Viscous Fluids — Stokesian Hydro.

→ So far ideal fluids. Platonic ideal → no friction.

→ Real, viscous fluids → internal friction.

so $\underline{\underline{\Pi}} = \rho \underline{v}\,\underline{v} + P\delta_{ij} + \Pi'_{ij}$

$\underline{\underline{\Pi}}$ → stress tensor

$\rho \underline{v}\,\underline{v}$ → Reynolds

$P\delta_{ij}$ → Pressure

Π'_{ij} → viscous stress, due internal friction

→ For internal friction:

- set by flow gradients

- $\sim c\, \nabla \underline{v}$, to lowest order. i.e. in l_{mfp}/L

- in particular, Π' must vanish for
  - $\underline{v} = \text{const.}$
  - solid body rotation

$$\underline{v} = \underline{\Omega} \times \underline{r}$$

so

$$\pi' \simeq \underbrace{\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}}_{\text{rate of strain}}$$

$$\text{strain} \sim \frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}$$

$$\underline{\xi} \equiv \text{displacement}$$

→ so, most general viscous stress tensor:

shear

$$\pi'_{ijk} = \eta\left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{jk}\frac{\partial v_\ell}{\partial x_\ell}\right)$$

(2/3 ensures vanish for compr)

$$+ \zeta\,\delta_{jk}\left(\frac{\partial v_\ell}{\partial x_\ell}\right)$$

→ compressive

$\eta \equiv$ shear viscosity

$\eta = \rho\,\nu$

ν: kinematic viscosity (diffusion)

$\zeta \equiv$ compressional / second

η $\rightrightarrows$ layers sliding

ζ  compression

So, general: (2nd order, contrast Euler)

Navier-Stokes Eqn.

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right] = -\underline{\nabla} p + \eta \nabla^2 \underline{v} + \left(\zeta + \frac{1}{3}\eta\right)\underline{\nabla}(\underline{\nabla}\cdot\underline{v})$$

$$\boxed{\underline{\nabla}\cdot\underline{v} = 0} \Rightarrow$$

$$\boxed{\left[\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right] = -\frac{\underline{\nabla} p}{\rho} + \frac{\eta}{\rho}\nabla^2 \underline{v}}$$

$Re \equiv VL/\nu$ $\Rightarrow$ Reynolds #!

$\rightsquigarrow$ ~ ratio of NL to diffusion

→ See notes:

Fluid ⇒ f near Maxwellian

$$f = f_{max} + \delta f$$

where, $\delta f \cong -\dfrac{\underline{v} \cdot \nabla f_0}{\nu}$

$$\Pi = \begin{cases} \Pi_{ideal} \rightarrow f = f_{max} \\ \Pi_{ideal} + \Pi_{visc} \rightarrow f = f_{max} + \delta f \end{cases}$$

deviation from pure local Maxwellian induces viscous stress

- Viscosity reflects diffusive relaxation to global Maxwellian.

- Calculate Π_{visc} from δf → notes.

→ will focus on both low and high Reynolds #

→ Vorticity

$$\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{v} - \underline{\omega}\, \nabla \cdot \underline{v} + \nu \nabla^2 \underline{\omega}$$

$$\partial_t \underline{\omega} = + \underline{\nabla} \times (\underline{v} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$$

for $\underline{\nabla} \cdot \underline{v} = 0$

$$\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{v} + \nu \nabla^2 \underline{\omega}$$

→ Apart Reynolds:

$$\Pi_{ij} = \delta_{ij} P + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

$$(\nabla \cdot \Pi)_i = - \partial_i P + \eta \nabla^2 v_i$$

Some #'s:

H_2O	$\nu = .01$	cm^2/sec
air	$\nu = .15$	" "
glycerine	$\nu = 6.8$	" "
Mercury	$\nu = .001$	" "

both $\left\{ \begin{array}{l} \text{viscous and pressure stress} \\ \text{contribute.} \end{array} \right.$

→ Low Re, (Quasi-steady)

Re <<< 1,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\frac{\underline{\nabla} p}{\rho} + \nu \nabla^2 \underline{v}$$

$$\underline{\nabla} \cdot \underline{v} = 0$$

⇒
$$\boxed{\begin{array}{l} \underline{\nabla} p = \mu \nabla^2 \underline{v} \\ \underline{\nabla} \cdot \underline{v} = 0 \end{array}}$$

Stokesian Hydrodynamics

$$\nabla^2 \underline{\omega} = 0.$$

(2D: $\nabla^2 \nabla^2 \phi = 0$
biharmonic)

N.B. → linear

→ non-evolutionary
i.e. no ∂_t

→ more concretely, reversible

i.e. $\underline{V} \to -\underline{V}$, $p \to c - p$ — recover Stokes Eqns.

