Fluids in Flatland - A Short Introduction

~ apologies to Edwin Abbott

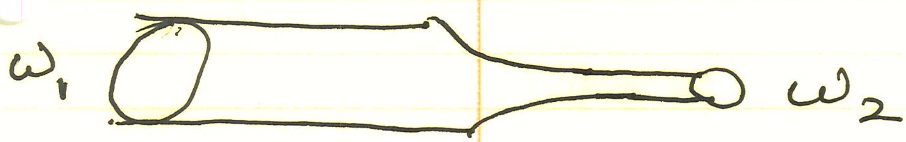
→ A Quick Look: (i.e. Why Notable)
(Sneak preview)

- what physical process underpins K41 cascade, etc. ?
  → vortex tube stretching

i.e. $\partial_t \underline{\omega} = \underline{\nabla} \times (\underline{v} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

$$\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v}$$

$\nabla \underline{v} \longrightarrow$



- 2D $\underline{\omega} \cdot \underline{\nabla}\, \underline{v} = 0$ $(\underline{\nabla} \cdot \underline{v} = 0)$

so $\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} = \nu \nabla^2 \underline{\omega} + \underset{\sim}{f_{ext}}$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi + \tilde{f}_{ext}$$

$\therefore$ two invsicid quadratic invariants:

$\int \frac{v^2}{2} \longrightarrow$ energy

$\int \frac{\omega^2}{2} \longrightarrow$ enstrophy (~ mean squared vorticity)

Refs:

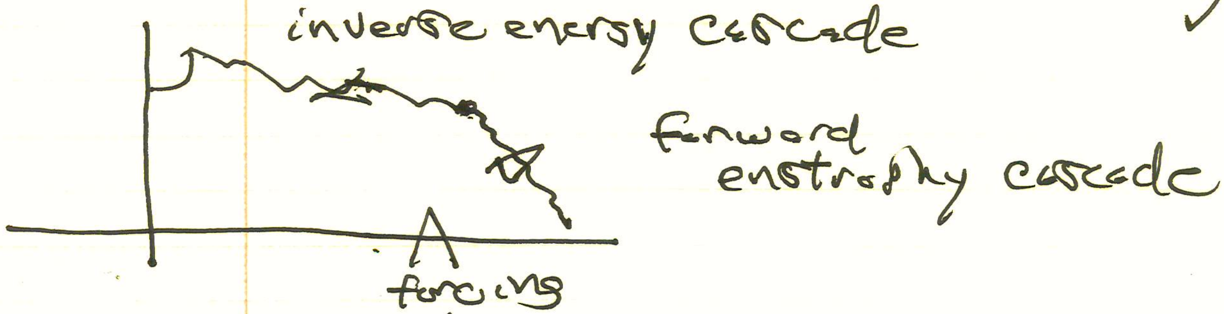
- R. Salmon: Notes on GFD
- G. Vallis: Atmospheric and Oceanic Fluids

+ Posted Notes from GFD Module
+ Posted References (both Module & Lectures)

N.B. Boffetta and Ecke review especially recommended.

Key element in 2D turbulence is constraint imposed on dynamics by dual conservation law.

Upshot: Dual Cascade (Kraichnan '67)



dual self-similarity ranges.

$\Rightarrow$ and drag $\partial_t \omega + \underline{v}\cdot\underline{\nabla}\omega + \mu\omega = \nu\nabla^2\omega + f$

<u>Why 2D ?</u> $\Rightarrow$ <u>Constrained Dynamics</u>.

- Recall Taylor - Proudman Theorem

$\rightarrow$ in rotating fluid, $(\underline{\omega} + 2\underline{\Omega})/\rho$ is 'frozen in'.

$\rightarrow$ $\Omega \gg$ other rates

$2\Omega\, \partial_z \underline{v} \cong 0$

$\Rightarrow$ <u>$\sim$2D dynamics</u>

Immediately realize that

~2D dynamics $\Leftrightarrow$

characteristic of ~2D dynamics

$Ro = V/L\Omega < 1$

Rossby #

– (V/L) → 'other rates'

i.e. $\underline{v}\cdot\nabla\underline{v}$ vs $2\underline{\Omega}\times\underline{v}$

– contrast Re

⇒ favors slow, large scale motion in (thin) rotating system i.e. atmosphere, ocean, etc.

Ways to 2D-ize:

- rotation, $Ro = V/L\Omega$
- stable stratification

$\# \sim V/LN$ $\qquad N^2 = \frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$

- strong magnetic field

$Ro \to V/L\Omega_{ci}$ ⇒ Hasegawa-Mima model

Ω_{ci}: cyclotron frequency

– Low Ro dynamics

Given $Ro \ll 1$, have fundamental relation between pressure and vorticity

→ includes centrifugal

$$\frac{d\underline{v}}{dt} = -\underline{\nabla}\left(p^*/\rho\right) - 2\underline{\Omega}\times\underline{v}$$

