Refs: Lechner, Chap 5
Jackson, 7.1-7
L&L, Chap 6, 46-48
Chaichian: Chap 4

EM waves

We consider time dependent (propagating) solutions of Maxwell's equations in free space (no sources).

Consider 1st the 4-vector potential. We have established that in free space

$$\partial^2 A_\mu + \partial_\mu (\partial\cdot A) = 0$$

We choose to impose the covariant gauge condition ("Lorentz gauge") $\partial\cdot A = 0$ so that

$$\partial^2 A_\mu = 0 \qquad \text{(wave equation)}$$

Plane-waves

A simple solution is

$$A_\mu(x) = a_\mu e^{-ik\cdot x} = a_\mu e^{i\vec{k}\cdot\vec{x} - ik^0x^0} = a_\mu e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad k^\mu = (k^0, \vec{k}) = (\tfrac{\omega}{c}, \vec{k})$$

where a_μ and k_μ are constants. Note that $\partial^2 A_\mu = 0$ requires $k^2 = 0$ or $(k^0)^2 = \vec{k}^2 \Leftrightarrow \omega = \pm c|\vec{k}|$

A_μ as written is complex. We always implicitly assume we take the real part:

$$A_\mu(x) = Re(a_\mu e^{-ik\cdot x}) = a_\mu e^{-ik\cdot x} + a_\mu^* e^{ik\cdot x}$$

Now, we insist that $\partial_\mu A^\mu = 0 \Rightarrow \partial_\mu (a^\mu e^{-ik\cdot x}) = -ik_\mu a^\mu e^{-ik\cdot x} = 0$

$$\Rightarrow k_\mu a^\mu = 0$$

For example, for wave propagation in $x^3 = z$ direction $k^\mu = (\frac{\omega}{c}, 0, 0, k) = k(1,0,0,1)$ and $k_\mu a^\mu = 0 \Rightarrow a^0 = a^3$

The $\vec{E}$ field is F^{i0}. Compute $F_{i0} = \partial_i A_0 - \partial_0 A_i = -i(k_i a_0 - k_0 a_i)\, e^{-ik\cdot x}$

or $E^i = F^{i0} = -i(k^i a^0 - k^0 a^i)\, e^{-ik\cdot x}$

For this example $k^\mu = k(1,0,0,1)$ and $a^3 = a^0$ so $E^3 = -i(k^3a^0 - k^0 a^3) = 0$

and $E^{1,2} = -i(k^{1,2}a^0 - k^0 a^{1,2})\, e^{-ik\cdot x} = ika^{1,2} e^{-ik\cdot x}$

Also $B^3 = -F^{12} = i(k^1a^2 - a^2k^1)\, e^{-ik\cdot x} = 0 \qquad B^1 = -F^{23} = i(k^2a^3 - k^3a^2)\, e^{-ik\cdot x} = -ika^2 e^{-ik\cdot x}$

and $B^2 = -F^{31} = i(k^3a^1 - k^1a^3) e^{-ik\cdot x} = ika^1 e^{-ik\cdot x}$

Note that $B^2 = E^1$, $B^1 = -E^2$. So $\vec{E} \perp \vec{B}$, $|\vec{E}| = |\vec{B}|$ and both $\perp$ $\vec{k}$: $\vec{B} = \hat{k}\times\vec{E}$

with $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$

Clearly only $a_{1,2}$ have physical meaning. Why do we still have $a^0=a^3$ in A_μ?
Recall if A_μ satisfies the covariant gauge condition $\partial\cdot A=0$
we can still make a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

with $\partial\cdot A'=0$ provided $\partial^2\Lambda=0$. So we can add to our solution

$$\Lambda(x) = i\lambda e^{-ik\cdot x} \quad \to \quad \partial_\mu\Lambda = k_\mu \lambda e^{-ik\cdot x}$$

so that $\quad a'_\mu = a_\mu + k_\mu\lambda$

or $\quad a'^\mu = (a^0+k\lambda,\, a^1, a^2,\, a^3+k\lambda)$

and choosing $\lambda=-a^0/k$ we obtain $a'^\mu=(0,a^1,a^2,0)$

So we may as well drop $a^0=a^3$, and drop the prime.

