) = \tilde{P}_2(0_-)$$

⇒ ϕ continuity

$$\text{i.e.} \quad -P = \rho\,\partial_t \phi + \rho\frac{(\nabla\phi)^2}{2}$$

$$\Rightarrow \quad \left.\frac{\partial\phi}{\partial z}\right|_{①} = \left.\frac{\partial\phi}{\partial z}\right|_{②}$$

$$\text{i.e.} \quad \partial_z^2\phi - k^2\phi = 0$$

$$\text{and} \int_{0_-}^{0_+} \partial_z^2\phi = 0 \Rightarrow \partial_z\phi\big|_+ - \partial_z\phi_- = 0$$

Now,

$$\partial_t \underline{v} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{\underline{\nabla} P}{\rho}$$

$$\rho_1\left(\partial_t \tilde{V}_{z_1} + V\partial_x \tilde{V}_{z_1}\right) = -\partial_z \tilde{P}_1$$

$$\rho_2\left(\partial_t \tilde{V}_{z_2}\right) = -\partial_z \tilde{P}_2$$

so $$\tilde{v}_{z_1} = \frac{-ik\tilde{P}_1}{\rho_1(kV-\omega)}$$

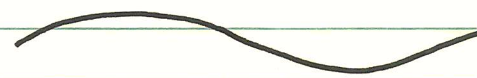
$$\tilde{v}_{z_2} = \frac{ik\tilde{P}_2}{-\rho_2\omega}$$

$$\partial_z P \begin{cases} -\ 1 \\ +\ 2 \end{cases}$$

(not related v_z)

Now, dynamic boundary:

$\eta = \eta(x,t)$ → displacement



$$\frac{d\tilde{\eta}}{dt} = \tilde{v}_{z_1}$$

and

$$\partial_t \tilde{\eta} + V\partial_x \tilde{\eta} = \frac{d\tilde{\eta}}{dt}$$

$$-i(\omega - kV)\tilde{\eta}_k = \tilde{v}_{z_1,k}$$

⇒

$$-i(\omega - kV)\eta_k = \frac{-ik\tilde{P}_1}{\rho_1(kV-\omega)}$$

$$\tilde{P}_{1_k} = -\rho_1 \frac{(kV-\omega)^2}{k}\eta_k$$

$$\tilde{P}_2 = \rho_2 \frac{\omega^2}{k} \tilde{\eta}_k$$

and $\tilde{P}_1 = \tilde{P}_2 \Rightarrow$

$$\boxed{-\frac{\rho_1}{k}(kv-\omega)^2 = \frac{\rho_2 \omega^2}{k}}$$

and finally,

$$\omega = kv\left(\frac{\rho_1 + i(\rho_1\rho_2)^{1/2}}{\rho_1+\rho_2}\right)$$

$$\Rightarrow \gamma \sim kv\frac{(\rho_1\rho_2)}{\rho_1+\rho_2}$$

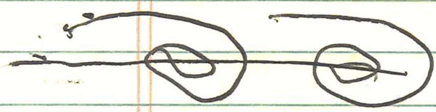
$$\Rightarrow \omega_{real} \sim kv\left[\rho_1 / (\rho_1+\rho_2)\right]$$

$\Rightarrow$ no exchange of stabilities here.

$\Rightarrow \rho_1 = \rho_2$, $\gamma \sim k\Delta V$

→ what happens ?

→ vortex roll-up, billows



(see Falkovich F2.3, F2.4)

→ vortex streets etc.

→ N.B. Vorticity concentrated on layer of interface.