$Ro \ll 1 \Rightarrow$ Geostrophic balance

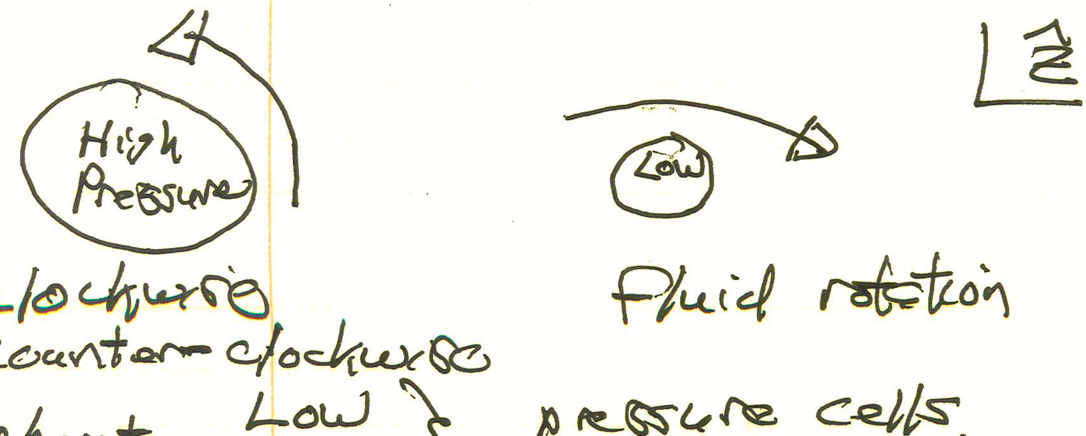
$$0 \cong -\nabla\left(p/\rho\right) - 2\underline{\Omega}\times\underline{v}$$

$$\Rightarrow \quad \underline{v}_\perp = \Omega\times\nabla\left(p^*/\rho\right)/\Omega^2$$

$$\boxed{\underline{v}_\perp = -\frac{\underline{\nabla}\left(p^*/\rho\right)\times\vec{z}}{\Omega}}$$

$p^*/\rho \leftrightarrow \phi$

Pressure-as-stream-function:

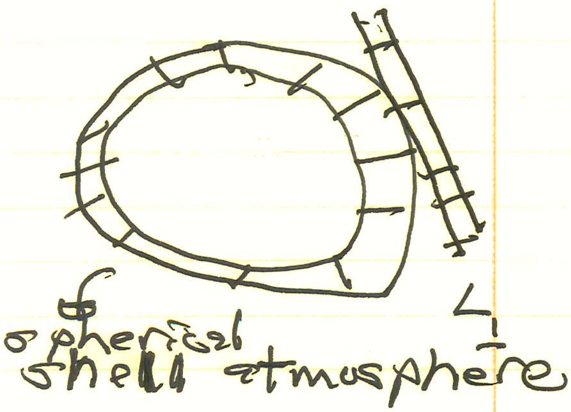


Fluid rotation

clockwise
counter-clockwise
about Low } pressure cells.
High }

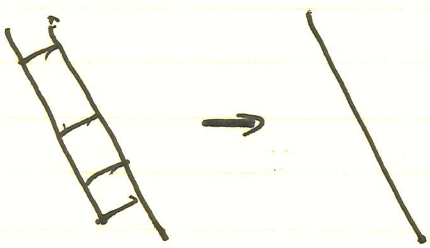
– β-plane Model

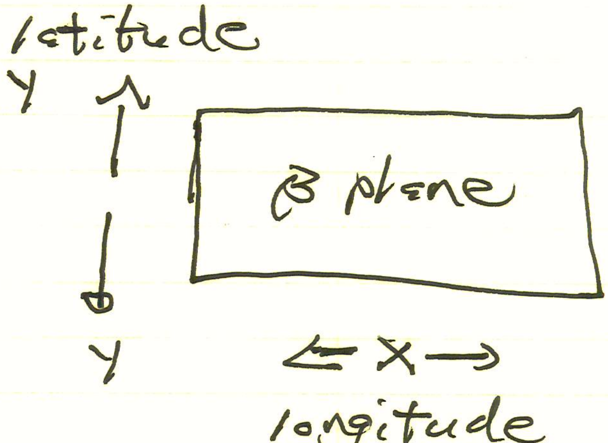
→ Quick derivation



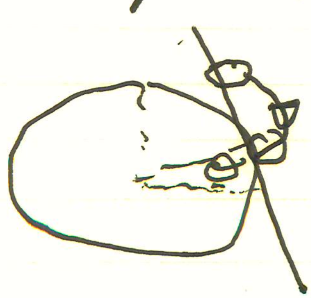
- β plane tangent to spherical shell atmosphere
- strong stable stratification on $L_\perp$.

so describe/approximate dynamics in 2D plane tangent to sphere i.e. β-plane





Now, consider displacement of fluid/vortex element:



- $\underline{\omega} + 2\underline{\Omega}$ frozen in
- $C = \int d\underline{a} \cdot (\underline{\omega} + 2\underline{\Omega})$

circulation conserved.

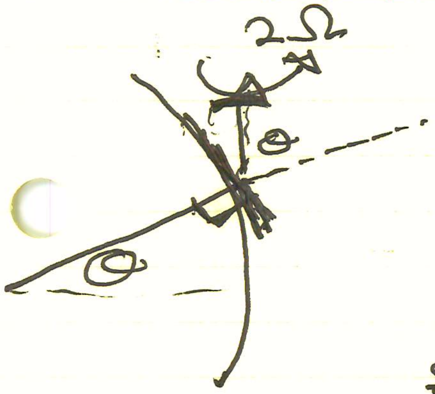
Point: displacing fluid element implies change in

$$\int d\underline{s} \cdot \underline{\Omega} \sim \hat{n} \cdot \hat{z} \sim \cos\Theta_p$$

$\Downarrow$ polar K

∴ there must be a change in fluid vorticity to conserve circulation

since planetary vorticity piece of circulation changed by displacement



$A \equiv$ area of vortex

$$\frac{d\Gamma}{dt} = 0$$

$$\frac{d}{dt}\left(A\omega + A\,2\Omega\sin\Theta\right) = 0$$

$\Downarrow$ projection factor

$\Rightarrow$

$$\frac{d\omega}{dt} = -2\Omega\cos\Theta\,\frac{d\Theta}{dt}$$

$$= -\frac{2\Omega}{R}\cos\Theta\,\frac{d}{dt}(R\Theta)$$

$$= -\beta V_y$$

$$\beta = \frac{2\Omega\cos\Theta}{R}$$

$\rightarrow$ ⊝ gradient in coriolis force.