It is convenient to define a unit vector $\vec\epsilon = \frac{\vec a}{|\vec a|}$. It satisfies $\vec k\cdot\vec\epsilon=0$. Also
$\epsilon^\mu=(0,\vec\epsilon)$ satisfies $k\cdot\epsilon=0$, $\epsilon^2=-1$

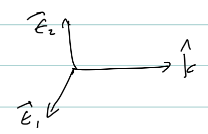
$\vec\epsilon$ is the "polarization" vector. We can write generally the equations found
for the z-propagation example:

$$\vec E = E_0\,\vec\epsilon\, e^{-ik\cdot x} \qquad \vec B = \hat k\times\vec E \qquad \vec k\cdot\vec\epsilon=0 \qquad \vec\epsilon^{\,2}=1$$

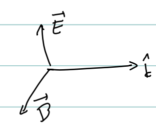
but now $\vec k$ is arbitrary.

Plane polarized wave: there are two independent solutions $\vec\epsilon_{1,2}$ to $\vec k\cdot\vec\epsilon=0$, $\vec\epsilon^{\,2}=1$.
that is two unit vectors in the plane $\perp$ to $\hat k$.
E.g. for $\hat k=(0,0,1)$ $\quad \vec\epsilon_1=(1,0,0)$ & $\vec\epsilon_2=(0,0,1)$

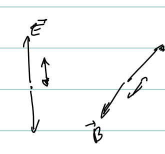
These, and any linear combination with relatively real coefficients are plane
polarized waves



, eg



At a fixed point in space, $\vec E = \underbrace{(E e^{i\vec k\cdot\vec x})}_{\text{fixed}}\vec\epsilon\, e^{-i\omega t}$ oscillates along $\vec\epsilon$:



For general $\hat k$ direction, choose a vector $\vec\epsilon_1$ + $\epsilon_1^2=1$ & $\hat k\cdot\vec\epsilon_1=0$. Then take $\vec\epsilon_2=\hat k\times\vec\epsilon_1$

L.L. 48
Jackson 7.2

Circular Polarization: In terms of $\vec{\epsilon}_{1,2}$ above, let

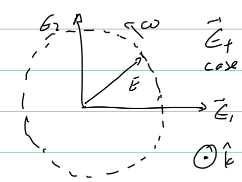
$$\vec{\epsilon}_{\pm} = \frac{\vec{\epsilon}_1 \pm i\,\vec{\epsilon}_2}{\sqrt{2}}$$

Then $\vec{\epsilon}_+ \cdot \vec{\epsilon}_+^* = \vec{\epsilon}_- \cdot \vec{\epsilon}_-^* = 1$ and $\vec{\epsilon}_+ \cdot \vec{\epsilon}_-^* = 0$, and $\vec{\epsilon}_-^* = \vec{\epsilon}_+$.

Now $\vec{E}(\vec{x},t) = E\,\vec{\epsilon}_+\, e^{+i\vec{k}\cdot\vec{x} - i\omega t}$

as seen at fixed point is a vector $\vec{E}$ of fixed magnitude rotating with angular speed ω in the $\vec{\epsilon}_1$-$\vec{\epsilon}_2$ plane

with $E = |E| e^{i\varphi}$ $\quad \mathrm{Re}\,\vec{E} = \frac{|E|}{\sqrt{2}}\left[\vec{\epsilon}_1 \cos(\omega t - i\vec{k}\cdot\vec{x} - \varphi) + \vec{\epsilon}_2 \sin(\omega t - i\vec{k}\cdot\vec{x} - i\varphi)\right]$



$\vec{\epsilon}_+$ case "right circular polarization"

for $\vec{\epsilon}_-$ case reverse direction (clockwise) "left circular polarization"

Elliptical polarization is a simple extension: $\vec{E} = (E_1\vec{\epsilon}_1 + i E_2 \vec{\epsilon}_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$ with E_1/E_2 = real. The $E_1 = E_2$ case is circular.

Exercise: for $\hat{k} = \hat{z}$, $\vec{\epsilon}_1 = \hat{x}$ verify that $\frac{(E^1(\vec{x},t))^2}{E_1^2} + \frac{(E^2(\vec{x},t))^2}{E_2^2} = 1$ which describes an ellipse.

Lechner: Chap 5
L&L 48
Jackson: 7.1 (partly)

Energy

Since $\vec{B} = \hat{k} \times \vec{E}$ we can get immediately

$$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = \frac{1}{4\pi} E^2 \qquad \text{energy density}$$

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{c}{4\pi} E^2 \hat{k} = cu\hat{k} \qquad \text{Poynting vector (energy flux)}$$

The momentum density $\vec{g}$ is related to the energy density (for plane waves!)

$$\vec{g} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$$

The relation $U = gc$ parallels $E_{kin} = \sqrt{(\vec{p}c)^2 + (mc^2)^2} \longrightarrow |p|c$ for the massless particle case. Such particle can move only at the speed of light, since

$$E_{kin} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

can stay non-zero as $m \to 0$ if we take $|\vec{v}| \to c$ simultaneously

So $\vec{g} = \frac{u}{c}\hat{k}$ can be interpreted as the momentum and energy make up of many non-interacting massless particles, all propagating at the common speed c. This interpretation is fleshed out in QM-version, the particles = photons.