→ More generally:

(g>0)

$$\gamma^2 = \underbrace{\frac{\rho_1\rho_2 k^2(V_2-V_1)^2}{(\rho_1+\rho_2)^2}}_{KH} + \underbrace{\frac{(\rho_1-\rho_2)}{\rho_1+\rho_2}(gk)}_{\text{Rayleigh-Taylor}} - \underbrace{\frac{T k^3}{(\rho_1+\rho_2)}}_{\text{surface tension}}$$

surface tension → (capillarity)

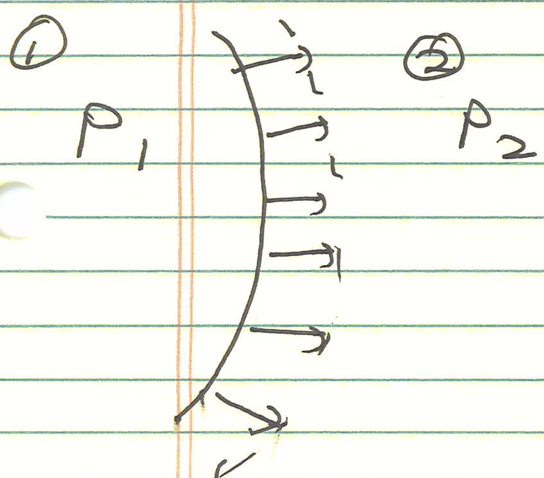
~ Threshold wind to excite waves ?

→ Surface Tension

- An important property of interfaces is surface tension

  i.e. – force due increase in surface area of interface

  – familiar from droplets, capillary waves, etc.



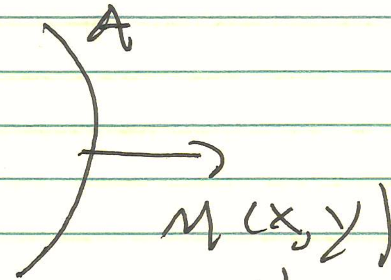
→ isothermal displacement, as ① expands

① expands into ②

$$\underset{\text{change in free energy}}{dF} = -P_1\,dV - P_2\underset{\text{② contracts}}{(-dV)} + \sigma\,dA$$

$$dV = dA\, d\eta$$

$d\eta \leftarrow$ displacement



so change in area:

$$dA = \int dx\,dy \left(1 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right)^{1/2}$$

$$- \int dx\,dy$$

i.e. – displacement expands area
– x, y parametrize surface

so, for small slope:

$$dA = \int dx \int dy \, \frac{1}{2}\left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]$$

IBP

$$= \int dx \int dy \, (-\nabla^2 \eta)\, d\eta$$

so

$$dF =$$

$$= \int dx \int dy \, d\eta \left[(\rho_2 - \rho_1) - \sigma \nabla^2 \eta\right]$$

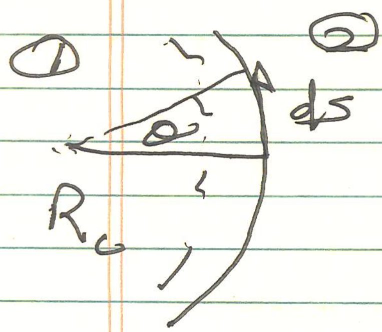
so, criterion for equilibrium:

$$\boxed{p_2 - p_1 = \sigma \nabla^2 \eta}$$

$$dF = (p_2 - p_1)\, dA\, d\eta + \sigma\, dA$$

More generally:

Now consider (i.e. not "weakly curved" interface)



$$ds = (R_c + d\eta)\, d\theta$$
$$= dl_0 \left(1 + \frac{d\eta}{R_1}\right)$$

radius curvature of interface, as shown

In general, surface parametrized by 2 radii curvature, R_1, R_2 (Gauss):

so

$$dA = \int dl_1\, dl_2 \left(1 + \frac{d\eta}{R_1}\right)\left(1 + \frac{d\eta}{R_2}\right) - \int dl_1\, dl_2$$
$$= \int dl_1\, dl_2 \left(1/R_1 + 1/R_2\right) d\eta$$

$dl_1 dl_2$

So

$$dF = \int dA_0 \left[(p_2 - p_1) + \sigma \left(1/R_1 + 1/R_2 \right) \right] d\eta$$

So, for equilibrium with interface (general)

$$\boxed{\sigma \left(1/R_1 + 1/R_2 \right) = p_1 - p_2}$$

Laplace's Law

N.B.:

