Electrodynamics:

Field-strength tensor $F^{\mu\nu} = -F^{\nu\mu}$

$$F_{0i} = -F^{0i} = E^i \qquad F_{ij} = F^{ij} = -\epsilon^{ijk} B^k$$

I do not think much is gained by writing this as a 4x4 matrix but most textbooks do:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & -B^3 & B^2 \\ \text{anti-} & & 0 & -B^1 \\ \text{sym} & & & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & -B^3 & B^2 \\ \text{anti-} & & 0 & -B^1 \\ \text{sym} & & & 0 \end{pmatrix}$$

Charge-current (or 4-current - much like 4-momentum for $p^\mu = (\frac{1}{c}E, p^i)$).

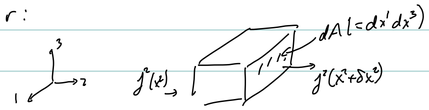
$$j^\mu = (c\rho, j^i)$$

Ex: Check the units (ie, engineering dimensions) $[c\rho] = [j^i]$.

Charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Reminder:



In this infinitesimal volume in time Δt

change in charge Δq = charge in − charge out

$= j^2(x^2) dA\, \Delta t - j^2(x^2+dx^2)\, dA\, \Delta t$

+ similar for other 2 pairs of faces

$= -\Delta t \left(\frac{\partial j^2}{\partial x^2} dx^2\, dx^1 dx^3 + 2 \text{ terms} \right) = -\Delta t\, dV\, \vec{\nabla} \cdot \vec{j}$

This is $\boxed{\partial_\mu j^\mu = 0}$

Maxwell's Equations

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} j^\mu \qquad \rightarrow \text{non-homogeneous} \quad \begin{cases} \text{Gauss} \\ \text{Ampere} \end{cases}$$

$$\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0 \qquad \rightarrow \text{homogeneous} \quad \begin{cases} \text{Faraday} \\ \text{Gauss for (absence of) magnetic "charge"} \end{cases}$$

Exercise: Verify this by writing the 4 equations corresponding to the 2 cases $\mu = 0$ and $\mu = i$ (x 2 equations).

Current conservation is required for consistency of Maxwell's equations:

$$\partial_\mu j^\mu = \partial_\mu (\partial_\nu F^{\nu\mu}) = \underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{F^{\nu\mu}}_{\text{anti-symmetric}} = 0$$

Note: Hom equation is, equivalently, $\partial_\nu F_{\lambda\sigma} + \partial_\lambda F_{\sigma\nu} + \partial_\sigma F_{\nu\lambda} = 0$

We are finally in a position to say something interesting in a trivial sort of way:

$$\boxed{\begin{array}{l}\text{Maxwell equations are form-invariant Lorentz transformations:}\\ F'^{\mu\nu}(x) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\lambda} F^{\rho\lambda}(\Lambda^{-1}x)\\ J'^{\mu}(x) = \Lambda^{\mu}{}_{\rho}\, J^{\rho}(\Lambda^{-1}x)\end{array}}$$

Exercise: although fairly explicit, you should know how to verify this statement!

We can explore some consequences by looking at components for specific transformations. We know everything (?) about rotations, so concentrate on boosts. Take boosts in $x^1 = x$-dir

$$-E'^{i} = F'^{0i} = \Lambda^{0}{}_{\mu}\Lambda^{i}{}_{\nu} F^{\mu\nu} \quad \text{with} \quad \Lambda = \begin{pmatrix} c & -s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$c = \cosh\eta$
$s = \sinh\eta$
($x' = cx - sx^0 = \gamma(x - vt)$
so $x' = 0$ is $x = vt$ $\Rightarrow \beta = \frac{v}{c}$ is vel of K' in K).

Skip the following tedious calculation in class:

$$-E'^{1} = \Lambda^{0}{}_{\mu}\Lambda^{1}{}_{\nu} F^{\mu\nu} = \Lambda^{0}{}_{0}\Lambda^{1}{}_{k} F^{0k} + \Lambda^{0}{}_{1}\Lambda^{1}{}_{0} F^{10} = c^2(-E^1) + s^2(E^1) = -E^1$$

$$-E'^{2} = \Lambda^{0}{}_{\mu}\Lambda^{2}{}_{\nu} F^{\mu\nu} = \Lambda^{0}{}_{0}\delta^{2}_{\nu} F^{0\nu} + \Lambda^{0}{}_{1}\delta^{2}_{\nu} F^{1\nu} = c(-E^2) + (-s)(-B^3) = -(cE^2 - sB^3)$$

$$-E'^{3} = \text{idem, note that } F^{13} = +B^2 \qquad = -(cE^3 + sB^2)$$

$$-B'^{3} = F'^{12} = \Lambda^{1}{}_{\mu}\Lambda^{2}{}_{\nu} F^{\mu\nu} = \Lambda^{1}{}_{0} F^{02} + \Lambda^{1}{}_{1} F^{12} = (-s)(-E^2) + c(-B^3) = -(cB^3 - sE^2)$$

$$B'^{2} = F'^{13} = \text{idem} \qquad = cB^2 + sE^3$$

$$-B'^{1} = F'^{23} = \Lambda^{2}{}_{\mu}\Lambda^{3}{}_{\nu} F^{\mu\nu} = F^{23} = -B^1$$

or

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B})$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \qquad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E})$$

where $\vec{\beta} = \frac{1}{c}$(velocity of unprimed system in primed one) $= -\frac{1}{c}$(vel. of primed system as seen from unprimed).

Note: $\vec{E}$ & $\vec{B}$ are clearly NOT components of some "E^{μ}" & "B^{μ}" 4-vectors.

Properties under discrete transformations P & T.

To determine transformation properties of $\vec{E}$ & $\vec{B}$, start from $\rho = j^0$. We know $\rho' = \rho$ under both P & T

So $j^\mu = (j^0 = \rho c, \vec{j})$ is a pseudo-tensor under T but not under P.

So we should have $\vec{j}' = -\vec{j}$, and this is physically sensible: P reverses current direction, as does T (running movie backwards).

Now ∂_μ is a vector (not-pseudo). So in

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

we'll better have that $F'^{\mu\nu} = \underline{\underline{(+1)}} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$ for any LT: $F^{\mu\nu}$ is also not-pseudo.

So what does this mean for $\vec{E}$ & $\vec{B}$?

$$E'^i = F'^{i0} = -F^{i0} = -E^i \text{ under P}, \qquad E'^i = E^i \text{ under T}$$

$$B'^i = \underset{\text{(cyclic)}}{F'^{jk}} = F^{jk} = B^i \text{ under P}, \qquad B'^i = -B^i \text{ under T}$$

In accord with intuition.

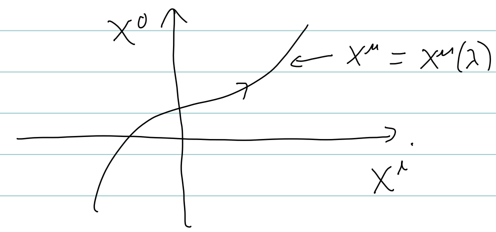
Exercise: verify that the Lorentz force equation is covariant under P & T

That laws of nature are invariant under any symmetries is an empirically determined fact. Could it be that they are only invariant under proper LT's but not under P nor T? YES, weak interactions are do not respect either.

But electromagnetic, strong, and gravitational interactions do respect both P & T. This will come handy in solving problems, modeling systems, etc.

Kinematics of Relativistic Particle

World-line: or particle trajectory

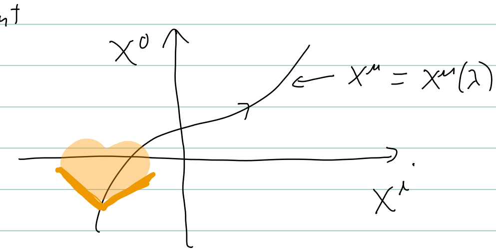


Not every collection of 4 functions $(x^0(\lambda), x^i(\lambda))$ is acceptable as describing the motion of a point particle:

(i) Move forward in time, or $\frac{dx^0}{d\lambda} > 0$

(ii) Move no faster than speed of light : $|\vec{v}| \leq c \iff \frac{\left|\frac{d\vec{x}}{d\lambda}\right|}{\frac{dt}{d\lambda}} \leq c \iff \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\eta_{\mu\nu} \geq 0$

(i)+(ii) ie, it lies in future light cone at each point of world line



Above we used condition (i) to implicitly invert $x^0 = x^0(\lambda)$ to give $\lambda = \lambda(x^0)$ (or $\lambda = \lambda(t)$) to write $\frac{d\vec{x}}{dt}(\lambda(t)) = \frac{d\vec{x}}{d\lambda} \cdot \frac{d\lambda}{dt}$.

We could have equally chosen a different parametrization of the same physical trajectory in the same coordinate frame:

Say the functions $x_1^\mu(\lambda_1)$ & $x_2^\mu(\lambda_2)$ are physically the same.

It must be that if we write λ_2 in terms of λ_1 then $x_2(\lambda_2)$ agrees with $x_1(\lambda_1)$:

$$x_2^\mu(\lambda_2(\lambda_1)) = x_1^\mu(\lambda_1)$$

That is, they both give the same trajectory $\vec{x}(t)$ as function of time $t = \frac{x^0}{c}$.

Two common preferred parameters:

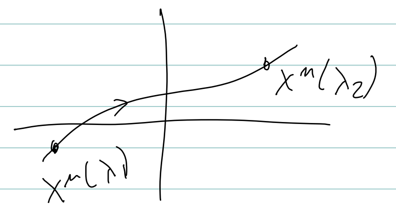
(i) $\lambda = x^0$, so $x^\mu(x^0) = (x^0, x^i(x^0))$

(ii) $\lambda = \tau$ (proper time) $x^\mu(\tau)$, so that $\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = 1$

(Note, I use τ in units of space; often elsewhere $d\tau^2 = \frac{1}{c^2}dx^\mu dx_\mu$).

To be clear $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. We often use the symbol $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ for general intervals, and reserve $d\tau$ for intervals that satisfy $d\tau^2 > 0$.

Given a parametrization $x^\mu = x^\mu(\lambda)$ we can give the proper distance between two events $x^\mu(\lambda_1)$ & $x^\mu(\lambda_2)$ along $x^\mu(\lambda)$ by



$$S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} ds$$

$$\text{or} \quad S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}\, d\lambda$$

$\frac{dx^\mu}{d\lambda}$ is a tangent vector to $x^\mu(\lambda)$

When x^μ is parametrized by τ we call this a 4-velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

It satisfies $u^2 = 1$

It's τ derivative is the 4-acceleration $a^\mu \equiv \frac{du^\mu}{d\tau}$

Since

$$\frac{d(u^2)}{d\tau} = \frac{d1}{d\tau} = 0 \qquad \text{we have} \quad a^\mu u_\mu = 0$$

A particle of mass m has 4-momentum $p^\mu \equiv mcu^\mu$

It satisfies $p^2 = m^2c^2$. m is a Lorentz invariant (since u^μ is a 4-vector and we want our definition of p^μ to also give a 4-vector). The zeroth component is called energy $p^\mu = (p^0, p^i) = \left(\frac{E}{c}, p^i\right)$

We often reserve the symbol v^i for a 3-vector that corresponds to the velocity of the particle: $v^i = \frac{dx^i}{dt}$

$$U^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} \qquad U^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} \quad \Rightarrow \quad U^i = \frac{v^i}{c}\, U^0$$

$$\text{Since } U^2 = 1 \qquad U^{02} - \vec{U}^2 = U^{02}\left(1 - \frac{\vec{v}^2}{c^2}\right) = 1 \quad \Rightarrow \quad U^0 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

It is customary (and sometimes confusing, beware!) to use the same symbols here as for Lorentz transformations: $\vec{\beta} = \vec{v}/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

so that

$$U^\mu = (\gamma, \gamma\beta^i) \quad \text{and} \quad p^\mu = (\gamma mc, \gamma\beta^i mc)$$

$$\text{or} \quad E = \frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} \quad \text{and} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}.$$

This Lorentz-boost notation is natural in that one can obtain these results from boosting to an arbitrary frame with velocity $-\vec{v}$ from the (possibly instantaneous) rest frame

$$U^\mu = (1, 0)$$

In the instantaneous rest frame, $\vec{p}=0$, $E = mc^2$ and $a^\mu = (0, a^i)$

Lorentz Force

We are ready to write $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$ in covariant form. It is easy to "derive" the covariant form.

Since $\frac{d\vec{p}}{dt} = \vec{F}$, we look for an equation with

$$c\frac{dp^\mu}{d\tau} = \text{4 vector made of } \vec{E}, \vec{B} \text{ and } \vec{v}$$

Now $\vec{E}, \vec{B}$ are encoded in the 2-index tensor $F^{\mu\nu}$, and $\vec{v}$ in the 4 vector U^μ. The 4-vector we search for must be linear in $F^{\mu\nu}$ (since $\vec{F}$ is linear in $\vec{E}, \vec{B}$); it is not necessarily linear in U^μ since $U^2 = 1$. Possibilities

$$F^{\mu\nu}U_\nu, \qquad F^{\sigma\nu}U_\sigma U_\nu U^\mu = 0, \qquad F^{\sigma\nu}\eta_{\sigma\nu}U^\mu = 0$$

(the last two vanish by $F^{\sigma\nu} = -F^{\nu\sigma}$, $U^\sigma U^\nu = U^\nu U^\sigma$, $\eta^{\sigma\nu} = \eta^{\nu\sigma}$).