Finally, $\sigma_{ij} = -\frac{1}{4\pi}\left(E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(\vec{E}^2 + \vec{B}^2) \right)$

For $\hat{k} = \hat{z}$ clearly vanishes unless $i = j = 3 = $ "z" for which $\sigma_{33} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = u$

Recall this is force on surface $\perp$ to "3" (ie to $\hat{k}$) along "3": if we put a perfectly absorbing plane parallel to $\hat{\epsilon}_1$-$\hat{\epsilon}_2$ plane, it will experience force/area $= u$ = radiation pressure

Time-averages

For most cases of interest $1/\omega$ is such a short time that measurements of energy density and other such quantities automatically average over many cycles. Then we really care about the average quantities:

$$\overline{f} = \frac{1}{T}\int_0^T dt\, f(t) \quad \text{where } T \text{ is either exactly one period } \frac{2\pi}{\omega}, \text{ or it's large } T \gg \frac{2\pi}{\omega}$$

Then if $\vec{E} = \frac{1}{2}(\vec{E}_0(\vec{x}) e^{-i\omega t} + c.c.)$, we have from $\overline{u} = \frac{1}{4\pi}\overline{\vec{E}^2}$ and $\vec{E}^2 = \frac{1}{4}\vec{E}_0^2 e^{-2i\omega t} + \frac{1}{4}\vec{E}_0^{*2} e^{2i\omega t} + \frac{1}{2}\vec{E}_0 \cdot \vec{E}_0^*$ so

that $\overline{\vec{E}^2} = \frac{1}{2}\vec{E}_0 \cdot \vec{E}_0^*$ and $u = \frac{1}{8\pi}\vec{E}_0 \cdot \vec{E}_0^*$ (The normalization is arbitrary: we could have chosen $\vec{E} = \frac{1}{\sqrt{2}}\vec{E}_0(x) e^{-i\omega t} + c.c.$).

Notes:

1. The waves above $\propto e^{-ik\cdot x}$ are monochromatic. Linearity of wave-equation means linear combinations are still solutions. Most general

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\left(a_\mu(k)\, e^{-ik\cdot x} + c.c.\right)$$

("c.c." = complex conjugate)

with $k^0 = |\vec{k}|$ and $k\cdot a = 0$. One may shift $a_\mu \to a_\mu + \lambda k_\mu$ freely, in particular to choose to make $a_0 = 0$, just as above. It is customary to expand in the two independent polarization vectors $\epsilon_\mu^{(i)}$, thus

$$A_\mu(x) = \sum_{i=1}^{2} \int \frac{d^3k}{(2\pi)^3}\left(a^{(i)}(k)\, \epsilon_\mu^{(i)}(k)\, e^{-ik\cdot x} + c.c.\right)$$

(In fact, for second quantization this is the starting point, except that there usually

$$A_\mu(x) = \sum_{i=1}^{2} \int \frac{d^3k}{2E(2\pi)^3}\left(\tilde{a}^{(i)}(k)\, \epsilon^{(i)}(k)\, e^{-ik\cdot x} + c.c.\right)$$

The factor of $2E$ is for Lorentz invariance of $\frac{d^3k}{E}$. To see this, recall $k^0 = |\vec{k}|$) so $\int d^4k\, \delta(k^2)\,\theta(k^0) = \int \frac{d^3k}{2E}$).

2. One can work directly from Maxwell's equations and get $\vec{E}$ & $\vec{B}$ waves: probably done in your UG course, but see next page.

3. For complex vectors, the magnitude is $\vec{a}\cdot\vec{a}^*$ or $a_\mu a^{*\mu}$. Beware of metric: $\epsilon_\mu \epsilon^{*\mu} = -1$!

Jackson 7.3
Scheck 4.4
Chaichian 4.4

Waves in media; reflection and refraction at interfaces; polarization at interfaces.

We will study E&M of/in media next quarter. For now we draw from previous knowledge. While we could still use a vector potential — since the homogeneous Maxwell equations are unchanged — the presence of a medium in which $\vec{E}$ and $\vec{B}$ propagate breaks Lorentz invariance — it defines a preferred frame.