- Given 2-phase equilibrium (separate domains), can use Laplace Law to estimate droplet size for immiscible liquids

i.e. $p_1 > p_2$, $\quad R \simeq \sigma / (p_1 - p_2)$

- For SW, R-T . . . . $\frac{\rho_L}{\rho_H}$ of g

$p \to p - \eta \gamma_T \nabla_h^2$

$$\omega^2 = kg \left(\frac{\rho_H - \rho_L}{\rho_H + \rho_L} \right) + \frac{\gamma_T k^3}{(\rho_H + \rho_L)}$$

For $\rho_H >> \rho_L$

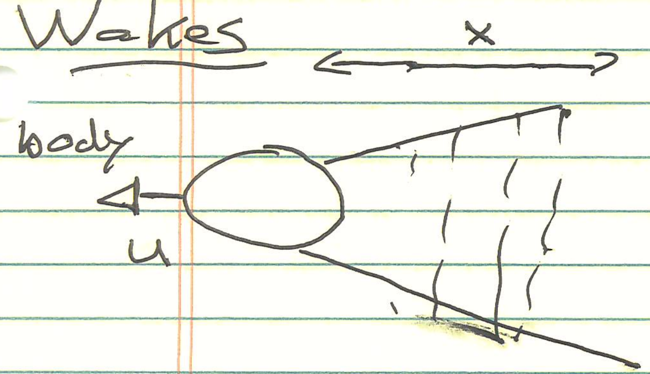
$$\omega^2 = \underbrace{gk}_{\substack{\text{gravity} \\ \text{wave}}} + \underbrace{\frac{\gamma_T}{\rho} k k^2}_{\text{cap.}} \Rightarrow \text{Grav-capillary wave}$$

- cross-over at few cm.
  Capillary effects important at $\lesssim 5$ cm.

→ Wake Structure

- Physics ideas → wake flow created by response to separation
- Link: Drag - asymmetry - wake flow
- Width: Laminar, Turbulent scalings
- [illegible] deficit and Sunshine re: N-S vs Euler

Wakes



region of wake limited in angular extent

- region behind body of departure from potential flow. Wake rotational.
- [illegible] wake breaks upstream-downstream symmetry of ideal flow

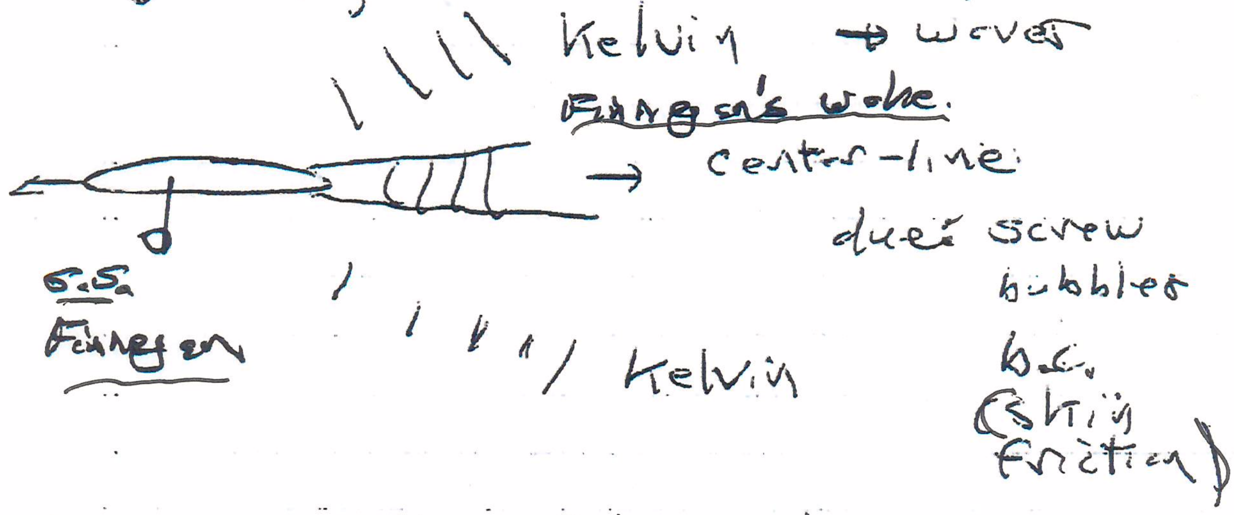
so

- wake is consequence of a body experiencing drag.