So $\frac{dp^\mu}{d\tau} \propto F^{\mu\nu}U_\nu$. The proportionality constant is q: $\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}U_\nu}$

It is understood that the field $F^{\mu\nu}(x)$ is at $x^\mu = x^\mu(\tau)$.

Note $$c\frac{dp^i}{d\tau} = q\left(F^{i0}U^0 - F^{ij}U^j\right)$$
$$= q\left(E^i U^0 + \epsilon^{ijk}B^k U^j\right)$$
$$= qU^0\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)^i$$

and since $c\frac{dp^i}{d\tau} = \frac{dp^i}{dt}U^0$ $\Rightarrow$ $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

What about $c\frac{dp^0}{d\tau} = \frac{1}{c}U^0\frac{dE}{dt}$? It is not really an independent equation

since $U^\mu a_\mu = 0$ $\Rightarrow$ $U^0\frac{dp^0}{d\tau} = U^i\frac{dp^i}{d\tau}$ $\Rightarrow$ $\frac{dE}{dt} = c\frac{U^i}{U^0}\frac{dp^i}{dt} = \vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{m}{2}v^2\right)$

The right hand side is $q\left(F^{0i}U_i\right) = q\left(-F^{0i}U^i\right) = q\vec{E}\cdot\left(\gamma\frac{\vec{v}}{c}\right)$

so $$\frac{dE}{dt} = q\,\vec{v}\cdot\vec{E}$$ (E on LHS is energy, $\vec{E}$ on RHS is Electric field).

This follows trivially from $\frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) = \vec{v}\cdot\vec{F}$ with $\vec{F} = \frac{d}{dt}(m\vec{v})$.

Note that $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$ is NOT $m\frac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

because $$\frac{d\vec{p}}{dt} = m\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1-v^2/c^2}}\right)$$

Exercise: show
$$m\ddot{\vec{x}} = q\sqrt{1-\vec{v}^2/c^2}\left[\vec{E} - \left(\frac{\vec{v}}{c}\cdot\vec{E}\right)\frac{\vec{v}}{c} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

Vector Potential, 4-vectorpotential,

Recall from electrostatics $\vec\nabla\times\vec E=0 \Rightarrow \vec E=-\vec\nabla\phi$ for some electrostatic potential $\phi=\phi(\vec x)$.

We'd like to generalize this to include $\vec B$ field & time dependence.

$$\vec\nabla\times\vec E+\frac{1}{c}\frac{\partial\vec B}{\partial t}=0 \qquad (3)$$

We can also use the other homogeneous equation

$$\vec\nabla\cdot\vec B=0 \qquad (4)$$

Start from (4): since $\vec\nabla\cdot(\vec\nabla\times\vec A)=0$ for any $\vec A$, we propose $\vec B=\vec\nabla\times\vec A$
Can this hold for any $\vec B$? Let (I am not balancing indices, using $\delta_{\mu\nu}$ for metric):

$$A^i=-\frac{1}{\nabla^2}(\vec\nabla\times\vec B)^i \quad\Rightarrow\quad (\vec\nabla\times\vec A)^i=-\epsilon^{ijk}\partial_j\left(\frac{1}{\nabla^2}\epsilon^{kmn}\partial_m B^n\right)$$
$$=-\frac{1}{\nabla^2}\left(\partial_i\vec\nabla\cdot\vec B-\nabla^2B^i\right)$$

So, provided $\vec\nabla\cdot\vec B=0$ and ∇^2 is invertible (ie $\vec B$ is well behaved) then $\vec B=\vec\nabla\times\vec A$ is always possible. It is not unique:

$$\vec B=\vec\nabla\times\vec A'=\vec\nabla\times\vec A \iff \vec\nabla\times(\vec A'-\vec A)=0$$

. We know the solution to this!
It is just like electrostatics: $\vec A'-\vec A=-\vec\nabla\omega$ for some scalar ω.

Now for (3): using $\vec B=\vec\nabla\times\vec A$

$$\vec\nabla\times\vec E+\frac{1}{c}\frac{\partial}{\partial t}(\vec\nabla\times\vec A)=0 \quad\text{or}\quad \vec\nabla\times\left(\vec E+\frac{1}{c}\frac{\partial\vec A}{\partial t}\right)=0 \Rightarrow \vec E+\frac{1}{c}\frac{\partial\vec A}{\partial t}=-\vec\nabla\phi \quad \text{as before.}$$

Summary: (3) & (4) insure we can write

$$\vec E=-\vec\nabla\phi-\frac{1}{c}\frac{\partial\vec A}{\partial t} \qquad \vec B=\vec\nabla\times\vec A$$

Names:
$\vec A$ "vector potential".

Note: if $\vec A\to\vec A'=\vec A-\vec\nabla\omega$ and $\phi\to\phi'=\phi+\frac{1}{c}\frac{\partial\omega}{\partial t}$ then $E\to E$ & $B\to B$

This is called "gauge invariance" of $\vec E$ & $\vec B$.

Relativistic extension

We can combine the above expressions as follows:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

Check: $F_{0i} = E^i$, $\quad F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \partial_i \phi$

works! provided $\phi = A^0 = A_0$; and $\vec{A}$ is A^i

$$F_{ij} = -\epsilon_{ijk} B^k = \partial_i A_j - \partial_j A_i = -(\partial_i A^j - \partial_j A^i)$$

or $\quad B^3 = \partial_1 A^2 - \partial_2 A^1$ etc ✓

Alternatively, $\quad \epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0 \iff F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(This is the 4D generalization of $\nabla \times \vec{B} = 0 \iff \vec{B} = \vec{\nabla}\times\vec{A}$).

A_μ is also called "vector potential", often also called a gauge field.

Gauge invariance: $\quad A_\mu \to A_\mu + \partial_\mu \omega$

Trivially seen directly: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \partial_\mu(A_\nu + \partial_\nu \omega) - \partial_\nu(A_\mu + \partial_\mu \omega) = \partial_\mu A_\nu - \partial_\nu A_\mu$

since $\quad \partial_\mu \partial_\nu \omega - \partial_\nu \partial_\mu \omega = 0$.

Non-homogeneous equations in terms of A_μ:

Now $\quad \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial\cdot A$

Here $\quad \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \vec{\nabla}^2$ is also "$\Box$" or "D^2" in textbooks

is the "d'Alembert operator", or "d'Alembertian" which appears prominently in the wave equation.

(Also $\partial\cdot A = \eta^{\mu\nu}\partial_\mu A_\nu$). So Maxwell (1) & (2) are

$$\partial^2 A_\mu - \partial_\mu(\partial A) = \frac{4\pi}{c} j_\mu \qquad (X)$$

An application of gauge invariance: since $F_{\mu\nu}$ is invariant under $A_\mu \to A'_\mu = A_\mu + \partial_\mu \omega$, given a solution A_μ of (X) above we can find another one, A'_μ, such that $\partial \cdot A' = 0$, since

in $\partial \cdot A' = \partial \cdot A + \partial^2 \omega = 0$ provided the solution to

$$\omega = -\frac{1}{\partial^2}\partial \cdot A$$ exists.

This is generally the case. So we can look for solutions to (X) that satisfy both (X) and $\partial \cdot A = 0$. But the latter means the former is

$$\partial^2 A_\mu = \frac{4\pi}{c} j_\mu \quad \Rightarrow \quad \boxed{A_\mu = \frac{4\pi}{c}\frac{1}{\partial^2} j_\mu}$$

The procedure above is called "gauge fixing", and the field A_μ that satisfies $\partial \cdot A = 0$ is said to be in "covariant" or "Lorentz" gauge.

Inverting differential operators. Green functions. Distributions.

We have seen a need for inverting ∇^2 & ∂^2. Let's!

For ∇^2 we may as well put it in the important context of solving the Poisson Equation. This arises in electrostatics (and in Newtonian gravitation):

since $\vec{E} = -\vec{\nabla}\phi$ for static fields, we have then

$$\text{Gauss's law } \vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow -\nabla^2\phi = 4\pi\rho \Rightarrow \phi = -4\pi\frac{1}{\nabla^2}\rho$$

Solve $\quad \nabla^2\phi = -4\pi\rho$

"Solve" means, given $\rho = \rho(\vec{x})$, determine $\phi(\vec{x})$; there are other related problems, in which we specify $\rho = \rho(\vec{x})$ and a boundary to the volume under consideration, and specify a boundary condition. In our case, we want $\rho(\vec{x})$ with compact support and $\phi = \phi(\vec{x})$ everywhere in space, up to solutions to $\nabla^2\phi = 0$.

ρ has compact support, and is sufficiently well behaved, so $\int d^3x\, \rho^2 < \infty \Rightarrow$ we can find its Fourier transform

$$\tilde{\rho}(\vec{k}) \equiv \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\rho(\vec{x}) \quad \Rightarrow \quad \rho(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\tilde{\rho}(\vec{k})$$

Do this for ϕ as well. Then

$$\nabla^2\phi = \frac{1}{(2\pi)^3}\vec{\nabla}^2\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3k\,(-\vec{k}^2)\tilde{\phi}(\vec{k})$$

$$\Rightarrow \quad -\vec{k}^2\tilde{\phi}(\vec{k}) = -4\pi\tilde{\rho}(\vec{k}) \quad \Rightarrow \quad \tilde{\phi}(\vec{k}) = -\frac{1}{\vec{k}^2}\left(-4\pi\tilde{\rho}(\vec{k})\right) = \frac{4\pi}{\vec{k}^2}\tilde{\rho}(k)$$

and $\quad \phi(\vec{x}) = \dfrac{4\pi}{(2\pi)^3}\displaystyle\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\frac{1}{k^2}\tilde{\rho}(\vec{k})$

We can express this back in terms of $\rho(\vec{x})$:

$$\phi(\vec{x}) = \frac{4\pi}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\frac{1}{k^2}\int d^3y\, e^{-i\vec{k}\cdot\vec{y}}\rho(\vec{y}) = \int d^3y\, G(\vec{x},\vec{y})\,\rho(\vec{y})$$

Here $G(\vec{x},\vec{y})$ is a Green function, satisfying $\nabla^2_{(x)}G(\vec{x},\vec{y}) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$

where $\nabla^2_{(x)}$ means it is the Laplacian with respect to $\vec{x}$ (to distinguish from $\vec{y}$), and $\delta^{(3)}(\vec{x})$ is the 3D Dirac-delta function.

$\delta^{(3)}(\vec{x}) = \delta(x)\,\delta(y)\,\delta(z)$, with $\delta(x)=0$ for $x\neq 0$ and $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$

so that $\int d^3x\, \delta^{(3)}(\vec{x}) = 1$

Just like $\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} = 2\pi\delta(t)$, we have $\int d^3k\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\,\delta^{(3)}(\vec{x})$

Now, we read off from above

$$G(\vec{x},\vec{y}) = \frac{4\pi}{(2\pi)^3}\int d^3k\; e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\frac{1}{\vec{k}^2} \qquad (G)$$

from which we learn that $G(\vec{x},\vec{y}) = G(\vec{x}-\vec{y})$ reflecting invariance under $\vec{x}\to\vec{x}+\vec{a}$ $\vec{a}$ = constant vector of Poisson equation.

Exercise: verify that G satisfies $\nabla^2 G = -4\pi\,\delta^{(3)}$ by direct computation

Note that we can add to this any solution to the associated homogeneous equation. if $\nabla^2 F = 0$ then $\nabla^2(G+F) = -4\pi\,\delta^{(3)}$ (provided $\nabla^2 G = -4\pi\delta^{(3)}$).

We'll take a brief look at $\nabla^2\phi = 0$ and boundary value problems in electrostatics later.

Physical interpretation: compare $\nabla_x^2\, G(x,y) = -4\pi\,\delta^{(3)}(\vec{x}-\vec{y})$ with $\nabla^2\phi = -4\pi\rho$

$\Rightarrow$ G is the electric potential due to a charge distribution $\rho(\vec{x}) = \delta^{(3)}(\vec{x}-\vec{y})$
That is $\rho = 0$ everywhere except at $\vec{x}=\vec{y}$, and $\int d^3x\,\rho = 1$
$\Rightarrow$ a unit charge at $\vec{y}$.

But we know $\phi = \frac{q}{r}$ for charge at origin: the $\propto \frac{1}{r}$ is from Coulomb's law
the constant of proportionality from $\int_V \vec{\nabla}\cdot\vec{E}\, dV = \int_{\partial V} \vec{E}\cdot\hat{n}\, da$ using $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$
so that LHS $= 4\pi q$ and $\vec{E} = -\nabla\phi$ with V = a sphere so RHS $= \int R^2 d\Omega \left(\frac{\partial}{\partial r}\left(-\frac{\kappa}{r}\right)\right)\Big|_{r=R} = 4\pi\kappa$.