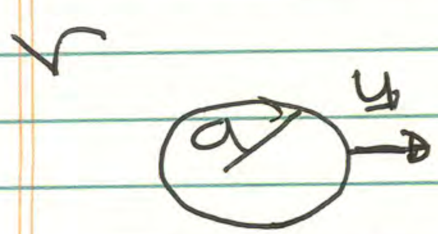
c to avoid negative pressure

leaves Stokes equations invariant

→ Stokes Drag.

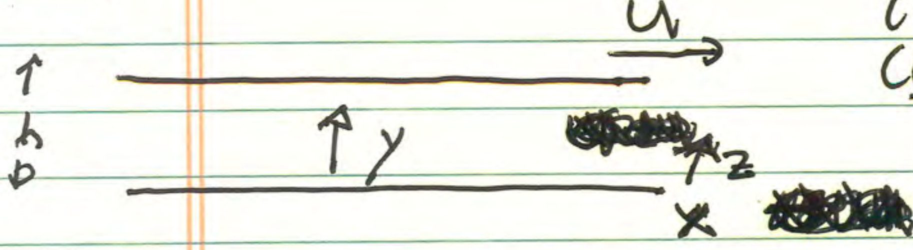
Consider sphere of radius a in steady motion at velocity $\underline{U}$ in Stokesian Fluid.

$Re \ll 1$



Calculate force of drag.

Simple Example: Viscous Pipe Flow
(Poiseuille Flow)
(Plane moves)



$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{V}$$

steady

$$V = v(y)\,\hat{x}$$

then

$$\nu\nabla^2 v(y)\,\hat{x} = 0$$

$$\frac{d^2 v}{dy^2} = 0$$

$$\frac{d}{dy}P = 0 \qquad , \qquad P = P_0$$

so $\quad V = yU/h \qquad V = 0$, fixed lower plate

$U = U$, upper

$$\langle v\rangle = \frac{1}{h}\int_0^h v\,dy = \frac{U}{2}$$

see LL: 48-49

For normal force on plane;

$$F_n/A = p$$

For tangential force; (on $y=0$ plane)

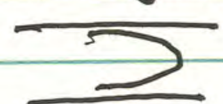
$$\left.\begin{matrix}\tau_{x,y} \\ \Pi'_{x,y}\end{matrix}\right|_0 = \eta \frac{\partial v_x}{\partial y} = \eta \frac{dV}{dy} = \frac{\eta U}{h}$$

(pushes)

$$\left.\begin{matrix}\tau_{x,y} \\ \Pi'_{x,y}\end{matrix}\right|_h = -\frac{\eta U}{h}$$

(opposes)

Now, consider fixed plates, with pressure gradient along flow:

 ← ∇p drives flow.

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{v \cdot \nabla v} = -\frac{\nabla p}{\rho} + \eta \nabla^2 \underline{v}$$

$$0 = -\frac{\partial p}{\rho \partial x} + \eta \frac{\partial^2}{\partial y^2} v_x$$

balance pressure drop by re-distribution of momentum.

so

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \quad ; \quad \frac{\partial p}{\partial y} = 0$$

so → p indep y

→ $p = p(x)$

→ $$\frac{\partial^2 v}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

$g(y)$ only — $f(x)$ only

Pressure drop → momentum input. Where go, for steady state?

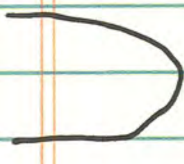
so both const, to be equal.