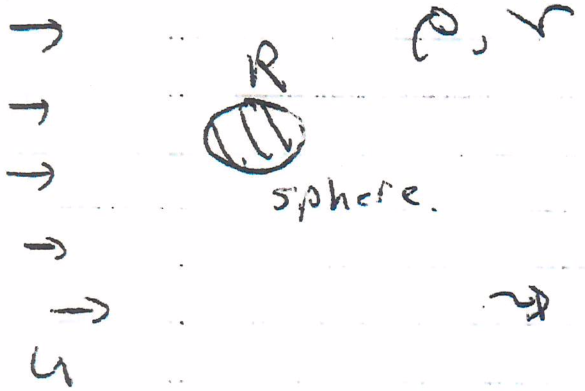
- Message: A little viscosity forces a global adjustment in flow structure

Note:

* – in general, wake multi-component



– here consider spherical cow of wake problems.



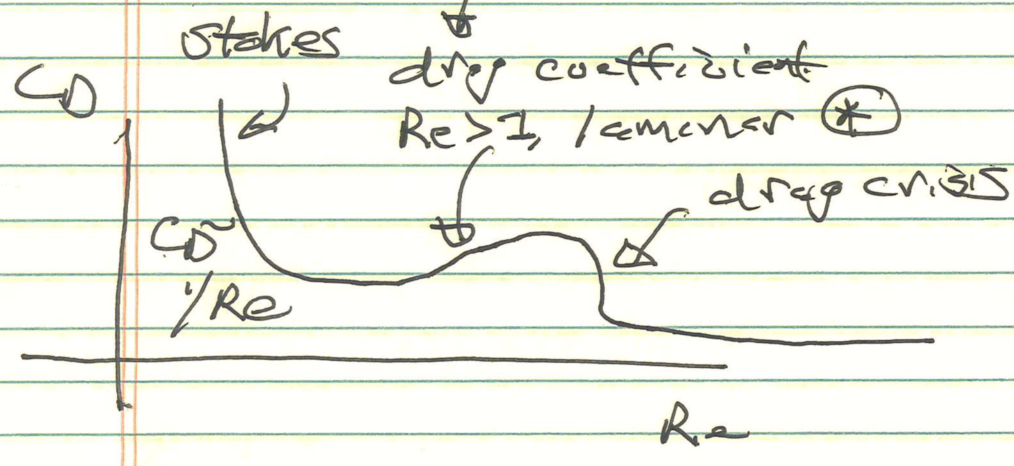
so $F_d \sim \rho u^2 R^2 f(Re)$

→ no surface effects.

→ How calculate wake structure?

Force of Drag $\equiv$ { Rate of Net Momentum Loss from Flow!

i.e. $F_d \sim C_D \rho A U^2$

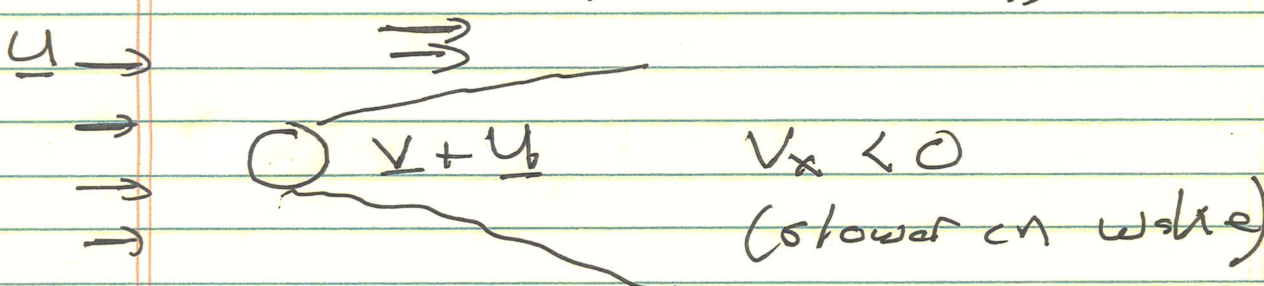


i.e. in laminar wake, flow not turbulent but inertia relevant

Furthermore:

- distances behind body $x \gg R$, → wake.