$\Rightarrow$ $\phi = \frac{q}{|\vec{x}-\vec{y}|}$ for charge q at $\vec{y}$. Setting $q=1$, $G(\vec{x}-\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}$

Exercise:
(i) Check $\nabla^2 \frac{1}{|x|} = 0$ for $\vec{x}\neq 0$, and $\int \nabla^2 \frac{1}{|x|}\, d^3x = -4\pi$

(ii) Verify by direct integration of (G) that $G(\vec{x}) = \frac{1}{|x|}$.

Now, let's try to do for ∂^2 what we did for ∇^2:

Let $\partial^2 \phi = 4\pi\rho$ (which is Gauss's Law in Lorentz gauge).

We try the same technique

$$\tilde{\rho}(k) = \int d^4x \; e^{ik\cdot x} \rho(x)$$ where k^μ is a 4-vector, $k\cdot x = k^\mu x_\mu$

Incidentally, $k\cdot x = k^0 x^0 - \vec{k}\cdot\vec{x} = \omega t - \vec{k}\cdot\vec{x}$ so $\omega = k^0 c$ has the interpretation of frequency (while $\vec{k}$ is still a wave-vector).

and $$\rho(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot x} \tilde{\rho}(k)$$

As before $\partial^2 e^{ik\cdot x} = -k^2 e^{ik\cdot x}$ with $k^2 = \eta_{\mu\nu} k^\mu k^\nu = k^\mu k_\mu = k^{0\,2} - \vec{k}^2$

and $$\phi(x) = \int d^4y \; G(x,y)\, 4\pi\rho$$

with $G(x,y) = G(x-y)$ and $\partial^2 G(x) = \delta^{(4)}(x)$.

But there is one interesting twist: while formally

$$G(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\left(-\frac{1}{k^2}\right) \Rightarrow \partial^2 G = \frac{1}{(2\pi)^4}\int d^4k \; e^{ik\cdot x}(-k^2)\left(-\frac{1}{k^2}\right) = \delta^{(4)}(x)$$

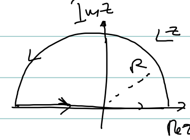
the actual integral is ill-defined:

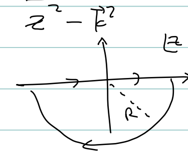
$$\frac{1}{k^2} = \frac{1}{(k^0)^2 - (\vec{k})^2}$$ diverges for $(k^0)^2 = (\vec{k})^2$

Consider the k^0 integration:

$$\int_{-\infty}^{\infty} dk^0 \; \frac{e^{-ik^0 x^0}}{(k^0)^2 - (\vec{k})^2}$$

Consider the complex variable integration $\displaystyle\int_C dz \; \frac{e^{-ix^0 z}}{z^2 - \vec{k}^2}$

where C is 

for $x^0 < 0$ and 

for $x^0 > 0$

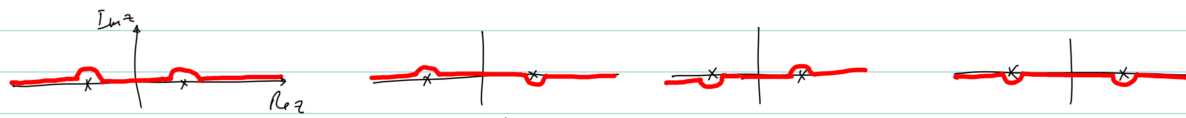
These choices guarantee $e^{-izx^0} = e^{-R|x^0|}$ on the circle and vanishes exponentially fast as $R \to \infty$.

This allows us to analyze our integral using Cauchy's theorem. The only problem is that there are poles at

$$z = +|\vec{k}| \quad \text{and} \quad z = -|\vec{k}|$$

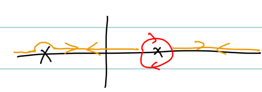
and the contour of integration goes through them.

So let's propose the following: G is defined by a choice of deformation of the contour to go just above or just below the poles. There are 4 possibilities:



(1) or G_{++} (2) G_{+-} (3) G_{-+} (4) G_{--}

By Cauchy's theorem the size of the deformation does not matter. So it can be infinitesimal, and the resulting integrals should all give Green functions $\Rightarrow$ the difference between any two should be a solution to the homogeneous equation. Easy to see: eg, take (1) − (2). Same as (1) followed by reversing direction of (2): cancels everywhere except at $z = +|\vec{k}|$



But this is an easy integral: $\displaystyle\int_{\substack{\text{little}\\ \text{circle}}} dz \frac{e^{-izx^0}}{(z-|k|)(z+|k|)} = -2\pi i \left.\frac{e^{-izx^0}}{z+|k|}\right|_{z=|k|} = \frac{-2\pi i\, e^{-i|k|x^0}}{2|k|}$

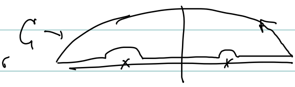
and $\displaystyle \Delta G = \frac{4\pi}{(2\pi)^3}\int d^3k \, \frac{e^{i\vec{k}\cdot\vec{x}-i|\vec{k}|x^0}}{2|\vec{k}|}$

This is a well defined integral and has $\partial^2(\Delta G) = 0$. In fact

Exercise: if $F(x) = \int d^3k \, f(\vec{k})\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$ is well defined, show that

$\partial^2 F = 0$ provided $\frac{\omega}{c} = \pm|\vec{k}|$

So all 4 Green functions are good candidates. But they have interesting physically distinguishing properties.

(1) G_{++}. Since there are no poles inside C for  corresponding to $x^0<0$ we have $G_{++}=0$ for $x^0<0$. This is called the "Retarded" Green fct.

(2) G_{--}. Similarly $G_{--}=0$ for $x^0>0$. "Advanced" Green functions

(3) Appears naturally in Quantum field theory, "Feynman" Green function and has the peculiar property that it can be deformed into an integral along imaginary axes. Not of interest to us for now.

Before we compute G_{Adv} and G_{ret} explicitly, note physical interpretation:
Since

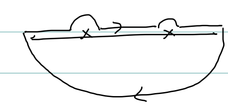
$$\phi(x) = \int d^4y\, G(x-y)\, \rho(y)$$

we have $\phi(\vec{x},x^0) = \int d^4y\, G_{ret}(x,y)\,\rho(\vec{y},y^0) = \int d^3y \int_{-\infty}^{x^0} dy^0\, G_{ret}(\vec{x}-\vec{y}, x^0-y^0)\,\rho(\vec{y},y^0)$

$\Rightarrow$ $\phi(\vec{x},t)$ depends on $\rho(\vec{x},t')$ only for times t' before t, ie, $t'<t$. This makes sense for future evolution: one must know the history of the charge distribution in the past to predict the future.

Similarly G_{Adv} gives $\phi(\vec{x},t)$ from knowledge of the future charge distribution which can be of interest in, say, scattering if we know the future outcome.

Explicit computation.



We have $G_{ret} = \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{x}} \int dz\, e^{-izx^0}\left(\frac{-1}{z^2-\vec{k}^2}\right) \longrightarrow$

$$= \frac{1}{(2\pi)^4}\int d^3k\, e^{i\vec{k}\cdot\vec{x}} \left[(-1)(-2\pi i)\left(\frac{e^{-i|k|x^0}}{2|k|} + \frac{e^{i|k|x^0}}{-2|k|}\right)\right.$$

$$= \frac{i}{2(2\pi)^3}\int d^3k\, \frac{1}{|k|}\, e^{i\vec{k}\cdot x}\left(e^{-i|k|x^0} - e^{i|k|x^0}\right)$$

Now $\int d^3k \, \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|} e^{\mp ikx^0} = 2\pi \int_0^\infty k^2 dk \int_{-1}^{1} d\cos\theta \, \frac{1}{k} e^{\mp ikx^0} e^{ikx\cos\theta}$ $x = |\vec{x}|$ for short

$$= 2\pi \frac{1}{(ix)} \int_0^\infty dk \, e^{\mp ikx^0} \left(e^{ikx} - e^{-ikx} \right)$$

Combining

$$G_{ret} = \frac{i}{2(2\pi)^3} \frac{2\pi}{i|x|} \int_0^\infty dk \left(e^{ikx} - e^{-ikx}\right)\left(e^{-ikx^0} - e^{+ikx^0}\right)$$

$$= \frac{1}{(2\pi)^2} \frac{1}{|x|} \cdot \frac{1}{4} \int_{-\infty}^{\infty} dk \left(e^{ikx} - e^{-ikx}\right)\left(e^{-ikx^0} - e^{ikx^0}\right)$$ since integrand is even under $k \to -k$.

$$= \frac{1}{(2\pi)^2} \frac{1}{|x|} \frac{1}{4} 2\pi \left[2\delta\left(|\vec{x}| - x^0\right) - 2\delta\left(|\vec{x}| + x^0\right) \right]$$

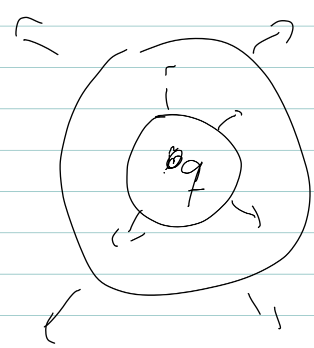
and since this is only for $x^0 > 0$, $\delta(|\vec{x}| + x^0) = 0$ so

$$\boxed{G_{ret}(\vec{x}, x^0) = \frac{1}{4\pi|\vec{x}|} \delta(x^0 - |\vec{x}|)}$$

Ex: show that

$$G_{adv}(\vec{x}, x^0) = -\frac{1}{4\pi|\vec{x}|} \delta(x^0 + |\vec{x}|)$$

A charge that appears for an instant and then dissapears (not physical?), at $t = 0$, at the origin, produces a potential field that propagates at the speed of light, spherically outward from the origin, with amplitude $\frac{1}{|x|}$ concentrated on that shell at $|\vec{x}| = ct$:



Cool! ☺ Moreover, easy to see real physical applications: start from charge distribution at rest and then have it start moving.

Action Integral, Lagrangian, Conservation Laws

Obtaining the equations of motion (EOM) as extrema of an action integral S, is useful in various ways: exploring symmetries, quantization, extension to other models of reality, and so on.

One of the nice things about it (not emphasized in kindergarten) is that covariance of EOM under some transformation corresponds to invariance of S. This is powerful stuff: if you have a good reason to suspect nature is symmetric under a group G of transformations, then make sure S is invariant under G.

(Digression: for those of you who know group theory, the set of physical transformations on a system forms a group, G, with multiplication given as one transformation followed by the other:

(i) there is an identity element $e \in G$, the "do nothing" transformation
(ii) for every transformation $g \in G$ there is an inverse $g^{-1} \in G$ $\ni$ $g^{-1}g = gg^{-1} = e$, the "undo" what you "did"
(iii) if $g_1 \in G$, $g_2 \in G \Rightarrow g_1 g_2 \in G$).

An example is the EOM for a relativistic free particle (we'll add EM interactions later). It has world-line $x^\mu(\lambda)$. We want a functional $S[x^\mu(\lambda)]$ that is invariant under Lorentz transformations. It should also be independent of the way in which we parametrize the world-line, ie, it should be reparametrization invariant. We know the interval ds is invariant, $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ and written this way it is independent of λ.

We therefore have as simplest invariant action integral

$$S[x^\mu(\lambda)] = \alpha \int ds \qquad \text{, with } \alpha \text{ some constant.}$$

If we want to be more careful we can specify that this is for paths between x_1^μ and x_2^μ so that $x^\mu(0) = x_1^\mu$ and $x^\mu(1) = x_2^\mu$ are fixed end-points and write

$$S[x(\lambda); x_1, x_2] = \alpha \int_{x_1}^{x_2} ds$$

but often we leave x_1 & x_2 as implicitly understood. Note that, eg, $\int ds^2$ makes no sense (the square of an infinitesimal integrates to zero).