Of course $\frac{d}{dt}(R\theta) = \frac{d}{dt} y = v_y$

$$\boxed{d\omega/dt = -\beta v_y}$$

+ add dissipation, forcing

Charney

$$\partial_t \omega + \underline{v} \cdot \underline{\nabla} \omega + \mu \omega = \nu \nabla^2 \omega + f$$

$$d/dt = \partial_t + \underline{v} \cdot \underline{\nabla}$$

$\hat{z} \perp \beta$ plane

$$\underline{v} = -\frac{\underline{\nabla}\phi}{2\Omega} \times \hat{z} \rightarrow \nabla\phi \times \hat{z}$$

$$\omega = \nabla_\perp^2 \phi / 2\Omega$$

$R \to \infty$, $\beta \to 0$

$$\boxed{\begin{gathered}\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2 \phi + \mu \nabla^2 \phi \\ = \nu \nabla^2 \nabla^2 \phi + f\end{gathered}}$$

2D fluid equation.

→ simplest incarnation of "2D Fluid"
⟺ 2D turbulence (eddies)

→ β-plane equation is next simplest ⟹ supports waves, eddys, zonal flows.

Observe:

Can re-write 2D inviscid equation as

$$\partial_t \omega_z + \{\omega_z, H\} = 0$$

$H = \phi$ — Conservative Hamiltonian evolution

similar to Liouville or Vlasov equation:

$$\partial_t f + \{f, H\} = 0$$

$H = \frac{p^2}{2M} + |e|\phi$, + Poisson's equation

$$\partial_t f + v\,\partial_x f + \frac{Z}{M} E\,\partial_v f = 0$$

i.e. $\omega_z \leftrightarrow f \Rightarrow \left\{ \begin{array}{l} \text{conserved (phase} \\ \text{space) density.} \end{array} \right.$

which brings us to:

$$\boxed{\text{Potential Vorticity}}$$

Observe can write equations in conservative form, i.e.

$$\frac{d}{dt}\omega = 0 \qquad \text{(pure 2D)}$$

$$\frac{d}{dt}(\omega + \beta y) = 0 \qquad (\beta\text{-plane})$$

ω: fluid vorticity

βy: planetary vorticity (l.o. in expansion)

$\omega + \beta y \equiv$ simple example of potential vorticity (PV)

- generalized vorticity akin to phase space density

"GFD = the study of Fluids with PV"

= "The Fluid Dynamics of PV"

More generally on PV:

- recall for rotating fluid:

$$\frac{d}{dt}\left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right) = \frac{(\underline{\omega} + 2\underline{\Omega})\cdot\underline{\nabla}\,\underline{V}}{\rho}$$

akin.

$$\frac{d}{dt}\,\delta\underline{l} = \delta\underline{l}\cdot\underline{\nabla}\,V$$

same eqn $\Rightarrow$ $\dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$ frozen in.

Now, consider conserved scalar field ψ

$$\frac{d}{dt}\psi = 0$$

$\int_1^2 \delta \underline{l}$

$$\frac{d}{dt}(\psi_1 - \psi_2) = 0$$

$$\delta\psi = \underline{\nabla}\psi \cdot \delta\underline{l}$$

so

$$\frac{d}{dt}\left(\underline{\nabla}\psi \cdot \delta\underline{l}\right) = 0$$

and $\delta l \longleftrightarrow \dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$

so if $\underline{\delta l}$ satisfies, $\dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$ must satisfy

$$\Rightarrow \underbrace{\boxed{\frac{d}{dt}\left(\frac{(\underline{\omega} + 2\underline{\Omega})\cdot\underline{\nabla}\psi}{\rho}\right) = 0}}_{\text{along trajectories}}$$

$\hookrightarrow$ general statement of PV conservation

$$\boxed{q = \frac{\underline{\omega} + 2\underline{\Omega} \cdot \nabla\psi}{\rho}}$$

PV (general) any $\underline{\nabla}\psi$

i.e.

- H-M: $\rho = \rho_0 + \tilde{\rho}$

$\hat{n} = (\text{e}\,\tilde{\phi}/T) n_0$

$\nabla\psi = \hat{z}$

- PV conservation $\leftrightarrow$ particle re-labeling symmetry
(i.e. particles can be re-labeled without changing thermodynamic state)

N.B. If consider finite thickness shell

$$q = \nabla_\perp^2 \phi + \beta y + \underbrace{\frac{f_0^2}{\rho} \partial_z \left(\frac{\rho}{N^2} \partial_z \phi \right)}$$

$f_0 = 2\Omega \sin\theta$ - rotation

$N^2 = g / L_\rho$ - buoyancy

Relevance of finite thickness?

Scale! $\rightarrow$ $1/L_\perp^2$ vs $\dfrac{f_0^2}{N^2 H^2}$

$H \rightarrow$ layer thickness

(deformation radius)$^{-2}$

$\sim 1/L_d^2$

So

$1/L^2 \sim 1/L_d^2$ → relative vorticity and deformation effects contribute equally

$$\begin{cases} \sim 100\,km & \text{ocean} \\ \sim 1000\,km & \text{atmosphere} \end{cases}$$

$L < L_d$ → β-plane.