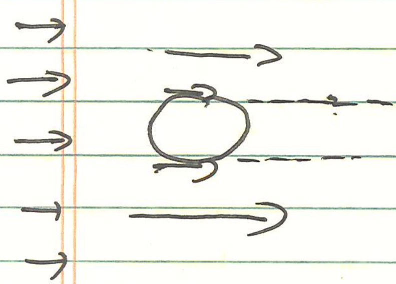
- if body speed U, then in frame where body stationary,



$\underline{V}$ differs zero in limited region

→ How limited? → as laminar, & signal propagation is diffusion, only.

→ How does wake form?



- no slip boundary condition 'slows down' fluid flowing past body

∴

- separation, discontinuity

but → ~~lift~~ results, mixing fluid into hole

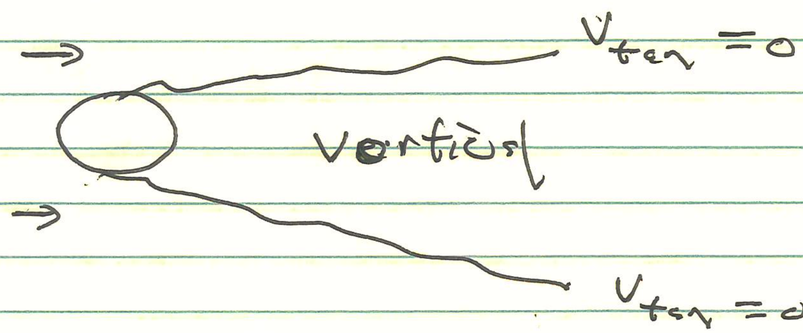
→ viscosity smooths out discontinuity

n.b. if turbulent wake turbulent mixing smooths discontinuity faster than viscous mixing.

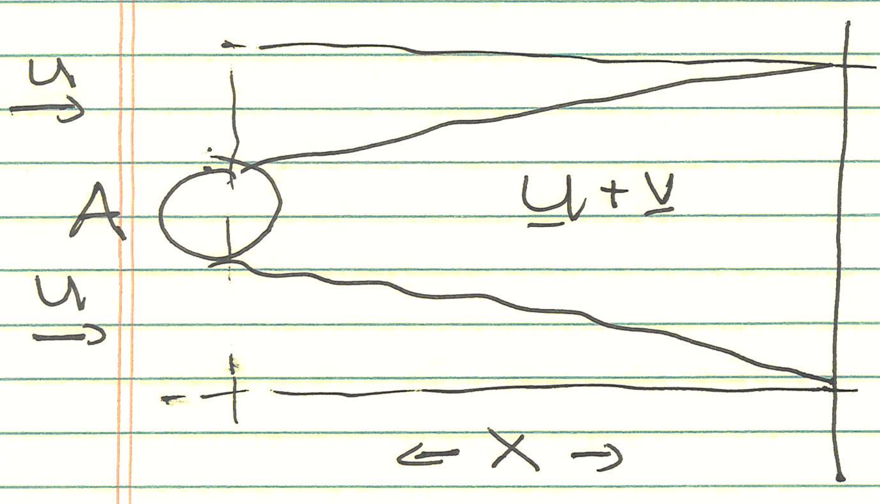
→ boundary of wake is traced by fluid particles

- passing close to body

- scattered by diffusion and turbulent mixing ⇒ expansion i.e



Now, to calculate rate of loss of momentum from flow, return to D'Alembert construction, but with asymmetry



Drag = difference influx, outflux

$$\underline{F}_d = \int_{A_0} d\underline{a} \cdot \underline{\underline{\Pi}}_{tot.}(0) - \int_{A_W} d\underline{a} \cdot \underline{\underline{\Pi}}_{Total}(x)$$

$$\underline{\underline{\Pi}} = p + \rho(\underline{u}+\underline{V})(\underline{u}+\underline{V})$$

$$\equiv p_{Tot.}$$

↳ momentum flux

i.e. $O(V)$

$$\Pi^{(0)}_{x,x} = P_0 + \rho U^2 \qquad \oint [p' \delta_{in} + \rho u_n V_i] da_n$$

$$\Pi_{xx}(x) = P_0 + p' + \rho U^2 + \rho U V_x \quad + h.o.$$

$$F_d \cong -\int_{A_w} da\, \rho U v_x$$

Now, can take conical symmetry, so

$$F_d \cong -\pi\, w(x)^2 \rho U v_x$$

What is width?