We introduce dimensionless permittivity ϵ $(\vec{D}=\epsilon\vec{E})$ and permeability μ $(\vec{B}=\mu\vec{H})$ so that the source free Maxwell equations read

$$\epsilon\,\vec{\nabla}\cdot\vec{E}=0 \quad (\leftarrow \text{was } \rho) \qquad (1) \qquad\qquad \nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \qquad (3)$$

$$\frac{1}{\mu}\vec{\nabla}\times\vec{B}-\frac{1}{c}\epsilon\frac{\partial\vec{E}}{\partial t}=0 \quad (\leftarrow \text{was } \vec{J}) \qquad (2) \qquad \text{and} \qquad \vec{\nabla}\cdot\vec{B}=0 \qquad (4)$$

Taking $\vec{\nabla}\times$ of (3) and using $\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{a})\right]^i=\epsilon^{ijk}\partial_j\,\epsilon^{kmn}\partial_m a^n$

$$=\left(\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm}\right)\partial_j\partial_m a^n$$

$$=\left[\vec{\nabla}(\vec{\nabla}\cdot a)-\nabla^2\vec{a}\right]^i$$

we get

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=0$$

and using (2) in this, $\vec{\nabla}\times\vec{B}=\frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$, and (1) $(\vec{\nabla}\cdot\vec{E}=0)$, obtain

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}-\nabla^2\vec{E}=0$$

This is the wave equation with velocity $v^2=\frac{c^2}{\mu\epsilon}$. With $n^2=\mu\epsilon$, this is $v=\frac{c}{n}$. n, we will see, is the index of refraction of the medium.

Similarly, taking $\vec{\nabla}\times$ (eq (2))

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}-\frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}=0$$

and using (4) and (3)

$$\Rightarrow \quad \frac{1}{v^2}\frac{\partial^2\vec{B}}{\partial t^2}-\nabla^2\vec{B}=0$$

again the wave equation with velocity $v=\frac{c}{n}$.

Plane wave solutions are as before,

$$\vec{E}(\vec{x},t)=\vec{E}_0\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}+\text{c.c.} \qquad \vec{B}(\vec{x},t)=\vec{B}_0\,e^{i(\vec{k}'\cdot\vec{x}-\omega' t)}+\text{c.c.}$$

where $\vec{k}^2-\frac{\omega^2}{v^2}=0$ and $\vec{k}'^2-\frac{\omega'^2}{v^2}=0$

These must satisfy (1)–(4):

$$(1)\Rightarrow\ \vec{\nabla}\cdot\vec{E}=0\ \Rightarrow\ \vec{k}\cdot\vec{E}_0=0 \qquad (4)\ \vec{\nabla}\cdot\vec{B}=0\ \Rightarrow\ \vec{k}'\cdot\vec{B}_0=0$$

(3) $\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \vec{k}\times\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} - \frac{\omega'}{c}\vec{B}_0 e^{i(\vec{k}'\cdot\vec{x}-\omega' t)} = 0$

which requires $\vec{k}'=\vec{k}$, $\omega'=\omega$ and $\vec{k}\times\vec{E}_0 = \frac{\omega}{c}\vec{B}_0 = \frac{v}{c}|\vec{k}|\vec{B}_0 = \frac{1}{n}|\vec{k}|\vec{B}_0$ or

$$\hat{k}\times\vec{E}_0 = \frac{1}{n}\vec{B}_0$$

(4) $\vec{\nabla}\times B - \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}=0 \Rightarrow \vec{k}\times\vec{B}_0 + \frac{\mu\epsilon}{c}\omega\vec{E}_0 = 0 \Rightarrow \vec{k}\times\vec{B}_0 = -n|\vec{k}|\vec{E}_0$

$$\hat{k}\times\frac{\vec{B}_0}{n} = -\vec{E}_0$$

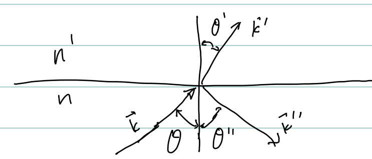
Summarizing, $\vec{E}_0$, $\vec{B}_0$ and $\hat{k}$ are mutually perpendicular, with $|\vec{B}_0| = n|\vec{E}_0|$ and $\frac{\vec{B}_0}{n}=\hat{k}\times\vec{E}_0$

In terms of polarization vectors

$$\vec{E} = E_0\,\vec{\epsilon}\, e^{i(\vec{k}\cdot x-\omega t)} \qquad \vec{B} = nE_0(\hat{k}\times\vec{\epsilon})\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

with $\vec{\epsilon}\cdot\vec{\epsilon}^*=1$ and $\vec{k}\cdot\vec{\epsilon}=0$.