For computations use $ds^2 = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2 > 0$

$$S[x(\lambda)] = \kappa \int_0^1 d\lambda \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

Now $S[x+\delta x] - S[x] = \kappa \int_0^1 d\lambda \left(\sqrt{\eta_{\mu\nu}(\dot{x}+\delta\dot{x})^\mu(\dot{x}+\delta\dot{x})^\nu} - \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \right)$ where $\dot{x} = \frac{dx^\mu}{d\lambda}$

$$= \kappa \int_0^1 dx \, \frac{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$$

$$= -\kappa \int_0^1 d\lambda \, \delta x^\nu \frac{d}{d\lambda}\left[\frac{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}}{\sqrt{\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right] + \text{boundary terms}$$

So that the EOM is
$$\frac{d}{d\lambda}\left[\frac{\kappa}{\sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\frac{dx^\mu}{d\lambda}\right] = 0 \qquad (*)$$

If we choose $\lambda = x^0$, then this is
$$\frac{d}{dt}\left(\frac{\kappa}{\sqrt{1-v^2/c^2}}\right) = 0 \qquad \frac{d}{dt}\left[\frac{\kappa}{\sqrt{1-\vec{v}^2/c^2}}\vec{v}\right] = 0 \qquad (@)$$

but no $\mu=0$ equation because there is no variation δx^0 of the trajectory (since $x^0(\lambda)=\lambda$ is fixed). Using $\lambda = x^0$ we can connect to Lagrangian mechanics; generally

$$S[q(t); t_1, t_2] = \int_{t_1}^{t_2} dt\, L(q,\dot{q})$$ ← Lagrangian, a function of two variables

and EOM's are $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$ and momentum $p = \frac{\partial L}{\partial \dot{q}}$

The relativistic particle has, from above
$$S = \kappa c\int_{t_1}^{t_2} \sqrt{1-\vec{v}^2/c^2}\, dt \quad . \text{ So } \quad L(x^i, v^i) = \kappa c\sqrt{1-\vec{v}^2/c^2}, \quad p^i = \frac{\partial L}{\partial v^i} = \frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{-\kappa v^i}{c}\right)$$

We see that with $\kappa = -mc$ this corresponds to momentum as found earlier (plus S then has the proper dimensions). For EOM we have
$$\frac{\partial L}{\partial v^i} = \frac{1}{\sqrt{1-v^2/c^2}} m v^i \qquad \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{m}{\sqrt{1-v^2/c^2}} v^i\right) = 0 \quad \text{as above.}$$

Note that this is just $\frac{d\vec{p}}{dt} = 0$

The non-relativistic limit $L = -mc^2\sqrt{1-v^2/c^2} = -mc^2\left[1 - \frac{1}{2}\frac{v^2}{c^2} + \dots\right] = -mc^2 + \frac{m}{2}v^2 + \cdots$

$-mc^2$ is an irrelevant, BUT INTERESTING, constant: if we write $L = T - V$ where $T = \frac{1}{2}mv^2$ is kinetic and V is potential energies, then $V = mc^2$, which the student should recognize as the rest energy of a particle in special relativity.

The Hamiltonian (or energy, since there is no explicit time dependence) is

$$H(q,p) = \dot{q}p - L$$

where $\dot{q} = \dot{q}(q,p)$ is obtained from $p = \frac{\partial L}{\partial \dot{q}}$.

In the present case ($H = H(q,p)$, so rewrite by solving for $\vec{v}$ in terms of $\vec{p}$)

$$H = v^i \frac{mv^i}{\sqrt{1-\vec{v}^2/c^2}} - \left(-mc^2\sqrt{1-\vec{v}^2/c^2}\right)$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}}\left[\left(1-\frac{\vec{v}^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{mc^2}{\sqrt{1-v^2/c^2}} = E$$

The Hamiltonian should be written in terms of p,

$$\frac{\vec{p}^2}{m^2c^2} = \frac{\vec{v}^2/c^2}{1-\vec{v}^2/c^2} \Rightarrow \frac{\vec{v}^2}{c^2}\left(1+\frac{\vec{p}^2}{m^2c^2}\right) = \frac{\vec{p}^2}{m^2c^2} \Rightarrow \frac{\vec{v}^2}{c^2} = \frac{\vec{p}^2}{m^2c^2+\vec{p}^2} \Rightarrow 1-\frac{\vec{v}^2}{c^2} = \frac{m^2c^2}{(mc)^2+\vec{p}^2}$$

So $E = \sqrt{(mc^2)^2 + \vec{p}^2c^2}$. No surprise!

Note that (@) above is $\frac{dE}{dt} = 0$.

The variation (functional derivative) defines the 4-momentum

$$\frac{\delta S}{\delta x^\mu(s)} = -p_\mu \frac{d}{d\lambda}\delta x^\mu$$

This makes it clear that p_μ transforms as a lower index 4vector. Using $\lambda = t$ the components of p^μ are what we saw above

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

so they form a 4-vector. Using the expression for E above, $p^2 = E^2/c^2 - \vec{p}^2 = (mc)^2$

Conservation Laws & Continuous Symmetries

Let us write generically

$$S[x^\mu] = \int_0^1 d\lambda \; \mathcal{L}\left(x^\mu, \frac{dx^\mu}{d\lambda}\right)$$

where $\mathcal{L}(x^\mu, v^\mu)$ is a function of these two variables. If the parameter where time $(\lambda = t)$ then $\mathcal{L}$ would correspond with the Lagrangian.

We can recover equations of motion by extremizing S:

$$\delta S = 0 \Rightarrow \int_0^1 d\lambda \left[\frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial v^\mu} \frac{d\delta x^\mu}{d\lambda} \right] = 0$$

Here $\frac{\partial \mathcal{L}}{\partial x^\mu}$ and $\frac{\partial \mathcal{L}}{\partial v^\mu}$ are understood to be evaluated on $v^\mu = \frac{dx^\mu}{d\lambda}$. Integrating by parts:

$$\frac{\partial \mathcal{L}}{\partial v^\mu} \delta x^\mu \Big|_0^1 + \int_0^1 d\lambda \left[\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) \right] \delta x^\mu = 0$$

The first term vanishes because $\delta x^\mu = 0$ at $\lambda = 0, 1$. The second term must then vanish, for any $\delta x^\mu(\lambda) \Rightarrow$ the $[\,..\,]$ must vanishes at each λ. This is a version of the Euler-Lagrange equation:

$$\boxed{\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) = 0} \qquad \text{(EOM)}$$

We define 4-momentum by

$$p_\mu = -\frac{\partial \mathcal{L}}{\partial v^\mu}$$

Symmetries: suppose that $x^\mu \to x'^\mu + \epsilon^\mu(x) + O(\epsilon^2)$ is a symmetry of S, that is $S[x'] = S[x]$

Then ↖ infinitesimal

$$0 = S[x'] - S[x] = \int d\lambda \left[\frac{\partial \mathcal{L}}{\partial x^\mu} \epsilon^\mu(x) + \frac{\partial \mathcal{L}}{\partial v^\mu} \frac{d\epsilon^\mu(x)}{d\lambda} \right]$$

Now, evaluating this for solutions of the EOM, $\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right)$ we have

$$0 = \int d\lambda \left[\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) \epsilon^\mu + \frac{\partial \mathcal{L}}{\partial v^\mu} \frac{d\epsilon^\mu}{d\lambda} \right] = \int d\lambda \frac{d}{d\lambda} \left[\frac{\partial \mathcal{L}}{\partial v^\mu} \epsilon^\mu(x) \right] = \frac{\partial \mathcal{L}}{\partial v^\mu} \epsilon^\mu \Big|_{in}^{fin} = -p_\mu \epsilon^\mu \Big|_{ini}^{fin}$$

or $p_\mu \epsilon^\mu(\text{final}) = p^\mu \epsilon_\mu(\text{initial})$

But "final" is arbitrary along world line $x^\mu(\lambda)$: this just means $\boxed{p_\mu \epsilon^\mu(x) = \text{constant along } x^\mu(\lambda)}$

Translations: suppose $x^\mu \to x^\mu + \epsilon^\mu$ with $\epsilon^\mu = $ constant is a symmetry.

Then $p_\mu \epsilon_\mu = $ constant. Since ϵ_μ is an arbitrary fixed vector this implies:

$p_\mu = $ constant : Conservation of momentum (as a result of translational invariance in space-time)

Angular-Momentum

Assume invariance under Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. The infinitesimal version is $x'^\mu = x^\mu + \epsilon^\mu{}_\nu x^\nu$ with $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ ($\epsilon_{\mu\nu} = \eta_{\mu\lambda}\epsilon^\lambda{}_\nu$)

$\Rightarrow \quad p_\mu \epsilon^\mu{}_\nu x^\nu = $ constant

$\Rightarrow \quad \epsilon_{\mu\nu} x^\mu p^\nu = $ constant

Since $\epsilon_{\mu\nu}$ is an arbitrary anti-symmetric tensor, it has 6 independent components that allow us to impose 6 independent conditions on $x^\mu p^\nu$. Since

$$x^\mu p^\nu = \tfrac{1}{2}(x^\mu p^\nu + x^\nu p^\mu) + \tfrac{1}{2}(x^\mu p^\nu - x^\nu p^\mu)$$

and $\epsilon_{\mu\nu} \cdot s^{\mu\nu} = 0$ for any $s^{\mu\nu} = s^{\nu\mu}$, it follows that the 6 components of

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

are constrained by the above, $M^{\mu\nu} = $ constant. This 2-index antisymmetric tensor contains angular momentum: $M^{ij} = x^i p^j - x^j p^i = \epsilon^{ijk} L_k$, where $L_k = \epsilon_{ijk} x^i p^k$ or $\vec{L} = \vec{r} \times \vec{p}$, as usual. But what is M^{0i}?

$$M^{0i} = x^0 p^i - x^i p^0 = \text{constant} \Rightarrow x^i = \left(\frac{p^i}{p^0}\right) x^0 + \text{constant}$$

In the non-relativistic limit: $p^0 = mc$ and $x^i = \frac{p^i}{m} t$, an elementary equation.

For many particles this is interesting:

$$M^{0i}_{tot} = \sum_n M^{0i}_n = x^0 \sum_n p^i_n - \sum_n x^i_n p^0_n = x^0 P^i_{TOT} - \sum x^i_n p^0_n$$

Define the center of energy (mass) as

$$X^i_{cm} = \frac{\sum_n x^i_n p^0_n}{\sum_n p^0_n} \quad \text{and total energy } P^0 = \sum_n p^0_n$$

Then $x^0 P^i = X^i_{cm} P^0$ or

$$X^i_{CM} = \left(\frac{P^i}{P^0}\right) x^0 = \frac{P^i}{E/c^2} t$$

This is the relativistic generalization of center of mass.

Interactions with E&M fields.

The non-relativistic limit of a point charge interacting with (i.e., moving under the effect of a) electric field described by a potential $\phi = A^0$ is described by a Lagrangian

$$L = \frac{1}{2} m \vec{v}^2 - q\,\phi(\vec{x}) \qquad (\text{recall } L = T - V)$$

Relativistic generalization: $\frac{1}{2}mv^2$, we have seen, is from $-mc\sqrt{1-\vec{v}^2/c^2}$.
Now, we want S to be invariant under Lorentz transformations. The kinetic term is explicitly invariant if we write again

$$S = -mc\int ds = -mc\int d\lambda\,\sqrt{\eta_{\mu\nu} U^\mu U^\nu}$$

with $U^\mu = \frac{dx^\mu}{d\lambda}$.

But $q\phi = qA^0$ is the 0-th component of a 4-vector. Now

$$U^\mu = \frac{dx^\mu}{ds} \quad \text{has} \quad U^0 = \frac{1}{\sqrt{1-\vec{v}^2/c^2}}, \quad U^i = \frac{v^i/c}{\sqrt{1-v^2/c^2}}$$

In the NR limit $U^0 \simeq 1$ and $U^i \simeq v^i$, so we can write

$$-q\,U^\mu A_\mu \simeq -q\phi + O(\vec{v})$$

The relativistic generalization is then

$$\int dt\,(-q\,\phi(\vec{x})) = \int dt\,\left(-q\,\frac{1}{c}\frac{dx^0}{dt}\phi\right) \to \int dt\left(-\frac{q}{c}\,\frac{dx^\mu}{dt}A_\mu\right)$$

or simply

$$S = \int -mc\,ds - \frac{q}{c}\,dx^\mu A_\mu$$

$$= \int d\lambda\left[-mc\sqrt{\eta_{\mu\nu}U^\mu U^\nu} - \frac{q}{c}U^\mu A_\mu\right]$$

Momentum conjugate; equation of motion (EOM)

Writing

$$S = \int dt\left[-mc^2\sqrt{1-v^2/c^2} - \frac{q}{c}\,\frac{dx^\mu}{dt}A_\mu\right] \Rightarrow L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + \frac{q}{c}\vec{v}\cdot\vec{A} - qA^0$$

we find the canonical momentum $\vec{P}$:

$$P^i = \frac{\partial L}{\partial v^i} = \underbrace{p^i}_{\text{as before}} + \frac{q}{c}A^i \qquad \text{or} \quad \boxed{\vec{P} = \vec{p} + \frac{q}{c}\vec{A}}$$

Now $H = \vec{P}\cdot\vec{v} - L = \vec{P}\cdot\vec{v} + mc^2\sqrt{1-v^2/c^2} - \frac{q}{c}\,\vec{v}\cdot\vec{A} + qA^0$

Writing $\vec{p} = \vec{P} - \frac{q}{c}\vec{A}$ this is

$$H = \vec{p}\cdot\vec{v} + mc^2\sqrt{1-\vec{v}^2/c^2} - qA^0(\vec{x},t)$$

The first two terms are as in the free case, so we can write immediately

$$H = \sqrt{\vec{p}^2c^2 + (mc^2)^2} + qA^0$$

or

$$\boxed{H = \sqrt{(\vec{P} - \frac{q}{c}\vec{A})^2 + (mc^2)^2} + qA^0}$$

At low velocity

$$L = \frac{1}{2}m\vec{v}^2 + \frac{q}{c}\vec{v}\cdot\vec{A} - qA^0$$

$$\text{and } H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + qA^0$$

For the EOM, we already have $\frac{\partial L}{\partial \vec{v}} = \vec{P} = \vec{p} + \frac{q}{c}\vec{A}$

but now we also need $\frac{\partial L}{\partial x^i}$. Recall that $A^0 = A^0(\vec{x})$ and $A^i = A^i(\vec{x})$, so we have

$$\frac{\partial L}{\partial x^i} = \frac{q}{c}v^j\frac{\partial A^j}{\partial x^i} - q\frac{\partial A^0}{\partial x^i}$$