$v_x < 0$
$F_d > 0$
$\to x$

Now, need $w(x)$ to get v_x!

→ Observe

- problem now reduced to one of scale.
- wake self-similar.

  $w \sim x^\alpha$, α?

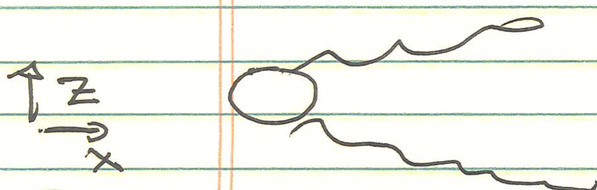
- wakes can be laminar or turbulent

(i) Laminar, $\frac{UR}{\nu} > 1$

but not $\gg 1$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} - \nu\nabla^2\underline{V} = -\frac{\nabla P}{\rho}$$

$$\underline{U}\cdot\underline{\nabla} V + \cancel{\underline{V}\cdot\underline{\nabla}\underline{V}} - \nu\nabla^2\underline{V} = -\frac{\nabla P}{\rho}$$



$\underline{\nabla}\cdot\underline{V} = 0$

narrow wake

$$U\,\partial_x V_z - \nu(\cancel{\partial_x^2} + \partial_z^2)\,V_z = -\frac{\partial_z P}{\rho}$$

$$U\,\partial_x V_x - \nu\,\partial_z^2 V_x = -\frac{\partial_x P}{\rho}$$

$\partial_x \sim 1/x$ downstream distance

$\partial_z \sim 1/w$ $\perp$ scale.

Obvious: $V \simeq \frac{1}{\left(\frac{\nu x}{U}\right)^{1/2}} \exp\left[\frac{-y^2}{\nu x/U}\right]$

$$\left(\frac{U}{X} - \frac{\nu}{W^2}\right) V_z \sim -\frac{p}{W\rho}$$

$$\left(\frac{U}{X} - \frac{\nu}{W^2}\right) V_x \sim -\frac{p}{X\rho}$$

$$\underline{\nabla} \cdot \underline{V} = 0 \quad \Rightarrow \quad \frac{V_x}{X} \sim \frac{V_z}{W}$$

as p negligible (will show) $\Rightarrow$

$$\frac{U}{X} \sim \frac{\nu}{W^2}$$

$$W \sim \left(\frac{\nu X}{U}\right)^{1/2}$$

→ diffusive spreading of momentum by ν

$\Rightarrow \sim (\nu t)^{1/2}$ with $t \sim x/U$

<u>So</u>

$$\boxed{\begin{gathered} W \sim (X/R)^{1/2} \left(\frac{\nu R}{U}\right)^{1/2} \\ W/R \sim \left(\frac{X}{R}\right)^{1/2} 1/Re^{1/2} \end{gathered}}$$

– akin Blasius ✓

check: $\tilde{z} \rightarrow$

is $\frac{P}{\rho w} \sim \frac{\nu V_z}{w^2}$

$\frac{V_x}{x} \sim \frac{\cancel{V_z}}{w}$

$P \sim \rho \nu \frac{V_x}{x}$

$\frac{P}{\rho x} \sim \frac{\nu V_x}{x^2}$

but eqn. $\underbrace{\left(\frac{u}{x} - \frac{\nu}{w^2}\right)}_{O(1/w^2)} V_x \sim \underbrace{\frac{-P}{\rho x}}_{O(1/x^2)}$

so drop P ✓

P likely important in V_z eqn.