→ 2D Turbulence

- issues : conservation of energy, enstrophy
- trends in constrained spectral evolution
- self-similarity ranges, inverse cascade
- Fate of energy

Issues:

- 2D turbulence is the generic problem of GFD
- $\beta \to 0$

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi - \nu \nabla_\perp^2 \nabla_\perp^2 \phi + \mu \nabla_\perp^2 \phi = \tilde{f}$$

$\tilde{f}$ → any scale

$\mu \nabla_\perp^2 \phi$ → drag → scale invariant damping → control large scale

2 inviscid invariants:

$\langle (\nabla\phi)^2 \rangle$ → energy

$\langle (\nabla^2\phi)^2 \rangle$ → enstrophy

N.B.:

- in 3D, enstrophy produced:

$$\frac{d}{dt}\langle \omega^2\rangle \sim \langle \underline{\omega}\cdot(\underline{\omega}\cdot\nabla\underline{v})\rangle$$

$$\langle \Omega(k)\rangle \sim k^2 k^{-5/3} \sim k^{1/3}$$

- 2D, $\underline{\omega}\cdot\underline{\nabla}\,\underline{v} \rightarrow 0$

→ all powers $\int d^2x\, \omega^n$ conserved

$\int d^2x\, \omega^2 \rightarrow \langle \omega^2\rangle$ conserved on finite box

∴ story incompatible with K41

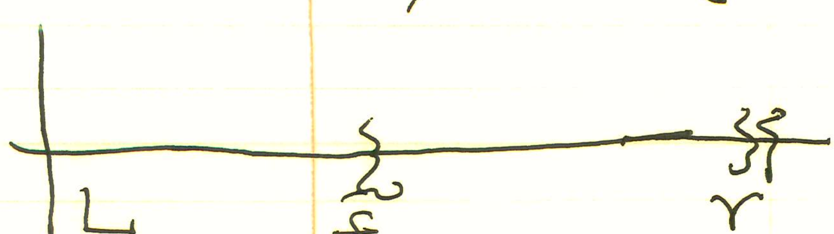
Problem of 2D Fluid:

- given forcing at any scale l_f s/t

$$L \geq l_f > l_v$$

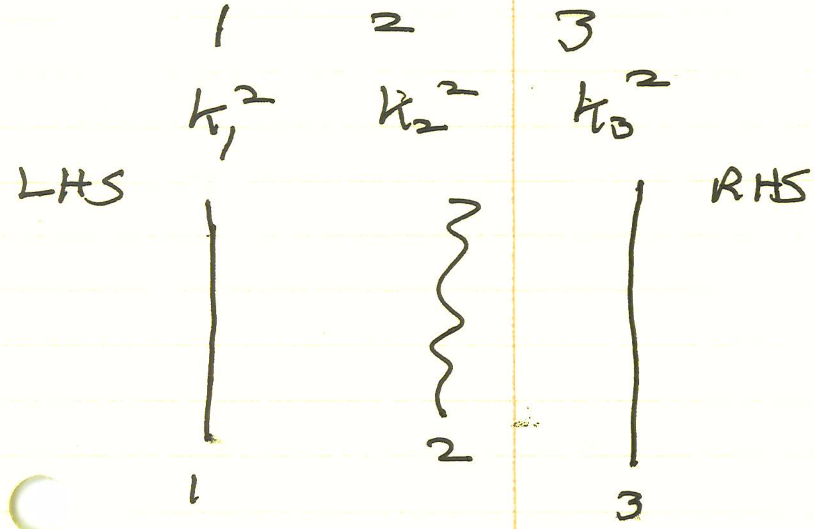
→ how does dual conservation of E, Ω constrain transfer?

→ self-similarity ranges?



Theoretical "clues":

— consider 3 modes, interacting
(3 to conserve quadratic quantity)



$$k_1^2 < k_2^2 < k_3^2$$

$$k_2^2 \leftrightarrow k_f^2$$

Conservation:

$$E_2 = E_1 + E_3$$

$$\Omega_2 = \Omega_1 + \Omega_3$$

but $\Omega(k) = k^2 E(k)$

$$\begin{cases} E_2 = E_1 + E_3 \\ k_2^2 E_2 = k_1^2 E_1 + k_3^2 E_3 \end{cases}$$

$$\therefore \quad E_1 = \left(\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}\right) E_2 \rightarrow E_1 \sim E_2$$

$$E_3 = \left(\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}\right) E_2 \rightarrow E_3 \sim \frac{k_2^2}{k_3^2} E_2$$

as $\Omega(k) = k^2 E(k)$

$E_1 \sim E_2 \Rightarrow$ energy transferred to large scale mode !?

$\Omega_3 \sim \Omega_2 \rightarrow$ enstrophy transferred to small scale mode !?

~ suggests: energy accumulates at large scale,
enstrophy accumulates at small scale.

$\Rightarrow$ $\underline{\underline{2}}$ self-similar transfer ranges in 2D turbulence !?