Two media and interface



Incident wave

$$\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \vec{B} = n(\hat{k}\times\vec{E}_0)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

Refracted wave

$$\vec{E}' = \vec{E}_0'\, e^{i(\vec{k}'\cdot\vec{x}-\omega' t)} \qquad \vec{B}' = n'(\hat{k}'\times\vec{E}_0')\, e^{i(\vec{k}'\cdot x-\omega' t)}$$

Reflected wave

$$\vec{E}'' = \vec{E}_0''\, e^{i(\vec{k}''\cdot x-\omega'' t)} \qquad \vec{B}'' = n(\hat{k}''\times\vec{E}_0'')\, e^{i(\vec{k}''\cdot\vec{x}-\omega'' t)}$$

Let the boundary be the plane $z=0$. The fields $\vec{E}$ & $\vec{B}$ are given by any of these at the boundary so we must have $\omega=\omega'=\omega''$ for common time dependence. Note that this means

$$|\vec{k}''|=|\vec{k}| = \frac{\omega}{v} = \frac{\omega}{c}n \qquad \text{and} \qquad |\vec{k}'| = \frac{\omega}{v'} = \frac{\omega}{c}n' = \frac{n'}{n}|\vec{k}|$$

Moreover, at $z=0$

$$\vec{k}\cdot\vec{x} = \vec{k}'\cdot\vec{x} = \vec{k}''\cdot\vec{x}$$

That is, the projection on the xy plane is equal:

$$|k|\sin\theta = |k'|\sin\theta' = |k''|\sin\theta''$$

It follows that $\theta''=\theta$ and $\frac{\sin\theta'}{\sin\theta} = \frac{|k|}{|k'|} = \frac{n}{n'}$ or $\boxed{n'\sin\theta' = n\sin\theta}$

Snell's Law of refraction.

Polarization: There is additional information coded in the boundary conditions. For this we need

(i) Normal components of $\vec{D}=\epsilon\vec{E}$ and $\vec{B}$ are continuous (from $\vec{\nabla}\cdot\vec{D}=0$ & $\vec{\nabla}\cdot\vec{B}=0$)

(ii) Tangential components of $\vec{E}$ and $\vec{H}=\frac{1}{\mu}\vec{B}$ are continuous (from $\vec{\nabla}\times\vec{E}$ & $\vec{\nabla}\times\vec{H}$ eqs).

These will be derived when we study continuous media. For now, use them: let $\hat{n}$ be normal to the boundary. Then

$$\hat{n}\cdot[\epsilon(\vec{E}_0+\vec{E}_0'') - \epsilon'\vec{E}_0'] = 0 \qquad (1)$$

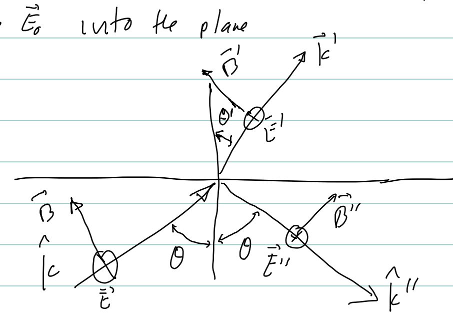
$$\hat{n}\cdot[n\hat{k}\times\vec{E}_0 + n\hat{k}''\times\vec{E}_0'' - n'\hat{k}'\times\vec{E}_0'] = 0 \qquad (2)$$

$$\hat{n}\times[\vec{E}_0+\vec{E}_0''-\vec{E}_0'] = 0 \qquad (3)$$

$$\hat{n}\times\left[\frac{n}{\mu}\hat{k}\times\vec{E}_0 + \frac{n}{\mu}\hat{k}''\times\vec{E}_0'' - \frac{n'}{\mu'}\hat{k}'\times\vec{E}_0'\right] = 0 \qquad (4)$$

Two cases (the general one is a linear combination of those):

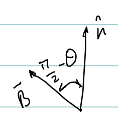
(i) Polarization vector $\vec{E}_0$ of incident wave (linearly polarized) perpendicular to plane of incidence that is plane of $\hat{k}$ and $\hat{n}$: take $\vec{E}_0$ into the plane



We have 1+1+3+3 equations for 3+3 unknowns ($\vec{E}_0'$ & $\vec{E}_0''$): the system is overdetermined. We look for solutions with the polarizations all into the plane (we could derive this). Then