<u>N. B.</u> $\frac{P}{\rho w} \sim \frac{\nu}{w^2} V_z$

→ Some Observations re: Wakes

– $$F_x = -\rho U \int_{wake} v_x \, dy\, dz$$

⟹

$$Q = \rho \int_{wake} v_x \, dy\, dz$$ → wake deficit

↓

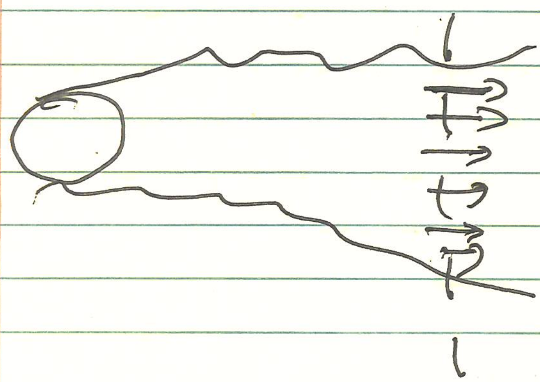
difference with, without body

↓

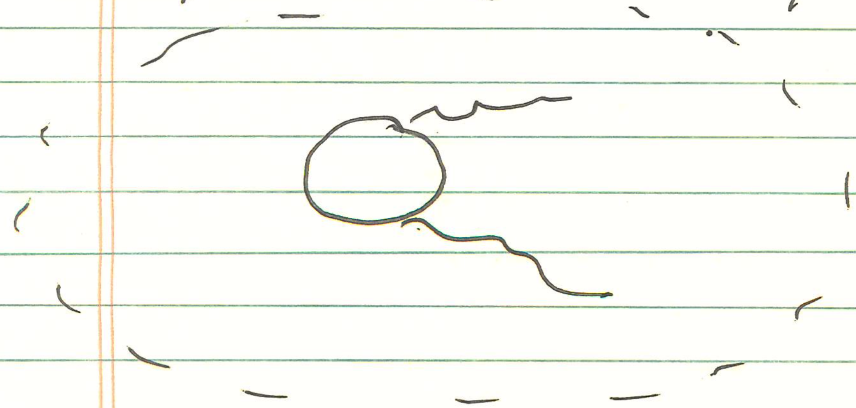
net mass flow thru wake area

Q is x independent:

i.e. $Q \sim F_x / U$ → const.



but, if encircle body:



have:

$$\oint_{tot} \underline{v} \cdot d\underline{a} = 0$$

i.e. no water lost

but

$$0 = \int_{wave} \underline{v}\, dy\, d\cancel{z} + \int_{pot\ flow} \underline{v} \cdot da$$

$$Q \sim V_x A$$

Q finite $\Rightarrow$

$$V_x \sim \frac{1}{A} \sim 1/r^2$$

so

$$\int_{pot\ flow} \underline{v} \cdot da = -Q$$

$$V_x A \sim -Q$$

$$V_x \sim 1/r^2 \rightarrow$$ potential flow

monopole.

$\underline{\underline{80}}$

— Euler equation

$$v \sim 1/r^3 \rightarrow \text{dipole}$$

but for N-S eqn.

$$v \sim 1/r^2 \rightarrow \text{monopole!}$$

$\Rightarrow$

- <u>global</u> adjustment in potential flow outside wake induced by viscosity and the wake.

- Message: A little ν forces a <u>global</u> adjustment in flow structure!

[redacted]

[redacted]

by analogy with K.T. gases

$$\underline{V}\cdot\nabla\underline{V} \to -\nu_T \nabla^2 \underline{V}$$

$$\nu_T \sim \tilde{w}\, l_{mix}$$

ii(a) Turbulent Wake $\qquad Re \sim UR/\nu \gg 1$

$$\underline{U}\cdot\nabla\underline{V} + \underline{V}\cdot\nabla\underline{V} - \cancel{\nu\nabla^2\underline{V}} = -\frac{\nabla p}{\rho}$$

ignore

$$\Rightarrow \quad \frac{U}{x} V_x \sim \frac{\tilde{V}_y}{W} V_x$$

[wake spreads by advection, not diffusion]

$$\boxed{W \sim \frac{\tilde{V}_y}{U} x}$$

$\tilde{V}_y \sim$ turbulent velocity

Take wake turbulence isotropic,

so $\tilde{V}_x \sim \tilde{V}_y$.