So we have $\frac{d}{dt}(p^i + \frac{q}{c}A^i) = \frac{q}{c}v^j\partial_i A^j - q\partial_i A^0$

We want an equation for $\frac{dp^i}{dt}$ (p^i is shorthand for $\frac{mv^i}{\sqrt{1-v^2/c^2}}$). Now $\vec{A} = \vec{A}(\vec{x}(t),t)$ so

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \frac{dx^j}{dt}\frac{\partial\vec{A}}{\partial x^j} = \partial_t\vec{A} + v^j\partial_j\vec{A}$$

$$\Rightarrow \frac{dp^i}{dt} = \frac{q}{c}\Big(\underbrace{v^j(\partial_i A^j - \partial_j A^i)}_{[\vec{v}\times(\nabla\times\vec{A})]^i = (\vec{v}\times\vec{B})^i}\Big)\underbrace{- \frac{q}{c}\partial_t A^i - q\,\partial_i A^0}_{qE^i}$$

$$\text{or } \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

(Check of curl: $\partial_i A_j - \partial_j A_i = \epsilon_{ijk}\epsilon_{kmn}\partial_m A_n = \epsilon_{ijk}(\nabla\times\vec{A})_k$. Then this times $v^j \Rightarrow \epsilon_{ijk}v^j(\nabla\times\vec{A})^k$)

One can check by direct computation $\frac{dE_{kin}}{dt} = q\vec{v}\cdot\vec{E}$ (we know from $V^\mu\frac{dp^\mu}{d\tau} = 0$) where $E_{kin} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

$$\left(\frac{dE_{kin}}{dt} = mc^2\frac{1}{(1-v^2/c^2)^{3/2}}\frac{1}{c^2}\vec{v}\cdot\frac{d\vec{v}}{dt}, \text{ and } \vec{v}\cdot\frac{d\vec{p}}{dt} = \frac{m}{\sqrt{1-v^2/c^2}}\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{v^2}{c^2}\frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}} \right.$$

$$\left. \text{so } \vec{v}\cdot\frac{d\vec{p}}{dt} = \left(1-\frac{v^2}{c^2}\right)\frac{dE_{kin}}{dt} + \frac{v^2}{c^2}\frac{dE_{kin}}{dt} \Rightarrow \frac{dE_{kin}}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} = q\vec{v}\cdot\vec{E} \right).$$

Exercise: Obtain the covariant form of the EOM from using

$$S = \int d\lambda \left[-mc\sqrt{U^2} - \frac{q}{c} U^\alpha A_\alpha \right]$$

in Euler-Lagrange.

Solution:

$$-p_\alpha = \frac{\partial \mathcal{L}}{\partial U^\alpha} = -mc\frac{U_\alpha}{\sqrt{U^2}} - \frac{q}{c} A_\alpha$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{q}{c} U^\beta \partial_\alpha A_\beta$$

$$\Rightarrow \quad \frac{d}{d\lambda}\left(\frac{mc\, U_\alpha}{\sqrt{U^2}} + \frac{q}{c} A_\alpha \right) = \frac{q}{c} U^\beta \partial_\alpha A_\beta$$

or

$$\frac{d p_\alpha}{d\lambda} = \frac{q}{c}\left(U^\beta \partial_\alpha A_\beta - \underbrace{\frac{d}{d\lambda} A_\alpha}_{= \frac{dx^\beta}{d\lambda}\partial_\beta A_\alpha = U^\beta \partial_\beta A_\alpha} \right)$$

$$\Rightarrow \quad \frac{d p_\alpha}{d\lambda} = \frac{q}{c} U^\beta \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) = \frac{q}{c} U^\beta F_{\alpha\beta}$$

as expected.

Motion in constant crossed $\vec{E}, \vec{B}$ fields

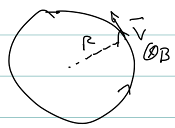
Warm-up: $\vec{E}=0$, $\vec{B}=$ constant

We know this from introductory courses: $\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$

except now $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$. However $\frac{dE_{kin}}{dt} = q\vec{v}\cdot\vec{E} = 0 \Rightarrow v^2 = $ constant

so $m\frac{d\vec{v}}{dt} = \sqrt{1-\frac{v^2}{c^2}}\; q\frac{\vec{v}}{c}\times\vec{B}$

$\Rightarrow$ $\vec{v}$ has a constant component along $\vec{B}$, plus it has a circular trajectory in the plane $\perp$ to $\vec{B}$.



$\frac{v_\perp^2}{R} = \sqrt{1-\frac{v^2}{c^2}}\;\frac{q}{mc}v_\perp B$ $\qquad v_\perp = \sqrt{1-\frac{v_\perp^2+v_\parallel^2}{c^2}}\;\frac{qRB}{mc}$

(or it gives R given initial $\vec{v}$).

— — — — . —

Now consider $\vec{E}$ & $\vec{B}$, with $\vec{E}\perp\vec{B}$, both uniform and constant. It is useful to 1st consider this in a frame moving in direction of $\vec{E}\times\vec{B}$. Let $\vec{E}=(0,E^2,0)$ & $\vec{B}=(0,0,B^3)$, and consider boost in $\vec{\beta}=(\beta,0,0)$ direction $\Lambda = \begin{pmatrix} c & -s \\ -s & c\end{pmatrix}$ with $s=-\gamma\beta$. We had computed

$$E'^1 = E^1 = 0,\quad E'^2 = cE^2 - sB^3,\quad E'^3 = cE^3 + sB^2 = 0$$
$$B'^1 = B^1 = 0,\quad B'^2 = cB^2 + sE^3 = 0\quad B'^3 = cB^3 - sE^2$$

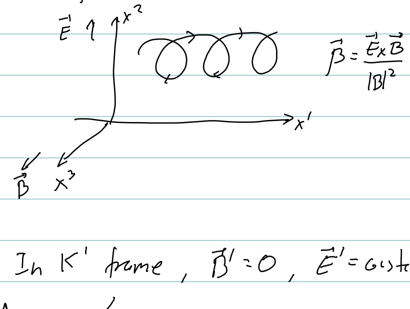
So $\vec{E}'$ & $\vec{B}'$ are still $\perp$, and along x^2 & x^3 axes, respectively. Now note that we can choose a frame

$E'^2 = 0$ provided $\beta = \frac{s}{c} = \frac{E^2}{B^3}$ which must satisfy $\beta<1$, ie, provided $|\vec{E}|<|\vec{B}|$

Similarly, $|\vec{B}'|=0$ in frame $\beta = \frac{B^3}{E^2}$, provided $|\vec{B}|<|\vec{E}|$.

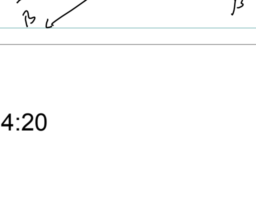
Motion

(i) $|\vec{E}|<|\vec{B}|$. In K' frame, $\vec{E}'=0$, $\vec{B}'=$ constant $\Rightarrow$ circular motion in plane $\perp$ to $\vec{B}$; in K frame we have the boost of this motion

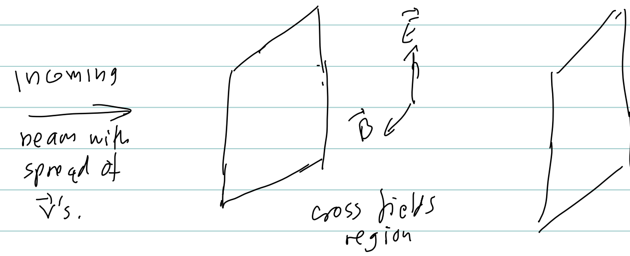


$\vec{\beta} = \frac{\vec{E}\times\vec{B}}{|B|^2}$ "$\vec{E}\times\vec{B}$ drift"
($\vec{\beta}$: "$E\times B$ drift velocity").

(ii) $|\vec{E}|>|\vec{B}|$. In K' frame, $\vec{B}'=0$, $\vec{E}'=$ constant. Rectilinear acceleration in K'



Velocity selector.



Consider one particle going into $E \times B$ region with velocity $\vec{v}$.
Let $\vec{U}_{/c} = \frac{\vec{E}\times\vec{B}}{|B|^2}$ (to differentiate from $\vec{v}$).

In K', where $\vec{E}'=0$, $\vec{v}' = \frac{\vec{v}-\vec{U}}{1-\frac{\vec{v}\cdot\vec{U}}{c^2}}$ or since all are in x-direction $v' = \frac{v-U}{1-vU/c^2}$.

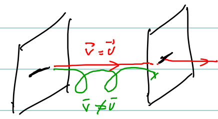
In K' this is circular motion with radius

$$R = \frac{mcv'}{\sqrt{1-v'^2/c^2}\,qB}$$

For $v'=0$, ie, $\vec{v}=\vec{U}$ the trajectory in K is a straight line ✓

For $v'\neq 0$ the trajectory is not straight, but

So use slits:



only $v = \frac{Ec}{B}$ goes through : velocity selector.

This followed by a magnetic spectrometer (pure $\vec{B}$ = constant), that selects $\vec{p}$ can be used for simultaneous $\vec{v}$ and $\vec{p}$ measurement $\Rightarrow$ measure mass and charge.

(The B = constant region selects $\frac{mv}{\sqrt{1-v^2/c^2}} = p = \frac{qB}{R}$ by measuring R.)

Elements of Classical Relativistic Field Theory

Before giving an action principle for $\vec{E}$ & $\vec{B}$, let's do a more general analysis — and simpler: consider first the continuum mechanics of a scalar field $\phi(x^\mu)$ ($\phi(x)$ for short).

$S[\phi]$ is a functional $\{\text{space of } \phi(x)\} \to \mathbb{R}$.

Lagrangian density $\mathcal{L}$:

$$S[\phi] = \int dt\, L = \int d^4x\, \mathcal{L}$$

with $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ a function of (5) variables, ϕ and $\partial_\mu\phi$.

(Again, think of this as a function $\mathcal{L}(\phi, a_\mu)$ which we evaluate at $a_\mu = \partial_\mu\phi$).

EOM (Euler-Lagrange equations):

$$\delta S[\phi] = 0 \Rightarrow \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\, \partial_\mu\delta\phi \right] = 0$$

(we have computed $S[\phi+\delta\phi] - S[\phi]$ for arbitrary function $\delta\phi(x)$).

There are implicit boundary conditions:

* initial/final: $\phi(\vec{x}, t_1) = \phi_1(\vec{x})$, $\phi(\vec{x}, t_2) = \phi_2(\vec{x})$ are initial and final field values.

* spatial infinity: assume $\phi(\vec{x}, t)$ is localized: $\phi(x^\mu) \to 0$ as $|\vec{x}| \to \infty$.

Integrate by parts and ignore boundary terms since $\delta\phi(x) = 0$ on boundaries (either $t = t_{1,2}$ or $|\vec{x}| = \infty$):

$$\int d^4x\, \delta\phi(x) \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right] = 0$$

$\delta\phi(x)$ is arbitrary:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0} \qquad \text{(EOM)}$$

Example:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi = \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

Then $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \eta^{\mu\nu}\partial_\nu\phi$ and $\frac{\partial\mathcal{L}}{\partial\phi} = 0$

$\Rightarrow$ EOM is $\partial_\mu(\eta^{\mu\nu}\partial_\nu\phi) = 0$ or $\partial^2\phi = 0$ ie ϕ satisfies wave equation.

Exercise: $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\mu^2}{2}\phi^2$

gives EOM $\partial^2\phi + m^2\phi = 0$ "Klein-Gordon" equation.

(often written $(\partial^2 + m^2)\phi = 0$).

Canonical momentum, Hamiltonian density

Define $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$ (∂_0 insted of ∂_t just rescales how we measure t).

and $\mathcal{H}(\pi,\phi) = \pi\partial_0\phi - \mathcal{L}$

Hamilton's equations follow from extrema of $S = \int d^4x\,(\pi\partial_0\phi - \mathcal{H})$

Ex: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\vec\nabla\phi)^2 - \frac{\mu^2}{2}\phi^2$

$$\pi = \partial_0\phi$$

$$\mathcal{H} = \pi\partial_0\phi - \left(\tfrac{1}{2}(\partial_0\phi)^2 - \tfrac{1}{2}(\vec\nabla\phi)^2 - \tfrac{\mu^2}{2}\phi^2\right)$$

or $$\mathcal{H} = \underbrace{\tfrac{1}{2}\pi^2}_{\text{"}K\text{"}} + \underbrace{\tfrac{1}{2}(\vec\nabla\phi)^2 + \tfrac{\mu^2}{2}\phi^2}_{\text{"}V\text{"}}$$

Note that $(\nabla\phi)^2$ is potential energy. The derivative is different in character that ∂_0: it is just the difference of two variables, $\phi(x+\epsilon)$ & $\phi(x)$, for $\epsilon = (0,\vec\epsilon)$, as $\vec\epsilon \to 0$. So $(\nabla\phi)^2$ belongs with $\mu^2\phi^2$!