∴ $$v = \frac{1}{2\mu}\frac{dp}{dx}y^2 + ay + b$$

b.c. $v(0) = v(h) = 0$

so

$$\boxed{v = \frac{-1}{2\mu}\frac{dp}{dy}\, y\,(y-h)}$$



parabolic profile

$$\bar{V} = \frac{-1}{12\mu}\, h^2 \,\frac{dp}{dx}$$

↓

pressure drop

$\Delta P/\ell$

Shear stress:

$$\tau_{xy}\Big|_0 = \mu \frac{\partial v}{\partial y}\Big|_{y=0} = \frac{1}{2}\, h\, \frac{dp}{dx}$$

Pipe (Circular) → HW.

Example

→ $\underline{F}_d = -6\pi\mu a \underline{u}$

i.e. ala' Aristotle → Friction !

→ $-6\pi\mu \ell \underline{u}$ (not effective inertia)

- easily recovered by dimensional analysis (up to 6π)

→ Key feature:

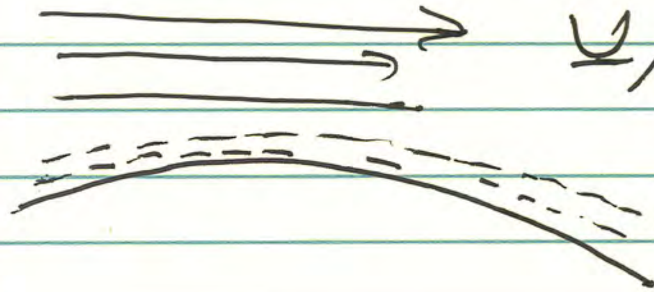
No slip boundary condition

$V_n = 0$ $\qquad V_T = 0$

→ tangential velocity of fluid at surface bounding the fluid vanishes

→ in addition to $V_n = 0$.

→ boundary layers ; "far field" → potential flow



$\underline{V}_1 \sim \underline{V}$ [Pot. flow]

boundary layer
→ V rises to match pot. flow
$V_t = 0$
→ vortex sheet

→ but <u>Stokesian</u> <u>fluid</u> is viscous dominated (i.e. all BL).

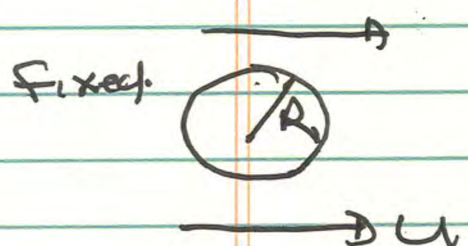
→ Calculating the Drag.

Recall

$$\eta \nabla^2 \underline{v} = \nabla p$$

$$\nabla \cdot \underline{v} = 0$$

$$\eta \nabla^2 \underline{\omega} = 0$$



$\underline{v}$ = flow.

Strategy: Compute flow and pressure fields, then integrated stress on sphere.

$$\nabla \cdot (\underline{v} - \underline{U}) = 0$$

$$\underline{v} = \underline{U} + \nabla \times \underline{A}$$

↓

axial

(curl A → polar)

↓

direction

Now,

- A must be axial
- A ~ U, linear

axial: not reversed under $x \to -x$

polar → reversed under $x \to -x$

scalar pot. $\qquad \vec{r}/|r|$

or $\qquad \underline{A} = \underbrace{f'(r)\,\hat{n}}_{\nabla f} \times \underline{u}$ $\qquad$ general form $\nabla \times \underline{A} \to \underline{u}$

& $\qquad \underline{V} = \underline{\nabla} \times (\underline{\nabla} f \times \underline{u}) + \underline{u}$

$$= \underline{u} + \underline{\nabla} \times \underline{\nabla} \times (f\underline{u})$$

$\underline{u}$ const.

Then,

$$\underline{\nabla} \times \underline{V} = \underline{\nabla} \times \underline{\nabla} \times \underline{\nabla} \times (f\underline{u})$$

$$= (\cancel{\nabla\nabla\cdot} - \nabla^2)\, \underline{\nabla} \times (f\underline{u})$$

$$= - \nabla^2 (\underline{\nabla} \times f\underline{u})$$

$$\underline{\omega} = - \nabla^2 (\underline{\nabla} \times f\underline{u})$$

$$\nabla^2 \underline{\omega} = 0$$

$\Rightarrow$

$$(\nabla^2)^2\, \underline{\nabla} \times (f\underline{u}) = (\nabla^2)^2 (\underline{\nabla} f \times \underline{u}) = 0$$

$$\nabla^2 \underline{\omega} = 0 = (\nabla^2)^2 (\underline{\nabla} f \times \underline{u})$$
$$= \left((\nabla^2)^2 \underline{\nabla} f\right) \times \underline{u} = 0$$

$$\boxed{(\nabla^2)^2 \underline{\nabla} f = 0}$$

biharmonic for scalar.

$$(\nabla^2)^2 f = \text{const.}$$
$$\to 0$$

as $\underline{v} \to \underline{u}$, so $\underline{v} - \underline{u} \to 0$ must vanish at ∞.