[Fair? Test]

$$W \sim x \tilde{V}_x / U$$

but from drag:

$$\tilde{V}_x \sim F_d / \rho U W^2$$

$\Rightarrow$

[redacted]

[redacted]

$$W \sim x \frac{F_d}{\rho U^2 W^2} \sim x \left(F_d / \rho U^2 W^2 \right)$$

$$W^3 \sim F_d\, x / \rho U^2$$

$$\Rightarrow \boxed{W \sim \left(F_d / \rho U^2 \right)^{1/3} x^{1/3} \sim \left(C_D R^2 \right)^{1/3} x^{1/3}}$$

then, comparing widths:

laminar: $w/R \sim (x/R)^{1/2}\, Re^{-1/2}$

$Re \sim UR/\nu$

turbulent: $w/R \sim (x/R)^{1/3}\, C_D^{1/3}$

interestingly, Laminar wake expands with downstream length more rapidly!

[redacted]

Why?

→ turbulence can relax ∇V behind object (due separation) rapidly, and faster than ν. Thus surrounding flow penetrates the dead water region more rapidly, less wake expansion.

Also observe: Wake Re drops with x.

⇒

$$Re \sim \frac{W V_y}{\nu} \sim \frac{\cancel{W} V_x}{\nu} \sim \frac{\cancel{D}}{\nu} \quad \frac{F_d}{\rho U W^2}$$

↑ y direction (spr) ↑ wake flow Re

$$Re \sim F_d / \rho U W \nu$$

$$\sim \frac{U^{\cancel{2}} R^2 \cancel{\rho_0} C_D}{\nu \cancel{\rho_0} \cancel{U} (C_D R^2)^{1/3} x^{1/3}}$$

$C_D \sim 1$

$$\sim \left(\frac{UR}{\nu}\right) \left(R/x\right)^{1/3}$$

[redacted] [redacted]

$$\boxed{Re(x) \sim Re_0 \, (R/x)^{1/3}}$$

and $Re(x) \to 1$ at

$$x_L \sim R \, (Re_0)^3$$

distance behind boat where turbulent wake transitions to laminar

i.e. akin l_d: transition from turbulent mixing to viscous mixing

N.B. [In wake, vortical/rotational region can expand into irrotational region, but never reverse !

i.e. would really violate H-Thm ...

Later discussion

Wakes – Supplement ... Sketch

→ Revisit turbulent wake, using turbulent viscosity, i.e.

$$w \sim (\nu x/u)^{1/2} \qquad (\nu \to D_T)$$

$$\to \left(D_T x/u\right)^{1/2}$$

i.e. width of turbulent wake set by turbulent diffn, following Blasius Law

but $D_T \sim w\tilde{v}$ ⇒ turbulent viscosity at mixing length level.

$$\sim w\left(F_d/\rho u w^2\right)$$

$$\sim F_d/\rho u w \sim \text{const}/w$$

⇒

$$w \sim \left(F_d x/\rho u^2 w\right)^{1/2}$$

$$w^{3/2} \sim \left(F_d/\rho u^2\right)^{1/2} x^{1/2} \sim (C_D R^2)^{1/2} x^{1/2}$$

$$\sim C_D^{1/2} R\, x^{1/2}$$

$$w \sim (C_D)^{1/3} R^{2/3} x^{1/3}$$

$$\Rightarrow \boxed{W/R \sim C_D^{1/3}\,(X/R)^{1/3}}$$ agrees ✓.

Now, $D_T \sim \tilde{\nu} W$

$$\sim \frac{(\tilde{\nu} W^2)}{W}$$

$$\sim \frac{\rho U \tilde{\nu} W^2}{\rho U W}$$

$$\sim \frac{Q}{W} \sim Q/R\,(X/R)^{1/3}$$

∴ – Point is that turbulent viscosity mixing drops downstream, relative to constant viscous mixing.

– follows from $\tilde{\nu} W \sim \overset{\rightarrow\,\text{const.}}{Q} / W$

– explains why turbulent wake spreads more slowly than laminar wake.