Now $S = \int d^4x\,(\pi\partial_0\phi - \mathcal{H})$

$\delta S = 0 \Rightarrow \partial_0\phi - \pi = 0$ & $\underset{\text{int by parts}}{-\partial_0\pi} + \nabla^2\phi - \mu^2\phi = 0$ using $\pi = \partial_0\phi$ from 1st in 2nd $-\partial_0^2\phi + \nabla^2\phi - \mu^2\phi = 0$ or $-(\partial^2 + \mu^2)\phi = 0$ Klein-Gordon ✓

Continuous symmetries and Noether's theorem

We'll do this by first looking at explicit symmetries and then generalizing.

Consider an action $S = \int d^4x\, \mathcal{L}$ with $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ that does not depend explicitly on x^μ. Then this is invariant under $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$. In this case this invariance holds for any ϵ^μ independent of x. But we will consider ϵ infinitesimal, so that $\delta\phi = \phi(x+\epsilon) - \phi(x) = \epsilon^\mu \partial_\mu \phi$

It is useful to consider $\epsilon^\mu = \epsilon^\mu(x)$, and only specialize to the case that ϵ is x-independent at the end. We still have

$$\delta\phi = \epsilon^\mu \partial_\mu \phi$$

In addition, since $\partial_\mu\phi$ is a function of x

$$\delta\, \partial_\mu \phi = \epsilon^\nu \partial_\nu (\partial_\mu \phi)$$

Similarly since $\mathcal{L}$ is implicitly a function of x, we have $\delta\mathcal{L} = \epsilon^\mu \partial_\mu \mathcal{L}$. So

$$\delta S = \int d^4x\, \epsilon^\mu \partial_\mu \mathcal{L} \qquad (*)$$

Now, for any variation $\delta\phi$, we have

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \delta \partial_\lambda \phi \right]$$

and using the EOM

$$\delta S = \int d^4x \left[\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \partial_\lambda \phi} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda\phi)} \delta\partial_\lambda \phi \right]$$

$$= \int d^4x \left[\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \delta\phi \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \left(\delta \partial_\lambda \phi - \partial_\lambda \delta\phi \right) \right]$$

Using $x \to x + \epsilon$

$$= \int d^4x \left[\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \epsilon^\mu \partial_\mu \phi \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \left[\epsilon^\mu \partial_\mu \partial_\lambda \phi - \partial_\lambda (\epsilon^\mu \partial_\mu \phi) \right] \right]$$

$$= \int d^4x\, \epsilon^\mu \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi \right)$$

Since this is equal to $(*)$ we have

$$\int d^4x\, \epsilon^\mu \left[\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi \right) - \partial_\mu \mathcal{L} \right] = 0 \qquad \text{for arbitrary } \epsilon^\mu \text{, or}$$

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi \right) - \partial_\mu \mathcal{L} = 0 \;\Rightarrow\; \partial_\lambda \left[\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi - \delta^\lambda_\mu \mathcal{L} \right] = 0$$

This means the tensor $T^\lambda{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi - \delta^\lambda_\mu \mathcal{L}$

defines 4 conserved currents: $\partial_\lambda T^{\lambda\mu} = 0$

We have raised the second index $T^{\lambda\mu} = \eta^{\mu\nu} T^\lambda{}_\nu = \eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\nu \phi - \eta^{\lambda\mu} \mathcal{L}$

Example: With $\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

Now

$$\partial_\mu T^\mu{}_\nu = \underbrace{\partial^2 \phi}_{\substack{= 0 \\ \text{by EOM}}} \partial_\nu \phi + \eta^{\mu\alpha} \partial_\mu \phi \partial_\alpha \partial_\nu \phi - \eta^{\alpha\beta} \partial_\alpha \phi \partial_\alpha \partial_\nu \phi = 0 \quad \checkmark\checkmark$$

Exercise: Find $T_{\mu\nu}$ for $\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\mu^2}{2} \phi^2$. Check explicitly, by use of EOM, that $\partial_\mu T^\mu{}_\nu = 0$.

The conserved "charges" are

$$P^\mu = \frac{1}{c} \int d^3x \, T^{\mu 0}$$

are interpreted as $P^0 = E/c = \text{energy}/c$ and $P^i = \text{momentum}$.

[Units $[S] = E \cdot T = P \cdot X$, $\int d^4x \, \mathcal{L}$ $[\mathcal{L}] = \frac{PX}{X^4} = \frac{P}{X^3}$ Same as $[T^{\mu\nu}]$]

Digression: (i) another approach

For spacetime symmetries, $S[\phi'(x')] = S[\phi(x)]$ involves $\int d^4x'$ vs $\int d^4x$

In comparing $\int d^4x'\, \mathcal{L}(\phi'(x'))$ to $\int d^4x\, \mathcal{L}(\phi(x))$ we want the same

integration region, so we use $\int d^4x \left[\left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(\phi'(x'(x))) - \mathcal{L}(\phi(x)) \right] = 0$

Now the Jacobian $\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det\left|\frac{\partial}{\partial x^\nu}(x^\mu + \epsilon^\mu(x))\right| = \det(\delta^\mu_\nu + \partial_\nu \epsilon^\mu)$

Using $\det A = e^{\mathrm{Tr} \ln A}$ (for any matrix $\det A = \prod \lambda_n$ with λ_n the eigenvalues, so taking the log, $\ln \det A = \ln \prod_n \lambda_n = \sum_n \ln \lambda_n = \mathrm{Tr} \ln A$), we have

$$\det(\delta^\mu_\nu + \partial_\nu \epsilon^\mu) = e^{\mathrm{Tr}\ln(\delta^\mu_\nu + \partial_\nu \epsilon^\mu)} \approx e^{\partial_\mu \epsilon^\mu} = 1 + \partial_\mu \epsilon^\mu$$

So

$$\int d^4x \left[\partial_\mu \epsilon^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \epsilon^\mu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \epsilon^\mu \partial_\mu \partial_\nu \phi \right] = 0$$

Since $\epsilon^\mu(x)$ is arbitrary, we will make it localized, and in particular $\epsilon^\mu = 0$ at the boundaries. So we integrate by parts the first term. Use the EOM for the second, and

$$\int d^4x\, \epsilon^\mu(x) \left[-\partial_\mu \mathcal{L} + \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \partial_\nu \phi \right] = 0$$

$$\int d^4x\, \epsilon^\mu(x)\, \partial_\nu \left[-\delta^\nu_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right] = 0$$

$$\Rightarrow \quad T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L} \quad \text{are } (\mu = 0, \dots 3) \text{ conserved.}$$

(ii) Since the EOM is derived from arbitrary variations, why do we get conservation laws from specific variations only after using the EOM?

ANS: For spacetime transformations (eg, translations $x \to x' = x + \epsilon$, or Lorentz Ts $x' = \Lambda x$) the transformation is not just $\phi'(x) = \phi(x) + \delta\phi(x)$; there is at least the Jacobian factor and there may also be a difference between $\delta \partial_\mu \phi$ and $\partial_\mu \delta\phi$, eg, for Lorentz trans's, $\partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi(\Lambda^{-1}x)$ but $\phi \to \phi(\Lambda^{-1}x)$ so $\delta \partial_\mu \phi$ includes $\epsilon_\mu{}^\nu \partial_\nu \phi$ but not so $\partial_\mu(\delta\phi)$.

For internal transformations we can work with constant ϵ, at the Lagrangian level, so clearly different. Also, internal transformations do not necessarily satisfy boundary conditions. For example, if at $t = t_{\text{initial}}$ $\phi_I(\vec{x}, t_{in}) = \phi_{I0}(\vec{x})$ then the transformed field $\phi'_I(\vec{x}, t) = R_{IJ} \phi_J(\vec{x}, t)$ does not have $\phi'_I(\vec{x}, t_{ini}) = \phi_{I0}(\vec{x})$.

Next consider Lorentz invariance: $\delta x^\mu = \epsilon^\mu{}_\nu x^\nu$. We have

$$\phi'(x) = \phi(\Lambda^{-1}x) = \phi(x^\mu - \epsilon^\mu{}_\nu x^\nu) = \phi(x) - \epsilon^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

Similarly $\delta \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}$

For $\partial_\mu \phi$ we must be careful that it is a vector, $a'_\mu(x) = \Lambda_\mu{}^\nu a_\nu(\Lambda^{-1}x)$ so that

$$\delta a_\mu = \epsilon_\mu{}^\nu a_\nu(x) - \epsilon^\lambda{}_\nu x^\nu \partial_\lambda a_\mu$$

Hence

$$\delta S = \int d^4x \left[-\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} \right]$$

$$\text{but also} = \int d^4x \left[\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} (-\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi) \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \left[(\epsilon_\lambda{}^\nu \partial_\nu \phi - \epsilon^\alpha{}_\nu x^\nu \partial_\alpha \partial_\lambda \phi) - \partial_\lambda(-\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi) \right] \right]$$

$$= \int d^4x \left[-\epsilon^\mu{}_\nu \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} x^\nu \partial_\mu \phi \right) \right]$$

With $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ arbitrary we have (use $\partial^\mu \phi = \eta^{\mu\nu} \partial_\nu \phi$ for short):

$$M^{\lambda\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} x^\nu \partial^\mu \phi - \eta^{\mu\lambda} x^\nu \mathcal{L} - (\mu \leftrightarrow \nu) = T^{\lambda\mu} x^\nu - T^{\lambda\nu} x^\mu$$

satisfy $\partial_\lambda M^{\lambda\mu\nu} = 0$ (6 conserved currents: 3 for rotations + 3 for boosts).

The conserved charges are

$$M^{\mu\nu} = \int d^3x\, M^{0\mu\nu}$$

with

$$M^{ij} = \int d^3x\, M^{0ij} = \int d^3x\, (T^{0i} x^j - T^{0j} x^i)$$

These have the form of $\vec{r} \times \vec{p}$, where $\vec{p}$ is a density then integrated over the density $\Rightarrow$ this is angular momentum.

$$M^{0i} = \int d^3x\, M^{00i} = \int d^3x\, (T^{00} x^i - T^{0i} x^0)$$
$$= \int d^3x (T^{00} x^i) - x^0 P^i$$

is analogous to particle case, giving

$$P^i = \frac{\int d^3x (T^{00} x^i)}{\int d^3x (T^{00})} \cdot \frac{\int d^3x\, T^{00}}{x^0} + C^i = \frac{x^i_{CM}}{x^0} \frac{E_{TOT}}{c} + C^i$$

Exercise: Compute $M^{\mu\nu\lambda}$ for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$, and check $\partial_\mu M^{\mu\nu\lambda} = 0$.

Sol: $T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi$

so $M^{\mu\lambda\nu} = T^{\mu\lambda}x^\nu - T^{\mu\nu}x^\lambda$

$= \partial^\mu\phi(\partial^\lambda\phi\, x^\nu - \partial^\nu\phi\, x^\lambda) - \frac{1}{2}(\eta^{\mu\lambda}x^\nu - \eta^{\mu\nu}x^\lambda)\partial_\alpha\phi\,\partial^\alpha\phi$

We have already checked $\partial_\mu T^{\mu\nu} = 0$. So

$\partial_\mu M^{\mu\lambda\nu} = \partial_\mu(T^{\mu\lambda}x^\nu - T^{\mu\nu}x^\lambda)$

$= (\partial_\mu T^{\mu\lambda})x^\nu + T^{\mu\lambda}\partial_\mu x^\nu - \partial_\mu T^{\mu\nu}x^\lambda - T^{\mu\nu}\partial_\mu x^\lambda$

$= T^{\mu\lambda}\delta^\nu_\mu - T^{\mu\nu}\delta^\lambda_\mu$

$= T^{\nu\lambda} - T^{\lambda\nu}$

$= 0$ since $T^{\mu\nu}$ is symmetric.

Note: in some cases $T^{\mu\nu}$ is not symmetric, and additional care is needed. One can define an improved $T^{\mu\nu}$ which is symmetric. Rather than doing the general case, we'll see this explicitly in the case of E&M.

"Internal" symmetries. Consider the case of two scalar fields

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2$$ where both $\phi_1(x)$ & $\phi_2(x)$ are scalar fields.

This is form invariant under the transformations:

$$\phi_1'(x) = \cos\alpha\,\phi_1(x) + \sin\alpha\,\phi_2(x)$$
$$\phi_2'(x) = -\sin\alpha\,\phi_1(x) + \cos\alpha\,\phi_2(x)$$

This is a continuous set of transformations since α can take on any value in $\mathbb{R}$. Note that this transformation has x^μ unchanged ($x^\mu \to x'^\mu = x^\mu$). It only "mixes" the fields ϕ_1 & ϕ_2; such symmetries are generically called "internal".