$(3) \Rightarrow \quad E_0 + E_0'' - E_0' = 0$

$(4) \Rightarrow$



$$\frac{n}{\mu}(E_0 - E_0'')\cos\theta - \frac{n'}{\mu'}E_0'\cos\theta' = 0$$

Solving for E_0' & E_0''

from (3) $E_0' = E_0 + E_0''$, from (4) $\frac{n}{\mu}(E_0-E_0'')\cos\theta - \frac{n'}{\mu'}(E_0+E_0'')\cos\theta' = 0$

$$\Rightarrow \frac{E_0''}{E_0}\left(\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'\right) = \frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}\cos\theta' \quad \Rightarrow \quad \boxed{\frac{E_0''}{E_0} = \frac{\frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}\cos\theta'}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'}}$$

$\vec{E}_0 \perp$ to $\hat{n},\hat{k}$ plane

$$\text{Then } \frac{E_0'}{E_0} = 1 + \frac{E_0''}{E_0} = \frac{2\frac{n}{\mu}\cos\theta}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'} \quad \Rightarrow \quad \boxed{\frac{E_0'}{E_0} = \frac{2n\cos\theta}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'}}$$

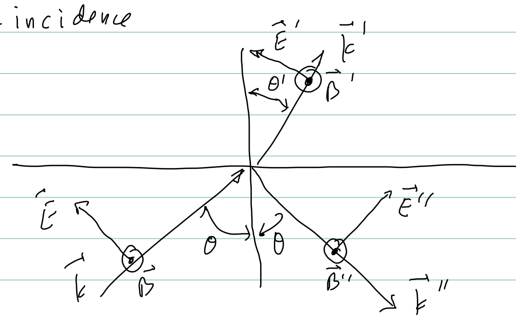
One often writes $\cos r = \sqrt{1-\sin^2\theta'} = \sqrt{1-\frac{n^2}{n'^2}\sin^2\theta} = \frac{1}{n'}\sqrt{n'^2-n^2\sin^2\theta}$ so that it is all expressed in terms of incident data.

What about (2)?

$\vec{B}$ $\hat{n}$ so $\hat{n}\cdot\vec{B} = \sin i\, B$ $\Rightarrow$ $n(\underbrace{E_0 + E_0''}_{E_0'})\sin\theta - n' E_0' \sin\theta' = 0$

$\Rightarrow$ $n \sin i = n' \sin r$ Snell's Law again

(ii) $\vec{E}_0$ parallel to plane of incidence



Now (3) and (4) give

$$(E_0 - E_0'')\cos\theta - E_0'\cos\theta' = 0$$

$$\frac{n}{\mu}(E_0 + E_0'') - \frac{n'}{\mu'}E_0' = 0$$

Algebra: (4): $E_0' = \frac{\mu'}{\mu}\frac{n}{n'}(E_0 + E_0'')$ in (3) gives

$$(E_0 - E_0'')\cos\theta - \frac{\mu'}{\mu}\frac{n}{n'}(E_0 + E_0'')\cos\theta' = 0 \quad \Rightarrow$$

$$\boxed{\frac{E_0''}{E_0} = \frac{\cos\theta - \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'}{\cos\theta + \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'} \qquad \frac{E_0'}{E_0} = \frac{2\frac{\mu'}{\mu}\frac{n}{n'}\cos\theta}{\cos\theta + \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'}}$$

$\vec{E}_0 \parallel$ to $\hat{k}$ & $\vec{n}$ plane

Then $\frac{E_0'}{E_0} = \frac{\mu'}{\mu}\frac{n}{n'}\left(1 + \frac{E_0''}{E_0}\right)$

Jackson: 7.4

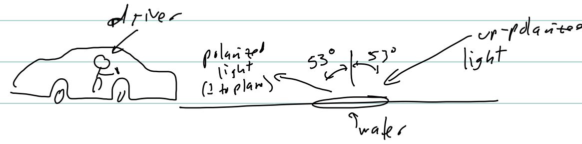
Brewster angle: in this case the reflected wave vanishes for $\cos\theta_B = \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta_B' = \frac{\mu'}{\mu}\frac{n}{n'}\sqrt{1 - \frac{n^2}{n'^2}\sin^2\theta_B}$