Symmetry $\to$ $f = f(r)$

$$\nabla^2 \nabla^2 f = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr}\right) \left[(\nabla^2 f)\right] = 0$$

$$\nabla^2 f = \frac{2a}{r} + C \qquad \text{(Poisson)}$$

$C \to 0$ if $\underline{v} \to \underline{u}$ at ∞.

so $\boxed{f = ar + b/r}$

so

$$\underline{V} = \underline{u} + \nabla \times \nabla \times (f\underline{u})$$

$$f = ar + b/r$$

so

$$\underline{V} = \underline{u} - a\left[\frac{(\underline{u} + \hat{n}(\underline{u}\cdot\hat{n}))}{r}\right] + b\left[\frac{3\hat{n}(\underline{u}\cdot\hat{n}) - \underline{u}}{r^3}\right]$$

Now, $\underline{V} = 0$ at $r = R$,
(+ ~~and~~ n)

$$-\underline{u}\left(\frac{a}{R} + \frac{b}{R^3} - 1\right) + \hat{n}(\underline{u}\cdot\hat{n})\left(\frac{-a}{R} + \frac{3b}{R^3}\right) = 0$$

all $\hat{n}$ → each cancel.

$$a = 3/4\ R$$

$$b = 1/4\ R^3$$

$$f = \frac{3}{4} Rr + \frac{1}{4} R^3/r$$

At last:

$$\boxed{\begin{aligned} v_r &= u\cos\theta\left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right] \\ v_\theta &= -u\cos\theta\left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right] \end{aligned}} \quad (1)$$

Velocity

→ velocity field.

For stress, need pressure:

$$\underline{\nabla} p = \mu \nabla^2 \underline{v}$$

$$= \mu \nabla^2 \left[\underline{\nabla}\times\underline{\nabla}\times(f\underline{u})\right]$$

$$= \mu \nabla^2 \left[\underline{\nabla}\left[\nabla\cdot(f\underline{u})\right] - \underline{u}\nabla^2 f\right]$$

$$(\nabla^2)^2 f = 0$$

$$\Rightarrow \underline{\nabla} p = \underline{\nabla}\left[\mu \nabla^2\left[\nabla\cdot(f\underline{u})\right]\right]$$

$$= \underline{\nabla}\left(\mu \underline{u}\cdot\nabla\nabla^2 f\right)$$

$$\underline{\nabla} p = \underline{\nabla}\left(\eta\, \underline{u} \cdot \underline{\nabla}\, \nabla^2 f\right)$$

$$\boxed{p = \eta\, \underline{u} \cdot \underline{\nabla}\, \nabla^2 f + p_0}$$

Plug in for f:

$$\boxed{p = p_0 - \frac{3}{2}\eta \frac{\underline{u}\cdot\hat{n}}{r^2} R}$$

Pressure (2)

(1), (2) tell story

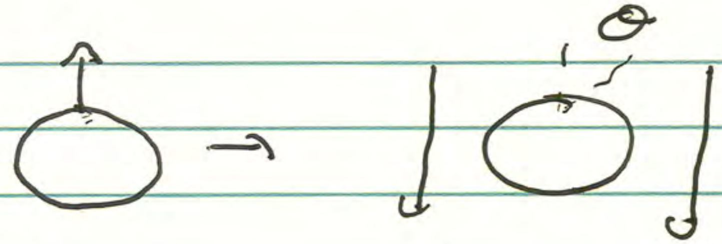
For force on sphere:

pressure

$$F_i = \int ds \left(p n_i - \Pi'_{ik} n_k\right)$$

viscous stress

(Re << 1 → no Reynolds contribution).