N.B. Analogy: Asymmetric Top.
(restricted)

Conserve: $\sum_i L_i^2 = L^2$

$E = \sum_i L_i^2 / 2I_i$

$L^2 = L_1^2 + L_2^2 + L_3^2$

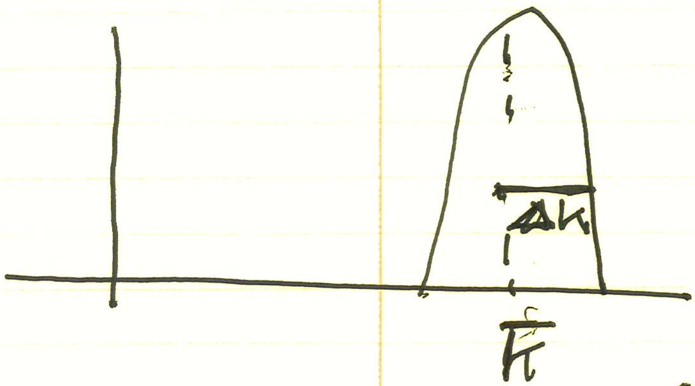
$E = \dfrac{L_1^2}{2I_1} + \dfrac{L_2^2}{2I_2} + \dfrac{L_3^2}{2I_3}$

$I \Leftrightarrow 1/k^2$, etc.

and energizing intermediate axis
$\rightarrow$ decay to 1, 3 etc.

→ But many D.O.F.'s ...

(Rhines; after the fact)



Consider a spectral 'slug' of turbulence, initialized.

How will $\bar{k}$ evolve, given $\partial_t \langle (\Delta k)^2 \rangle > 0$?
i.e. assume spectrum spreads ...

N.B. Does $\langle \Delta k^2 \rangle$ exist?

$$\langle (\Delta k)^2 \rangle = \int dk\, (k-\bar{k})^2 E(k) \Big/ \int dk\, E(k)$$

$$= \int dk\, (k^2 - 2k\bar{k} + \bar{k}^2)\, E(k) \Big/ \int dk\, E(k)$$

$$= \int dk\, \left(k^2 E(k) - 2k\bar{k} E(k) + \bar{k}^2 E(k) \right) \Big/ \int dk\, E(k)$$

$$= \left(\Omega_0 - 2\bar{k}^2 E_0 + \bar{k}^2 E_0 \right) / E_0$$

$$= \Omega_0 / E_0 - 2\bar{k}^2$$

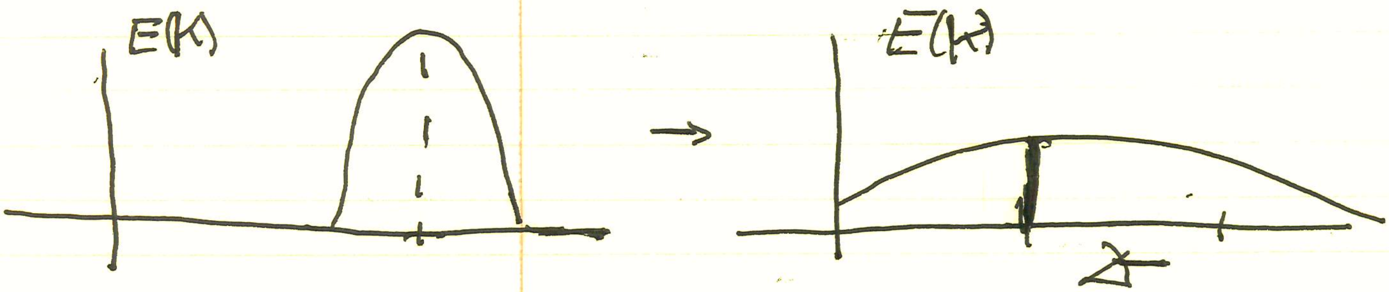
$$= (\Omega_0 / E_0) - \bar{k}^2$$

Here: $\int dk\, E(k) = E_0 \to$ const

$\int dk\, k^2 E(k) = \Omega_0 \to$ const

$\int dk\, k\, E(k) = \bar{k}\, E_0 \to$ defines centroid

so $\partial_t \langle \Delta k^2 \rangle > 0 \Rightarrow \partial_t \bar{k} < 0$



- spectrum broadens but also shifts toward large scales

- energy content shuffled / coupled to larger scale

$\Rightarrow$ suggestive of inverse energy cascade

$\to$ similar story for enstrophy $\Rightarrow$ forward cascade!

$\therefore$ Enter the Dual Cascade!

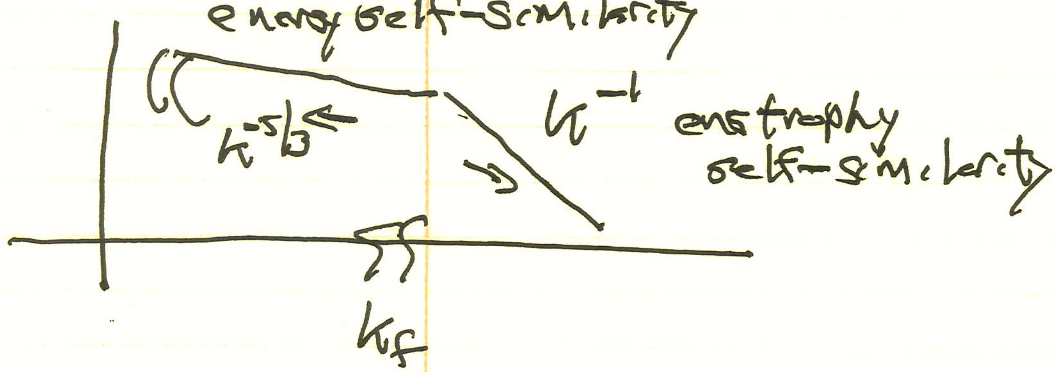
Dual cascade (Kraichnan '67):

from forcing, system supports $\underline{\underline{2}}$ self-similarity ranges:

- forward enstrophy range / cascade ($k > k_f$)
  → no forward energy flux
  → no energy dissipation by viscosity ($Re \to \infty$)

- <u>inverse</u> energy cascade
  → no inverse energy flux
  → no viscous power dissipation
  → damping by drag, etc.
  → not stationary

Cascade ≡ range self-similar transfer



$$\eta = \frac{d}{dt}\langle \omega^2 \rangle \sim \left(v_f / l_f \right)^3$$

$$\epsilon = \frac{d}{dt}\langle v^2 \rangle \sim v_f^3 / l_f \quad \to \text{forcing rate, not dissipation}$$

of course $k_f^2 \epsilon \sim \eta$.