More generally, consider a set of fields ϕ_I, $I = 1,\dots,N$, with a Lagrangian density $\mathcal{L}(\phi_I, \partial_\mu\phi_I)$ that is invariant under

$$\delta\phi_I(x) = \epsilon(D\phi)_I(x)$$

where ϵ is an infinitesimal parameter, $(D\Phi)_I$ is a linear transformation on Φ and we indicate explicitly that x is unaffected. That $\mathcal{L}$ is invariant means

$$\sum_I \frac{\partial\mathcal{L}}{\partial\phi_I}\epsilon D\phi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\epsilon\partial_\lambda D\phi_I = 0$$ (we are taking ϵ to be x independent)

$\sum_I$ understood throughout

EOM: $\frac{\partial\mathcal{L}}{\partial\phi_I} = \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\right) \Rightarrow \epsilon\left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\right)D\phi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\partial_\lambda D\phi_I\right] = 0$

Hence $\partial_\lambda\left[\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}D\phi_I\right] = 0$ or $\boxed{\partial_\lambda J^\lambda = 0 \text{ for } J^\lambda = \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}D\phi_I}$

"Noether's Theorem"

Example: The 2 field case above has ($\alpha = \epsilon$, infinitesimal)

$$\delta\phi_1 = \epsilon\,\phi_2 \quad \delta\phi_2 = -\epsilon\phi_1 \quad \text{So } (D\phi)_1 = +\phi_2 \quad (D\phi)_2 = -\phi_1$$

Also $\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)} = \eta^{\lambda\nu}\partial_\nu\phi_I = \partial^\lambda\phi_I$. So $J^\lambda = \partial^\lambda\phi_1\,\phi_2 + \partial^\lambda\phi_2(-\phi_1) = \phi_2\partial^\lambda\phi_1 - (\partial^\lambda\phi_2)\phi_1$

Note that $\partial_\lambda J^\lambda = \phi_2\partial^2\phi_1 - \partial^2\phi_2\,\phi_1$ vanishes by EOM.

Exercise: Consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V(\phi_1,\phi_2)$ for some function V.
(i) Give examples of V that are polynomial in ϕ_I and invariant under $\delta\phi_1 = \phi_2$, $\delta\phi_2 = -\phi_1$.
(ii) Including V, compute J^λ.

Action integral for Electromagnetic Field

The Euler–Lagrange equations for the $\vec{E}$ & $\vec{B}$ fields should correspond to Maxwell's equations, the source-less form if the action integral does not include the interaction with particles (which we'll ignore at first — easy to include later).

The problem is that Maxwell's equations are first order in derivatives. But this is not what follows normally from Euler-Lagrange. To see this, consider what can be written as possible forms for the Lagrangian. It should be quadratic in the field variables $\vec{E}$ & $\vec{B}$, invariant under rotations (assuming Lorentz invariance is even more constraining — see further below). So we can have terms

2nd pass:

$$\vec{E}^2, \ \vec{B}^2, \ \underbrace{\vec{E}\cdot\vec{B}}_{P \,\&\, T \text{ odd}}$$

and derivatives on this

$$\underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{1}{2}\frac{\partial \vec{E}^2}{\partial t},\ T\text{ odd}} \qquad \underbrace{\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}}_{\text{idem}} \qquad \underbrace{\vec{E}\cdot\frac{\partial\vec{B}}{\partial t}}_{P \text{ odd}} \qquad \underbrace{\frac{\partial\vec{E}}{\partial t}\cdot\vec{B}}_{\text{idem}}$$

$$(\vec{\nabla}\cdot\vec{E})^2, \ (\vec{\nabla}\times\vec{E})^2, \ \text{etc}\ldots$$

The terms in the last line is quadratic in derivatives $\rightarrow$ will give 2-derivative EOM's.

The first line has E^2 & B^2. In fact energy density $\propto \vec{E}^2+\vec{B}^2$ and it would make sense that, since $L = T - V$, that it involves this. But then the EOMs would have no derivatives.

There is an easy workaround. If we use the potentials ϕ & $\vec{A}$ (or $A^\mu = (A^0, \vec{A})$) as the fundamental fields (fundamental in the sense that then $\vec{E}$ & $\vec{B}$ are derived) then if $\mathcal{L} \propto (\partial A)^2$ the EOM will involve $\partial\cdot\partial A \sim \partial(E \text{ or } \vec{B})$ which is precisely what we need.

Additionally, we want $\mathcal{L}$ to be gauge invariant (so that Maxwell equations are too).
$\Rightarrow$ Write $\mathcal{L}$ in terms of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
As a bonus, the homogeneous (sourceless) equations follow automatically.

Possible terms for $\mathcal{L}$

(i) Made of $F_{\mu\nu}$
(ii) Quadratic in fields
(iii) Lorentz invariant
(iv) P and T invariant

Possibilities are limited:

$$F_{\mu\nu}F^{\mu\nu}\,, \quad \underbrace{\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}}_{P,T \text{ odd}}$$

So we conclude

$$\mathcal{L} = \tfrac{1}{4}\kappa\, F_{\mu\nu}F^{\mu\nu}$$

with some constant κ to be determined.

Euler-Lagrange

Write $\mathcal{L} = \frac{\kappa}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \kappa(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad \frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \quad \Rightarrow \quad \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \kappa(\partial^2 A_\nu - \partial_\nu \partial\cdot A) = 0$$

Compare with Maxwell's equations: $\partial^2 A_\nu - \partial_\nu(\partial\cdot A) = \frac{4\pi}{c} j_\nu$; since we have $j_\nu = 0$ (for now), these agree! ✓✓

To determine κ, we now add a charge particle and insist in getting the correct Maxwell's equations

$$S = \int d^4x\, \kappa F_{\mu\nu}F^{\mu\nu} + \int d\lambda\left(-mc\sqrt{U_\alpha U^\alpha} - \frac{q}{c}U^\alpha A_\alpha\right)$$

Then $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ is as above, but now $\frac{\partial \mathcal{L}}{\partial A_\mu}$ no longer vanishes. To make the coupling term a lagrangian density, choose $\lambda = x^0$, so $U^\alpha \to \frac{dx^\alpha}{dx^0} = (1, \vec{v}/c)$ and include $\int d^3x'\, \delta^3(\vec{x}' - \vec{x}(\lambda))$ to replace $A_\mu(x^0, \vec{x}(x^0))$ by $A_\mu(x)$. Then

$\underset{\text{dummy}}{x'}$ $\underset{\text{world-line}}{\vec{x}(\lambda)}$

$$-\frac{q}{c}\int d\lambda\, U^\alpha A_\alpha = -\frac{q}{c}\int d^4x'\, \delta^{(3)}(\vec{x}' - \vec{x}(x^0))\,\frac{dx^\mu}{dx^0}A_\mu(x')$$

$$\Rightarrow \mathcal{L}_q = -\frac{q}{c}\delta^3(\vec{x}' - \vec{x}(x^0))\frac{dx^\alpha}{dx^0}A_\alpha = -\frac{1}{c^2}j^\alpha A_\alpha \quad \text{since } j^\alpha = cq\frac{dx^\alpha}{dx^0}\delta^3(\vec{x}' - \vec{x}(x^0)) \quad \text{and } \frac{\partial \mathcal{L}_q}{\partial A_\mu} = -\frac{1}{c^2}j^\mu$$

$$\text{So } \kappa(\partial^2 A_\nu - \partial_\nu \partial\cdot A) = -\frac{1}{c^2}j^\nu \quad \Rightarrow \kappa = -\frac{1}{4\pi c} \quad \Rightarrow \quad \boxed{S_{E\&M} = \int d^4x\left(-\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\right)}$$

Leaving out particle dynamics, it is convenient to factor out $4\pi c$:

$$S = \frac{1}{4\pi c}\int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{4\pi}{c}J^\mu A_\mu\right]$$

Note that $F^{\mu\nu}F_{\mu\nu} = 2F^{0i}F_{0i} + F^{ij}F_{ij} \Rightarrow \mathcal{L}_{free} = \frac{1}{2}(E^2 - B^2)\cdot\frac{1}{4\pi c}$

<u>Canonical Momentum and Hamiltonian</u>

π^ν the momentum conjugate to A_ν: $\pi^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\nu)} = \frac{-1}{4\pi c}(\partial^0 A^\nu - \partial^\nu A^0) = -\frac{1}{4\pi c}F^{0\nu} = \begin{cases} 0 & \nu = 0 \\ \frac{E^i}{4\pi c} & \nu = i\end{cases}$

Note $\pi^0 = 0$. There is no canonical momentum associated with A_0. This is because $\mathcal{L}$ contains no time derivative of A_0. A_0 is not dynamical, it's "EOM" is an equation of constraint.

Now $(4\pi c)\pi^i = -\partial^0 A^i + \partial^i A^0$ so $\partial^0 A^i = \partial^i A^0 - (4\pi c)\pi^i$

So $\mathcal{H} = \pi^i \partial_0 A_i - \mathcal{L}$

$$= \pi^i\left(-(4\pi c)\pi_i + \partial_i A_0\right) - \frac{1}{2}\frac{\left((4\pi c)\pi^2 - B^2\right)}{4\pi c}$$ (where $\vec{B}$ is short for $\vec{\nabla}\times A$).

$$= \frac{(4\pi c)\pi^2}{2} + \frac{1}{2}\frac{B^2}{4\pi c} + \pi^i\partial_i A_0 = \frac{4\pi c}{2}\pi^2 + \frac{1}{2}\frac{(\vec{\nabla}\times\vec{A})^2}{4\pi c} + \pi^i\partial_i A_0$$

The first two terms $\frac{1}{8\pi c}\left[(4\pi c\,\pi^i)^2 + B^2\right] = \frac{1}{8\pi c}(E^2 + B^2)$ are interpreted as energy density/c

Now, in $\quad S = \int d^4x\,(\pi^i\partial_0 A_i - \mathcal{H})$

The variation w.r.t. A_0 gives $\partial_i\pi^i = 0$, or $\vec{\nabla}\cdot\vec{E} = 0$ (Gauss's law).

This has no time dependence, it is an equation of constraint.

If we retain source term, $\Delta S = \int d^4x\,\frac{1}{c^2}J^\mu A_\mu$

then $\frac{\delta S}{\delta A_0} = -\partial_i\pi^i + \frac{1}{c^2}J^0 = 0 \qquad \vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c}J^0 = 4\pi\rho$

so Gauss's Law is an equation of constraint (makes sense, no $\frac{\partial}{\partial t}$'s here!)

Energy-Momentum tensor $\quad T^{\lambda}{}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi - \delta^\lambda_\mu \mathcal{L}$

Use this with $\phi \to A_\nu$

$$(4\pi c)\, T^{\lambda}{}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)} \partial_\mu A_\nu - \delta^\lambda_\mu \mathcal{L}$$

↑ to avoid repeating in each term

$$= -(\partial^\lambda A^\nu - \partial^\nu A^\lambda)\partial_\mu A_\nu - \delta^\lambda_\mu \left(-\tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right)$$

$$= -F^{\lambda\nu} \partial_\mu A_\nu + \tfrac{1}{4} \delta^\lambda_\mu F_{\alpha\beta} F^{\alpha\beta}$$

or $(4\pi c)\, T^{\lambda\mu} = -F^{\lambda\nu} \partial^\mu A_\nu + \frac{1}{4} \eta^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}$

This tensor is conserved, but neither symmetric nor gauge invariant.

We can try to improve this by noting that we may add to $T^{\lambda\mu}$ another tensor $t^{\lambda\mu}$

(i) $\partial_\lambda t^{\lambda\mu} = 0$

(ii) $\int d^3x\, t^{0\mu} = 0$

so that $\tilde{T}^{\lambda\mu} = T^{\lambda\mu} + t^{\lambda\mu}$ is conserved and defines the same "charge" as $T^{\lambda\mu}$. If we find such tensor then we may try to adjust it so that $\tilde{T}^{\lambda\mu} = \tilde{T}^{\mu\lambda}$ and is gauge invariant.

Consider $\quad (4\pi c)\, t^{\lambda}{}_{\mu} = F^{\lambda\nu} \partial_\nu A_\mu$. It has

$$4\pi c\, \partial_\lambda t^{\lambda}{}_{\mu} = \partial_\lambda F^{\lambda\nu} \partial_\nu A_\mu + F^{\lambda\nu} \partial_\lambda \partial_\nu A_\mu$$

Since we are considering the sourceless case, $\partial_\lambda F^{\lambda\nu} = 0$. The second term vanishes by symmetry of $\partial_\lambda \partial_\nu$ times antisymmetry of $F^{\lambda\nu}$. Also

$$\int d^3x\, F^{0\nu} \partial_\nu A_\mu = \int d^3x\, F^{0i} \partial_i A_\mu = -\int d^3x\, \partial_i F^{0i} A_\mu = 0 \quad \Big(\underset{\text{(no source)}}{\partial_\lambda F^{0\lambda} = 0}\Big).$$

So we adopt a symmetric and gauge invariant expression for the energy-momentum tensor (droping tilde):

$$T^{\mu\nu} = -\frac{1}{4\pi c}\left[F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

Interpretation

$$4\pi c\, T^0{}_0 = -F^{0i}F_{0i} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

$$= \vec{E}^2 + \frac{1}{2}(-\vec{E}^2+\vec{B}^2) = \vec{E}^2 - \frac{1}{2}(\vec{E}^2-\vec{B}^2) = \frac{1}{2}(\vec{E}^2+\vec{B}^2)$$

so, energy density $u = \frac{1}{8\pi}(\vec{E}^2+\vec{B}^2)$ $\qquad \left(T^{00} = \frac{u}{c}\right)$

$4\pi c T^{i0}$ = energy flux

$$4\pi c\, T^{i0} = -F^{i\lambda}F^0{}_\lambda = F^{ij}F^{0j} = (-\epsilon^{ijk}B^k)(-E^j)$$

$$= (\vec{E}\times\vec{B})^i$$

Let $\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$ "Poynting vector" $\left(T^{i0} = \frac{1}{c^2}S^i\right)$.