Consider simpler (but often found) case $\mu'/\mu = 1$. Then

$$\left(\frac{n'}{n}\cos\theta_B\right)^2 = 1 - \frac{n^2}{n'^2}\sin^2\theta_B = 1 - \frac{n^2}{n'^2} + \frac{n^2}{n'^2}\cos^2\theta_B \quad \Rightarrow \quad \cos^2\theta_B = \frac{1 - n^2/n'^2}{\frac{n'^2}{n^2} - n^2/n'^2} = \frac{1 - \frac{n^2}{n'^2} + n'^2/n^2 - n'^2/n'^2}{n'^2/n^2 - n^2/n'^2} = 1 - \frac{n'^2/n^2 - 1}{n'^2/n^2 - n^2/n'^2} = 1 - \sin^2\theta_B$$

$$\tan^2\theta_B = \frac{n'^2/n^2 - 1}{1 - n^2/n'^2} = \left(\frac{n'}{n}\right)^2 \qquad \boxed{\theta_B = \tan^{-1}(n'/n)}$$

For water $n = 1.33$ and air $n = 1.00$, $\theta_B = 0.93$ or 53°.



Jackson 7.4

Total internal reflection.

You know from elementary courses that for (using Snell's Law)

$$\sin\theta' = \frac{n}{n'}\sin\theta \geq 1 \qquad \text{critical at incident angle } \sin\theta_T = \frac{n'}{n}$$

There is no refracted wave. This is the phenomenon of "total internal reflection". We can understand some aspects of this in more detail with the tools developped.

The refracted wave is $\vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$

Now $\vec{k}'\cdot\vec{x} = |\vec{k}'|\left(z\cos\theta' + x_{\parallel}\sin\theta'\right)$

with $\cos\theta' = \sqrt{1-\sin^2\theta'} = i\sqrt{\sin^2\theta' - 1} = i\sqrt{\left(\frac{\sin\theta}{\sin\theta_T}\right)^2 - 1}$

so $e^{i\vec{k}'\cdot\vec{x}} = e^{-|\vec{k}'|\sqrt{\left(\frac{\sin\theta}{\sin\theta_T}\right)^2 - 1}\, z} \; e^{i|\vec{k}'|x\frac{\sin\theta}{\sin\theta_T}}$

$\Rightarrow$ the wave is exponentially damped into the interior of the medium n'.

There is no energy flux into n': $\vec{S} \propto \vec{E}\times\vec{B}$ (actually $\vec{E}\times\vec{H} = \vec{E}\times\frac{1}{\mu}\vec{B}$)

So

$$\hat{n}\cdot\vec{S} \propto \hat{n}\cdot(\vec{E}_0'\times\vec{B}_0'^{*}) + \text{c.c.} \;\propto\; 2\,\mathrm{Re}\left[\hat{n}\cdot\vec{E}_0'\times(\vec{k}'\times\vec{E}_0')^{*}\right] = 2\,\mathrm{Re}\left[\hat{n}\cdot\left(\vec{k}'|\vec{E}_0'|^2 - \vec{E}_0'\,\cancel{(\vec{k}\cdot\vec{E}_0'^{*})}^{\,0}\right)\right]$$

$$= 2\,\mathrm{Re}(\hat{n}\cdot\hat{k}')\,|\vec{E}_0'|^2$$

But $\hat{n}\cdot\hat{k}' = \cos\theta'$ is purely imaginary for $\theta > \theta_T \Rightarrow \underline{\underline{\hat{n}\cdot\vec{S} = 0}}$

Note also that $|\vec{E}_0''/E_0| = 1$ for each polarization, eg, for $\vec{E}_0 \perp$ to plane of incidence

$$\frac{E_0''}{E_0} = \frac{\frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}i|\cos\theta'|}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}i|\cos\theta'|}$$

The reflected wave has same intensity, but experiences a phase shift. The phase shift is different for each of the two plane polarizations, so this can be used to polarize waves.

L&L: 50
Jackson: 7.2

Degree of Polarization

(See time averages, p. 4 of these notes)

For $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} + c.c.$

the polarization tensor ρ_{ij} is defined by

$$\rho_{ij} \equiv \frac{E_{0i} E_{0j}^*}{|\vec{E}_0|^2}$$

The rank of this matrix is 2, since $\vec{E}_0$ has $\hat{k}\cdot\vec{E}_0 = 0$. Let's specify $\hat{k} = \hat{z}$ so that we can write ρ as a 2×2 matrix. Note that

$$\mathrm{Tr}\,\rho = 1, \qquad \rho^\dagger = \rho \quad \text{(hermitian)}. \qquad (A)$$

$$\text{and} \quad \det\rho = 0 \qquad (B)$$

For linear polarization, eg $E_{0y} = 0$ or $E_{0x} = 0$ we have $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