Now 

normal

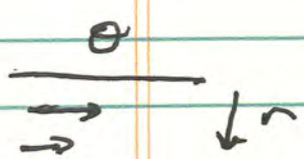
$$F = \text{[redacted]} \int ds \Big[\underbrace{-p\cos\theta}_{\text{pressure head}} + \pi'_{rr}\cos\theta - \pi'_{r\theta}\sin\theta \Big]$$

viscous

$$\pi'_{r,r} = 2\mu \frac{\partial v_r}{\partial r}$$ normal stress

$$\pi'_{r,\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \overbrace{\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}}^{\text{sb}} \right)$$

tangential stress



(r flux θ momentum)

$$\pi'_{r,r}\Big|_R = 0$$

$$\pi'_{r,\theta}\Big|_R = -\frac{3\mu}{2R}\, u \sin\theta$$

so

$$\boxed{F_d = \frac{3\eta U}{2R}\int dS \quad = \quad 6\pi\eta R U.}$$

More generally,

$$F_i = \eta U_{k} \, a_{ik}$$

order U

symmetric, due $F \sim U$.

Example

→ Scallop Thm and Related

→ Mechanical fish

→ water → swims

→ corn syrup → no net motion !

Why ?

Difference ? → ν and thus Re
fixed

$$Re = VL/\nu$$

$Re_{cs} \ll 1$

$Re_{H_2O} > 1$

Why ?

Recall :

$$\begin{cases} \eta \nabla^2 \underline{V} = \underline{\nabla} P \\ \underline{\nabla} \cdot \underline{V} = 0 \end{cases}$$

$$\nabla^2 \underline{\omega} = 0$$

→ Point:

- time does not appear

↓

- equations not evolutionary.

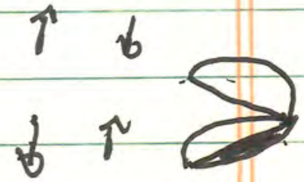
- pattern motion same, whether forward or back in time

Now:

- to swim, need <u>reconfigure body shape</u> to propel self.

- at low Re,

  "flapping" reversible → back flap undoes flap → <u>no motion</u>



<u>i.e.</u> $\nu \nabla^2 \underline{V} = \nabla p$

$\underline{\nabla} \cdot \underline{V} = 0$

then $\underline{V} \to -\underline{V}$ $\left\{ \begin{array}{l} \text{reversible} \\ \text{flow} \end{array} \right.$

$$p \to \underbrace{c}_{\text{Lpinned}} - p(\underline{x})$$

also solves Stokes eqns.

so $\underline{V}_{\text{Boundary}} = \underline{f}(x)$ flap

$$\to \underline{V}$$

then if reverse

$$\underline{V}_{\text{Boundary}} = -\underline{f}(\underline{x}) \quad \text{back-flap}$$

$$\to -\underline{V} \quad \text{solution}$$

$\therefore$ Back-flap undoes flap!

[Reversal boundary conditions $\to$ reversal flow.]

Scallop thm consequence of "reversible" structure of Stokes equations.

→ why? Uniqueness Thm.
↳ 1 solution, per bndry config.

— viscous fluid (Stokes)

— S $\underline{V} = \underline{V}_B(\underline{X})$ on S

surface S

then $\exists$ solution

Proof: (contradiction)

$\underline{V}^*$ s/t $\underline{V}^* = \underline{V}_B(\underline{X})$ on S

$\exists$ another (hypothetical) solution?
p^*

$\underline{u} = \underline{V} - \underline{V}^*$ ≡ difference flow.

$\hat{p} = p^* - p$.

linear equations ⇒

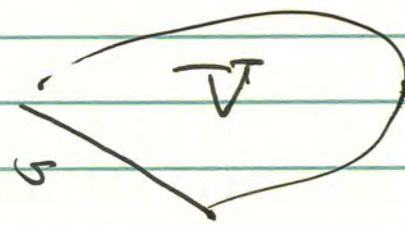
$$0 = -\nabla p + \eta \nabla^2 \underline{v}$$

$$0 = -\underline{\nabla} p^* + \eta \nabla^2 \underline{v}^*$$

so

$$0 = -\underline{\nabla} p + \eta \nabla^2 \underline{u}$$

$$\underline{\nabla} \cdot \underline{u} = 0$$



$\underline{u}$ *

$$0 = -\underline{u} \cdot \underline{\nabla} p + \eta\, \underline{u} \cdot \nabla^2 \underline{u}$$

$$= -\underline{\nabla} \cdot (p\underline{u}) + \eta (\nabla^2 \underline{u}) \cdot \underline{u}$$

so integrating over flow volume,

$$0 = -\int_S \cancel{p\, \underline{u} \cdot \hat{n}\, dS}$$ $u = 0$ on S (both $\underline{v}, \underline{v}^*$ sat b.c.)