- Forward Enstrophy

$$\frac{\Omega(\ell)}{T(\ell)} \sim \eta \qquad \frac{v(\ell)}{\ell} \sim T(\ell)^{-1}$$

$$\sim \omega(\ell)$$

$$\omega(\ell)^3 \sim \eta \qquad \Omega(\ell) \sim \omega(\ell)^2$$

$$\omega^2(\ell) \sim \eta^{2/3}$$

but $\omega^2(\ell) \sim k\, \Omega(k)$

$$\Rightarrow \boxed{\begin{array}{l} \Omega(k) \sim \eta^{2/3}/k \\ E(k) \sim \Omega(k)/k^2 \sim \eta^{2/3}/k^3 \end{array}}$$

energy spectrum in enstrophy range

- no forward energy flux in enstrophy range
- observe $1/\tau_\ell \sim \eta^{1/3} \rightarrow$ const, here

vs K41 $1/\tau_\ell \sim \frac{\epsilon^{1/3}}{\ell^{2/3}} \rightarrow$ faster for smaller

⇒ tip-off that since all scales transfer at same rate, non-local transfer of enstrophy can occur

⇒ Corrections? [Logarithmic due straining]

— Inverse Energy

$\epsilon = v(\ell)^2 \frac{v(\ell)}{\ell}$, as before but $\epsilon \equiv$ energy stirring rate

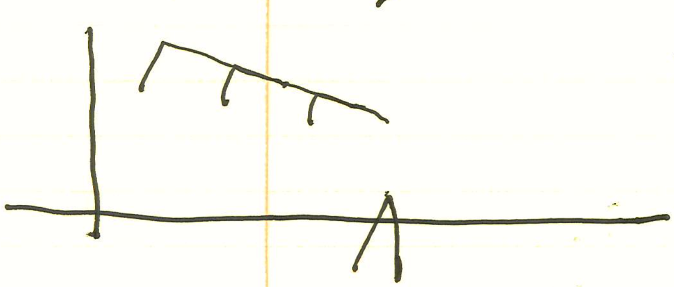
⇒

$E(k) = \epsilon^{2/3} k^{-5/3}$ ⇒ inverse energy cascade range

→ akin 3D, but upscale

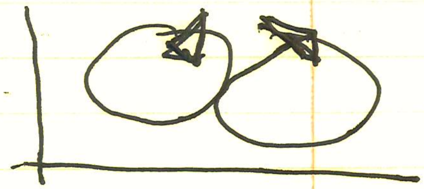
→ $1/\tau(\ell) \sim \frac{v(\ell)}{\ell}$ → cascade slows as larger scales approached.

→ not stationary state



largest scale slowest, keeps evolving

→ eventually encounters drag, boundary, etc.



→ straining (non-local) effects in scale !

→ structure of large scale.

→ no inverse enstrophy flux in energy range.

→ no forward energy flux, P_d by viscosity → 0.
⇓
dissipated power

→ so, where does the energy go ?
- friction μ
- boundary effects
- straining on small scales (Boffetta, et al. posted)

→ $\langle \delta u_l(l)^3 \rangle = + 3/2\, \epsilon\, l$

~ analogue of 4/5 for inverse energy range

~ + for $l > l_f$

~ ϵ here is stirring rate.

→ (static) vorticity contours in inverse energy range exhibit statistics of percolation cluster !

→ no "conventional" intermittency in inverse cascade. Deviations from Gaussian occur

→ what of particles, particle dispersion?

Revisiting Richardson:

$\ell_{1,2}$ → energy range

$$\frac{d\ell_{1,2}}{dt} = V(\ell_{1,2}) = \epsilon^{1/3}\ell^{1/3}$$

$$\ell_{1,2}^2 \sim \epsilon t^3 \quad ; \quad \text{as before}$$

$\ell_{1,2}$ → enstrophy

$$\frac{d\ell}{dt} = v(\ell) = [\Omega(\ell)]^{1/2}\ell = \eta^{1/3}\ell$$

⇒ separation grows exponentially in enstrophy range

N.B. : Dual cascade used to justify selective decay - minimum enstrophy

$$\Omega = \int d^2x\,(\nabla^2\phi)^2 + \lambda\int d^2x\,(\nabla\phi)^2$$

$$\delta\Omega = 0.$$

→ β-Plane: Turbulence, Waves, Flows

Recall:

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi - \nu \nabla^2\nabla^2\phi$$
$$+ \mu \nabla^2\phi = -\beta V$$ [redacted] $$y + \tilde{f}$$

Ignoring: ν, μ, $\tilde{f}$

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi = -\beta V_y$$

⇒ Waves

$$\omega_k = -\beta k_x / k^2, \qquad V_{gy} = \frac{2\beta k_x k_y}{(k^2)^2}$$

→ Rossby wave

and

⇒ flows

how does large scale order emerge?

$$\left.\begin{array}{l} k_x \to 0 \\ k_y \text{ finite} \\ \omega_k \to 0 \end{array}\right\}$$ Zonal Mode

jets, belts, jet stream

2 new players → waves, flows.

Numerous questions:

(2) → how do zonal flows form?
why
⇒ many ways!