Now $\partial_\lambda T^{\lambda 0} = 0 \Rightarrow$ the "current" $(\frac{u}{c}, \vec{S})$ is conserved $\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$

(Check nagging "c"s $\partial_\lambda T^{\lambda 0} = 0 \Rightarrow$

$\rightarrow \partial_0 T^{00} + \partial_i T^{i0} = 0 \rightarrow \frac{1}{c}\frac{\partial}{\partial t}\frac{u}{c} + \partial_i\left(\frac{1}{c^2}S^i\right) = 0$ ✓✓).

(Dimensions: $[T^{i0}] = \frac{P}{L^3}$ $\quad [S^i] = \left(\frac{L}{T}\right)^2\frac{P}{L^3} = \frac{\frac{L}{T}\cdot P}{L^2 T} = \frac{E}{L^2\cdot T}$ = energy flux ✓).

T^{0i} : momentum density $= T^{i0}$ $\Big\}$ see below
T^{ij} : momentum flux

Note: $\partial_\lambda T^{\lambda\mu} = 0$ holds even in the presence of charges IF we add to $T^{\mu\nu}_{EM}$ the contribution from the rest of the system PROVIDED the whole action is invariant under space-time translations.

Alternatively $\partial_\lambda T^{\lambda\mu}_{EM} = -\partial_\lambda T^{\lambda\mu}_{rest} \neq 0$

Rather than giving the rest of the Lagrangian and computing $T^{\mu\nu}_{rest}$, which depends on the details of "rest", we can compute directly $\partial_\lambda T^{\lambda\mu}_{EM}$ assuming $\partial_\lambda F^{\lambda\nu} = \frac{4\pi}{c}j^\nu$ (ie, with sources). The result (checked in next page) is

$$\partial_\lambda T^{\lambda\mu}_{EM} = -\frac{1}{c^2}F^{\mu\nu}j_\nu$$

For $\mu=0$ $(\times c^2)$ $\quad \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{E}\cdot\vec{j}$

Interpretation of $\vec{j}\cdot\vec{E}$: Consider $\int d^3x\, \vec{j}\cdot\vec{E}$ and use $\vec{j} = q\vec{v}\,\delta^3(\vec{x}-\vec{x}(t))$

Then $\int d^3x\, \vec{j}\cdot\vec{E}(x) = q\vec{v}\cdot\vec{E}(\vec{x}(t))$

and recall $\frac{d\vec{p}}{dt} = q\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)$ so $\vec{v}\cdot\frac{d\vec{p}}{dt} = q\vec{v}\cdot\vec{E}$

Moreover $\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = m\frac{d\vec{v}}{dt}\frac{1}{\sqrt{1-v^2/c^2}} + m\vec{v}\frac{\vec{v}\cdot\frac{d\vec{v}}{dt}}{c^2(1-v^2/c^2)^{3/2}}$

so $\vec{v}\cdot\frac{d\vec{p}}{dt} = \frac{m\,\frac{1}{2}\frac{d\vec{v}^2}{dt}}{(1-v^2/c^2)^{3/2}} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{dE_{kin}}{dt}$

Combining with the above

$$\frac{d}{dt}\Bigg[\underbrace{\int_V d^3x\,u + E_{kin}}_{\text{total energy in volume } V}\Bigg] = -\int_V d^3x\,\vec{\nabla}\cdot\vec{S} = \underbrace{-\oint_{\partial V} ds\,\hat{n}\cdot\vec{S}}_{\text{energy escaping } V \text{ through boundary } \partial V}$$

Note: $\vec{S}$ may be non-zero even for static fields! For example, see previous example of static "$E\times B$" field. In this case $\frac{\partial u}{\partial t}=0$ so $\vec{\nabla}\cdot\vec{S}=0$, and while not vanishing it does not add energy to any region of space (and $\vec{S}=\vec{\nabla}\varphi$ for some φ?).

Calculation for previous page:

$$(-4\pi c)\,\partial_\lambda T^{\lambda\mu} = \partial_\lambda\left(F^{\lambda\alpha}F^{\mu}{}_{\alpha} - \frac{1}{4}\eta^{\lambda\mu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

$$= \frac{4\pi}{c}j_\alpha F^{\mu\alpha} + X^\mu$$

where

$$X^\mu = F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} - \frac{1}{2}F_{\alpha\beta}\partial^\mu F^{\alpha\beta}$$

$$= F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} - \frac{1}{2}F_{\alpha\beta}\left(-\partial^\alpha F^{\beta\mu} - \partial^\beta F^{\mu\alpha}\right) \quad \text{by homogeneous equation}$$

$$= F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} + F_{\alpha\beta}\partial^\beta F^{\mu\alpha}$$

$$= 0$$

Hence

$$\partial_\lambda T^{\lambda\mu} = -\frac{1}{c^2}F^{\mu\alpha}j_\alpha$$

Often defined: $\vec{g} = T^{0i} = \frac{1}{c^2}\vec{S}$ = density of momentum

and let $\sigma^{ij} = c\,T^{ij}$, the "Maxwell stress tensor"

($T^{\mu\nu}$ is often called the "stress-energy tensor", another name for "energy-momentum tensor").

Then $\partial_\lambda T^{\lambda i} = \frac{1}{c}\frac{\partial}{\partial t}T^{0i} + \partial_j T^{ji} = \frac{1}{c}\left(\frac{\partial g^i}{\partial t} + \partial_j \sigma^{ji}\right) = 0$

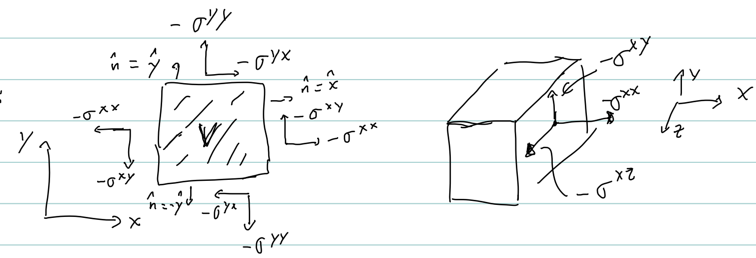
Dimensions: $[T^{ij}] = \frac{P}{L^3}$ $\quad [cT^{ij}] = \frac{P}{L^2 T}$ = momentum flux = force/area.

Since $\frac{d}{dt}\int_V d^3x\, g^i = \frac{d}{dt}(\text{momentum})^i = \text{force}^i$ (force on V)

$$= -\oint_S \sigma^{ji}\,\hat{n}^j\,dS$$

$\Rightarrow \sigma^{ji}$ is the i-th component of force <u>into</u> the volume on the surface element dS, with normal $\hat{n}^j$

Forces:



In terms of $\vec{E}$ & $\vec{B}$:

$$\sigma^{ij} = -\frac{1}{4\pi}\left[F^{i\lambda}F^{j}{}_{\lambda} - \frac{1}{4}\eta^{ij}F^{\alpha\beta}F_{\alpha\beta}\right]$$

$$= -\frac{1}{4\pi}\left[F^{i0}F^{j0} - F^{ik}F^{jk} + \frac{1}{2}\delta^{ij}\left(-E^2 + B^2\right)\right]$$

$$= -\frac{1}{4\pi}\Big[E^i E^j - \underbrace{\epsilon^{ikm}\epsilon^{jkn}B^m B^n}_{\delta^{ij}B^2 - B^i B^j} + \frac{1}{2}\delta^{ij}\left(-E^2 + B^2\right)\Big]$$

$$= -\frac{1}{4\pi}\left[E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}\left(E^2 + B^2\right)\right]$$

So, for example: $\sigma^{xx} = -\frac{1}{8\pi}\left(E_x^2 - E_y^2 - E_z^2 + B_x^2 - B_y^2 - B_z^2\right)$

$$\sigma^{xy} = -\frac{1}{4\pi}\left(E_x E_y + B_x B_y\right)$$

Note: This is the same convention for σ^{ij} as Landau-Lifshitz, opposite sign for Jackson.

With sources: we've seen above

$$\partial_\lambda T_{EM}{}^{\lambda}{}_{\mu} = -\frac{1}{c^2} F_{\lambda\nu} J^\nu$$

We will integrate this over space and compare with $\partial_\lambda T_{rest}{}^{\lambda}{}_{\mu}$ $\quad T_{rest}^{\lambda\mu}$ = charged particles contribution

$$-\frac{1}{c^2}\int d^3x\, F_{\lambda\nu} J^\nu \to -\frac{1}{c^2}\int d^3x\, F_{\lambda\nu}\, q\frac{dx^\nu}{dt}\delta^3(\vec{x}-\vec{x}(t))$$

$$= -\frac{1}{c^2} qF_{\lambda\nu}(\vec{x}(t))\frac{dx^\nu}{dt}$$

Recall $\displaystyle \frac{dp_\mu}{dt} = \frac{q}{c}\frac{dx^\nu}{dt}F_{\nu\mu}$

so $\displaystyle -\frac{1}{c^2}\int d^3x\, F_{\lambda\nu} J^\nu = -\frac{1}{c}\frac{dp_\mu}{dt}$

$$\Rightarrow \int d^3x\, \partial_\lambda T^\lambda{}_\mu + \frac{1}{c}\frac{dp_\mu}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{c}\int_V d^3x\, T^{0\mu} + \frac{1}{c}p^\mu\right) = -\oint_{\partial V} T^{i\mu}\,\hat{n}^i\, ds$$

The $\mu=0$ component is as before: $T^{00} = \frac{u}{c}$, $T^{i0} = \frac{1}{c^2}S^i$

so, as before, we have $\displaystyle \frac{d}{dt}\Big(\int d^3x\, u + \underbrace{cp^0}_{=E_{rest}}\Big) = -\oint \vec{S}\cdot\hat{n}\, ds$ ← area element (lowercase)

↑ Poynting vector (uppercase) → sorry!

That is $\displaystyle \frac{d}{dt}\Big(\underbrace{\text{Total energy}}_{\text{in volume } V}\Big)$ = (energy flow <u>into</u> volume V through ∂V per unit time).

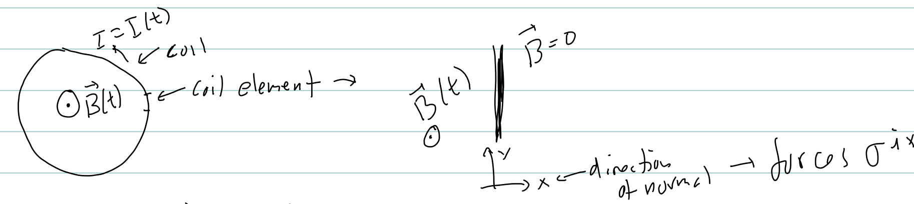
For $\mu = i$

$$\frac{d}{dt}\left(\frac{1}{c}\int d^3x\, g^i + \frac{1}{c}p^i\right) = -\oint \frac{1}{c}\sigma^{ij}n^j ds$$

or $$\frac{d}{dt}\left(\int d^3x\, g^i + p^i_{rest}\right) = -\oint \sigma^{ij} n^j ds$$

or $\displaystyle \frac{d}{dt}\begin{pmatrix}\text{Total momentum}\\ \text{in volume } V\end{pmatrix}$ = (momentum flow <u>into</u> volume V through ∂V / time)

Example:



$$\sigma^{xx} = \frac{1}{8\pi}\left(B_z^2 + 0 \dots\right) = \frac{1}{8\pi} B_z^2 = \frac{1}{8\pi} B^2$$

$$\sigma^{yx} = \frac{1}{8\pi}\left(B_x B_y + \dots\right) = 0$$

$$\sigma^{zx} = 0$$

So the $\frac{\text{force}}{\text{area}}$ on the coil has magnitude $\frac{B^2}{8\pi}$, pointing radially outward.

(The force on the field, $-\int \sigma \cdot n \, ds$ points in, then Newton's 3^{rd} gives force on coil).

Of course we can do $|d\vec{F}| = |dq \frac{\vec{v}}{c} \times \vec{B}| = \frac{1}{c} I B \, dL$ over length dL of wire.

Now Ampere's law: $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$ $\quad \oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$ or $B = \frac{4\pi}{c} I / 2\pi R = \frac{2I}{Rc}$

or $dF = \frac{I}{c} B \, dL = \frac{1}{c}\left(\frac{cRB}{2}\right) B \cdot dL = (2\pi R \, dL) \frac{B^2}{4\pi}$ or $\frac{F}{\text{area}} = \frac{B^2}{4\pi}$ ouch.