$$+ \eta \int_V d^3x \left[\cancel{\frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j} \right)} - \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right]$$

$u_i = 0$ on S.

so $\int d^3x \left(\frac{\partial u_i}{\partial x_j}\right)^2 = 0$

→ u_i constant (all i, j).

→ $u_i = 0$ on S → $u_i = 0$.

so no other solution.

Corollary → back flow
— reversal flow.

N.B.: Thm. — linearity
— reversibility.

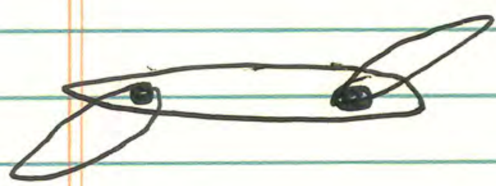
Begs questions:

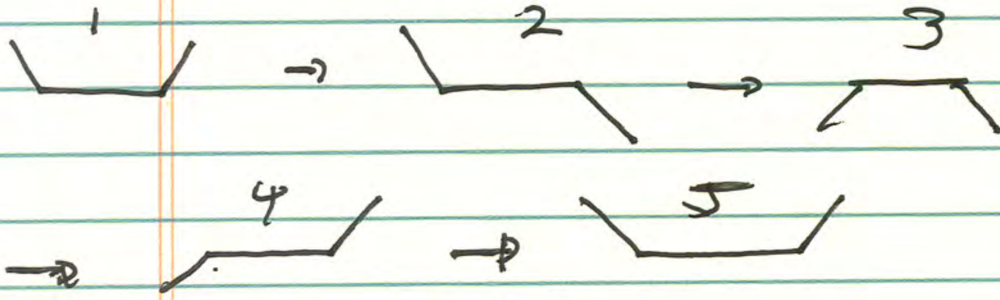
- how escape the scallop thm.?

- how do micro-organisms swim.
($Re \ll 1$).

Some examples:

- i) two flappers!

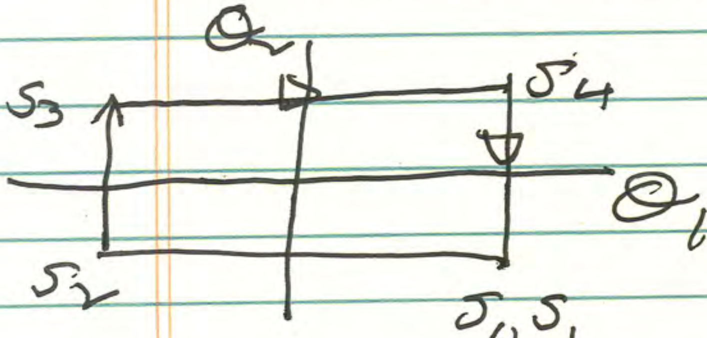




back stroke
different forward.

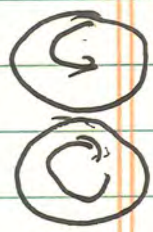
→ animal configuration
→ $\underline{2}$ angles

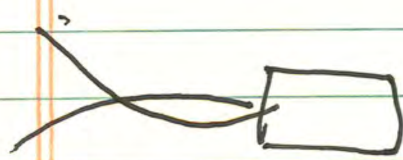
→ traverses loop



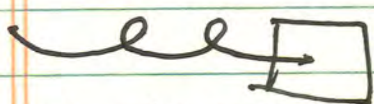
direction
→ clock vs
counter-clock

ii.) Toroidal blob



iii.) 

flexible oar (deformed bndry)

iv.) 

<u>Corkscrew</u>

<u>rotates</u>

(common for microorganisms).