(1) → how do {waves, flows} modify, interact with inverse cascade?

(3) → scale of zonal flows?

(4) → implications for atmospheric phenomenology

On zonal flows:

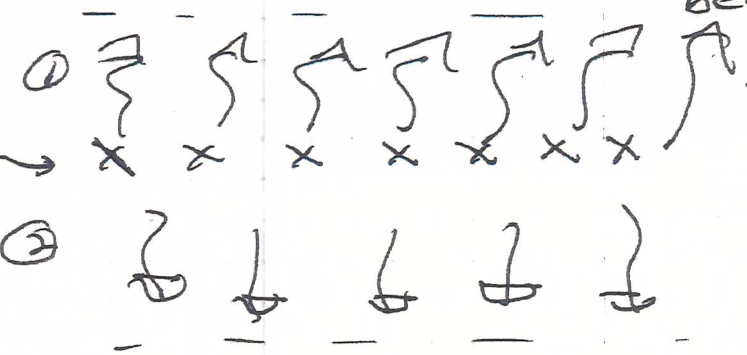
Reynolds stress

- ZFs ubiquitous
- Flows produced by momentum transport
- simplest perspective ⇔ wave propagation!

(a)

(Linear) wave propagation can account for ZF formation!

recall:

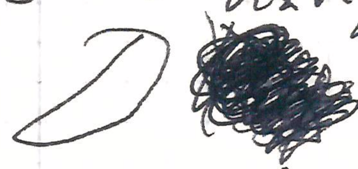


$$\underline{S} = v_{gy}\, \mathcal{E}\, \hat{y} = \frac{2 k_x k_y \beta \mathcal{E}}{(k^2)^2}\, \hat{y}$$

beach (absorber) (useful cartoon)

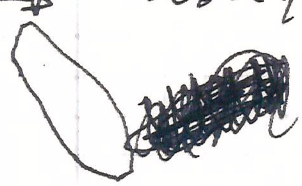
① outgoing waves $\rightarrow k_x k_y > 0$

$S \sim (+)\, \hat{y}$



$(\omega > 0)$

② $S \sim -\hat{y} \rightarrow k_x k_y < 0$



$(\omega > 0)$

eddy tilt changes

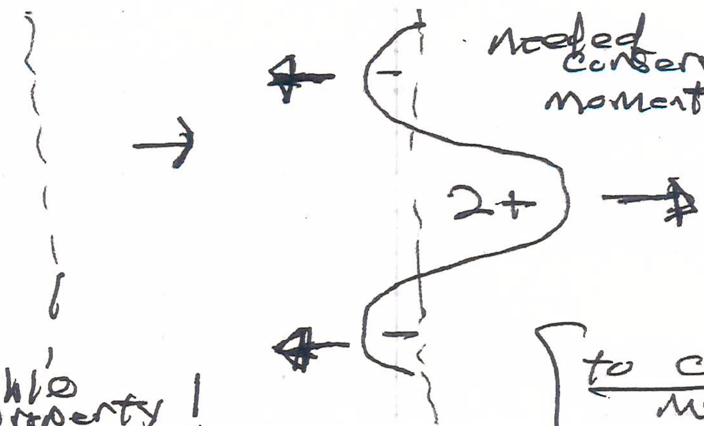
but

$$\langle \tilde{v}_y \tilde{v}_x \rangle = \sum_k -k_x k_y |\phi_k|^2$$

so ① $\rightarrow \Pi_{y,x} < 0$

② $\rightarrow \Pi_{y,x} > 0$

point: outgoing wave energy density flux generates incoming momentum flux



needed conserve momentum

spun up by incoming flux

[to conserve momentum.]

Remarkable property!

→ beautiful example of:

... " the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region "
(stirring → spin-up!)

Isaac Held ('01)

Flows ⟺ energy stirring

→ Wave mechanism requires separation of excitation and dissipation (beach) regions.

→ Requires:
- waves
- vorticity/momentum transport in space
- * irreversibility → outgoing waves
- symmetry breaking, ⇒ has direction
- sep. forcing/damping

⇒ Useful to investigate wave theorems for flow production → {something general}

→ key observation: PV flux (inhomogeneous PV mixing) why?

$\langle \tilde{v}_y \tilde{q} \rangle_z$ ⇒ zonal avg.

$q = \beta y + \nabla^2 \phi$

recall essence of PV conservation forces planetary – flow vorticity exchange.

$= \langle \tilde{v}_y \nabla^2 \phi \rangle_z$

$= \langle (-\partial_x \phi)(\partial_x^2 \phi + \partial_y^2 \phi) \rangle_x$

but: $\langle \partial_x \phi \, \partial_x^2 \phi \rangle = \langle \partial_x \left[\frac{(\partial_x \phi)^2}{2} \right] \rangle_x = 0$

symmetry!

$\langle \tilde{v}_y \tilde{q} \rangle_z = -\langle (\partial_x \phi) \, \partial_y^2 \phi \rangle$

$= -\partial_y \langle \partial_x \phi \, \partial_y \phi \rangle_x + \langle \partial_x \partial_y \phi \, \partial_y \phi \rangle$

$\langle \partial_x \partial_y \phi \, \partial_y \phi \rangle = \langle \partial_x \frac{(\partial_y \phi)^2}{2} \rangle_x = 0.$

$= \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_y$

Taylor Identity

$$\boxed{\langle \tilde{v}_y \tilde{q} \rangle_z = \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_z}$$

(comment 3D) – EP.

z dropped hereafter.

→ Reynolds force drives flow!