<u>Model</u> — Thin Flexible Sheet

sheet
S

$$x = x_s$$
$$y = a \sin(kx - \omega t)$$

time and direction

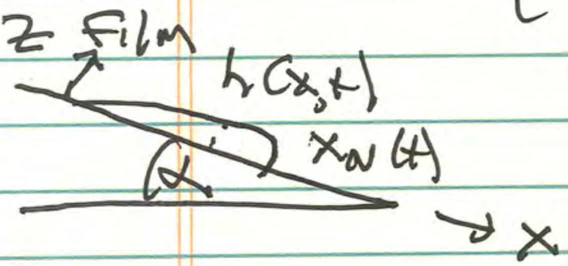
→ y motion of sheet reversible

→ time reversal → wave reversed in x. !

⇒ steady flow.

→ HW

→ Example   Thin Film on Slope (Free surface)



[Flow down slope]

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + g \sin\alpha \qquad \text{along}$$

↳ thin.

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\cos\alpha \qquad \text{perp}$$

Integrating second;

$$p = -\rho g z \cos\alpha + f(x,t)$$

$$z = h(x,t), \quad p = p_0 \qquad \text{(free surface)}$$

$$\boxed{p = \rho g \left[h(x,t) - z \right] \cos\alpha + p_0}$$

Free surface → tangential stress vanishes $\eta \dfrac{\partial u}{\partial z} = 0$.

Free surface:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + g \sin\alpha$$

$$= \rho g \cos\alpha \frac{\partial h}{\partial x} + g \sin\alpha + \nu \frac{\partial^2 u}{\partial z^2}$$

small.

then,

$$\nu \frac{\partial^2 u}{\partial z^2} = -g \sin\alpha$$

$$\boxed{u = \frac{g \sin\alpha}{\nu} \left(hz - z^2/2 \right)}$$

(no slip $z = 0$)

$$\frac{\partial w}{\partial z} = \frac{\partial v_z}{\partial z} = -\frac{\partial u}{\partial x} = -\frac{g \sin\alpha}{\nu} z \frac{\partial h}{\partial x}$$

$$\boxed{w = -\frac{g \sin\alpha}{2\nu} \frac{\partial h}{\partial x} z^2}$$

Finally: (kinematic free surface condition)

$$\boxed{w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}} \quad \text{on } z = h(x,t)$$

$$u = \frac{g \sin\alpha}{\nu} \left(hz - z^2/2 \right)$$

$$u(h) = \frac{g \sin\alpha}{\nu} \frac{h^2}{2}$$

$$w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}$$

$$u\Big|_h = \frac{g \sin\alpha}{\nu} \frac{h^2}{2}$$

$$w = -\frac{g \sin\alpha}{2\nu} \frac{\partial h}{\partial x} z^2 \Big|_h$$

$$= -\frac{g \sin\alpha}{2\nu} \frac{\partial h}{\partial x} h^2 \Big|_h$$

$$-\frac{g \sin\alpha}{2\nu} \frac{\partial h}{\partial x} h^2 = \frac{\partial h}{\partial t} + \frac{g \sin\alpha}{2\nu} h^2 \frac{\partial h}{\partial x}$$

so

$$\boxed{\frac{\partial h}{\partial t} + \frac{g \sin\alpha}{\nu} h^2 \frac{\partial h}{\partial x} = 0}$$

NL wave.

$$h = f\left(x - \frac{g \sin\alpha}{\nu} h^2 t\right)$$ → similarity ??!

i.e. compare:

$$\frac{\partial \alpha}{\partial t} + c\,\frac{\partial \alpha}{\partial x} = 0.$$

$$\alpha = \alpha(x - ct)$$

Try: $h = h(x/t^\alpha)$

$\Rightarrow$

$$-\frac{x}{t^{\alpha+1}}\,h' + g\,\frac{\sin\alpha}{\nu}\,h^2\,\frac{h'}{t^\alpha} = 0$$

$$\frac{h'}{t^\alpha}\left[-\frac{x}{t} + g\,\frac{\sin\alpha}{\nu}\,h^2\right] = 0$$

$$\boxed{h \sim x^{1/2}/t^{1/2}}$$ $\rightarrow$ similarity soln.

How fast does it spread downhill?

note

$$\int_0^{x_N(t)} h(x,t) = A$$

$$\rightarrow x_N \sim t^{2/3}.$$