⇒ Look at potential enstrophy balance.

$\Rightarrow$ Zonally averaged Latitudinal PV flux = zonally averaged Latitudinal Reynolds force $\Rightarrow$ drives flow.

As Reynolds stress controls flow:
i.e.

$$\rho\left(\frac{\partial V_x}{dt} + \underline{V}\cdot\underline{\nabla}U_x\right) = \cancel{-\partial_x P} - \cancel{(2\underline{\Omega}\times\underline{V})_x}$$

cancel $\rightarrow$ geostrophic balance

so

$$\left[\;\frac{\partial\langle U_x\rangle}{\partial t} = -\partial_y\langle\tilde{U}_y\tilde{U}_x\rangle + \nu\,\partial_y^2\langle U_x\rangle - \mu\langle U_x\rangle\right.$$

then PV evolution / Potential Enstrophy } necessarily control flow.

$\Rightarrow$ What are essential to ZF generation:

- inhomogeneous PV mixing / transport in space
- translation symmetry in direction of the flow.

Now, consider P.E balance:

3/n (forcing)

$$\frac{d}{dt} q - \nu \nabla^2 q = 0$$

$$\frac{\partial}{\partial t} \tilde{q} + \underline{V} \cdot \nabla \tilde{q} - \nu \nabla^2 \tilde{q} = - \tilde{v}_y \frac{d\langle q \rangle}{dy}$$

potential enstrophy evolution

so

$$\frac{\partial}{\partial t} \left\langle \frac{\tilde{q}^2}{2} \right\rangle + \partial_y \left\langle \tilde{v}_y \frac{\tilde{q}^2}{2} \right\rangle + \nu \left\langle (\nabla \tilde{q})^2 \right\rangle$$

Flux of potential enstrophy.

$D_\Omega \rightarrow$ dissipation

$$= - \langle \tilde{v}_y \tilde{q} \rangle \frac{d\langle q \rangle}{dy}$$

[potential enstrophy production, (flux - gradient)

$$\left(\frac{d\langle q \rangle}{dy}\right)^{-1} \left[\partial_t \left\langle \frac{\tilde{q}^2}{2} \right\rangle + \partial_y \left\langle \tilde{v}_y \frac{\tilde{q}^2}{2} \right\rangle + \nu \left\langle (\nabla \tilde{q})^2 \right\rangle \right]$$

$$= - \langle \tilde{v}_y \tilde{q} \rangle = - \langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle$$

but Mean (zonal) flow

$$\partial_t \langle v_x \rangle = - \frac{\partial}{\partial y} \langle \tilde{v}_y \tilde{v}_x \rangle - \mu \langle v_x \rangle$$

$$= - \langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle - \mu \langle v_x \rangle$$

$$-\langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle = -\left(\partial_t \langle V_x \rangle + \mu \langle V_x \rangle\right)$$

$\therefore$

WAD

$$\partial_t \left\{ \langle V_x \rangle + \frac{\langle \tilde{\xi}^2 \rangle}{2\, d\langle \xi \rangle / dy} \right\} = -\nu \frac{\langle (\nabla \tilde{\xi})^2 \rangle}{d\langle \xi \rangle / dy} - \partial_y \frac{\langle \tilde{v}_y \tilde{\xi}^2 / 2 \rangle}{d\langle \xi \rangle / dy} - \mu \langle V_x \rangle$$

Wave Activity Density

WAD

pseudomomentum to

$$\partial_t \left\{ \langle V_x \rangle - \left(\frac{-k_x \langle \tilde{\xi}^2 \rangle}{2 k_x\, d\langle \xi \rangle / dy} \right) \right\}$$

$$= -\mu \langle V_x \rangle - \delta \langle \tilde{\xi}^2 \rangle \Big/ \frac{d\langle \xi \rangle}{dy}$$

absent
- drag
- damping
- mixing (3^{rd} order)

$\Rightarrow$ Flow ~~locked~~ to wave momentum density (Charney – Drazin Thm.)

non-acceleration thm!

ZF's $\Leftrightarrow$ wave momentum density

~~Cannot~~ accelerate (or maintain vs drag) zonal flow without changing (decreasing) wave intensity.

Note: $$\frac{-k_x \langle \tilde{q}^2 \rangle}{2 k_x \, d\langle q \rangle / dy}$$

$$q = \nabla^2 \phi + \beta y$$

absent mean flow,

$$\frac{d\langle q \rangle}{dy} = \beta \; !$$

$$\frac{\langle \tilde{q}^2 \rangle}{2} = k^2 \mathcal{E}$$

$$\frac{-k_x k^2 \mathcal{E}}{k_x \frac{d\langle q \rangle}{dy}} = \frac{k_x \mathcal{E}}{\frac{-k_x \Omega}{k^2}} = \frac{k_x \mathcal{E}}{\omega_k}$$

→ Action Density

$$= k_x \, N_k$$

$$= P_W.$$

i.e. a/a/ Adiabatic Invariant

⇒ wave momentum density

$$\partial_t \left\{ \langle V_x \rangle - P_W \right\} = -\mu \langle V_x \rangle - \delta \langle \tilde{q}^2 \rangle \Big/ \frac{\partial \langle q \rangle}{\partial y} + \cdots$$

$$\underline{u} \sim \frac{v(\ell)}{\ell} \sim \epsilon^{1/3} \underline{\ell}^{-2/3}$$

$$\ell^{2/3} \sim \cancel{u/\epsilon^{1/3}} \quad \epsilon^{1/3}/u$$

$$\underline{\ell \sim \epsilon^{1/2}/u}^{3/2} < L$$