For $\vec{E}_0 = E_0 \vec{\epsilon}_\pm = \frac{1}{\sqrt{2}} E_0 (\hat{x} \pm i\hat{y})$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$$

Realistic light is not exactly monochromatic: consists of a superposition of frequencies $\omega \pm \Delta\omega$ with $\Delta\omega \ll \omega$. For example,

$$\int_{-\infty}^{\infty} d\omega'\, e^{-i\omega' t}\, e^{-(\omega'-\omega)^2/2(\Delta\omega)^2} = e^{-i\omega t}\int_{-\infty}^{\infty} d\omega'\, e^{-i\omega' t}\, e^{-\omega'^2/2(\Delta\omega)^2}$$

completing the squares $\frac{\omega'^2}{2(\Delta\omega)^2} + i\omega' t = \frac{1}{2(\Delta\omega)^2}\left(\omega' + i(\Delta\omega)^2 t\right)^2 + \frac{(\Delta\omega)^2 t^2}{2} \Rightarrow \propto \underbrace{e^{-i\omega t}}_{\text{fast}}\, \underbrace{e^{-\frac{(\Delta\omega)^2 t^2}{2}}}_{\text{slow}}$

At one point in space this is a wave packet that moves through the point.
One can generally write (at one point in space)

$$\vec{E}(t) = \vec{E}_0(t)\, e^{-i\omega t} \qquad \text{with } \left|\frac{d\vec{E}_0}{dt}\right| \sim \Delta\omega |\vec{E}_0|, \text{ ie, } \vec{E}_0 \text{ varies slowly}$$

ρ is more generally defined as

$$\rho_{ij} = \frac{\overline{E_{0i} E_{0j}^*}}{\overline{|\vec{E}_0|^2}} \qquad \text{where } \overline{f(t)} = \frac{1}{T}\int_0^T dt\, f(t) \quad \text{the time average}$$

For example, above, with $T = \frac{2\pi}{\omega}$

$$\frac{1}{T}\int_0^T e^{-i\omega t - \frac{(\Delta\omega)^2 t^2}{2}}\, dt = \frac{1}{T}\int_0^T e^{-i\omega t}\left(1 - \frac{(\Delta\omega)^2 t^2}{2} + \cdots\right) dt = 0 - \frac{(\Delta\omega)^2}{\omega^2} + i\pi\frac{(\Delta\omega)^2}{\omega^2} + O\!\left(\frac{(\Delta\omega)^4}{\omega^4}\right)$$

where we used $\int dt\, e^{-i\omega t} t^2 = \left(\frac{1}{-i}\frac{\partial}{\partial\omega}\right)^2 \int dt\, e^{-i\omega t} = -\frac{\partial^2}{\partial\omega^2}\left(i\frac{e^{-i\omega t}}{\omega}\right) = -i\frac{\partial}{\partial\omega}\left[e^{-i\omega t}\left(-\frac{1}{\omega^2} - \frac{it}{\omega}\right)\right] = i e^{-i\omega t}\left[\frac{2}{\omega^3} - \frac{2it}{\omega^2} - \frac{t^2}{\omega}\right]$

Now the conditions (A) still hold, but (B) does not hold generally.
$\det\rho = 0$ is the condition for complete polarization. It is necessary and sufficient.

$$(\overline{E_0^2})^2 \det\rho = \overline{E_{0x}E_{0x}^*}\;\overline{E_{0y}E_{0y}^*} - \overline{E_{0x}E_{0y}^*}\;\overline{E_{0y}E_{0x}^*}$$

Schwarz inequality: a & b functions of time, λ a parameter,

$$\overline{|a+\lambda b|^2} \geq 0 \quad\Rightarrow\quad \overline{|a|^2} + |\lambda|^2\overline{|b|^2} + \lambda^*\overline{ab^*} + \lambda\overline{a^*b} \geq 0 \qquad \text{any } \lambda$$

with $\lambda = -\overline{ab^*}/\overline{|b|^2}$ $\Rightarrow$ $\overline{|a|^2}\,\overline{|b|^2} \geq \overline{ab^*}\;\overline{a^*b}$

and the equality is only if $a+\lambda b = 0$ or $a(t) = -\lambda b(t)$ (λ is time independent).

So $\det\rho \geq 0$ with $\det\rho = 0$ only for $E_{0x} + \lambda E_{0y} = 0$, ie E_{0y} proportional to E_{0y} with constant (in time) proportionality constant: for λ real this is plane polarization for $\lambda = \pm i$, circular polarization, others $\Rightarrow$ ellyptical.

Stoke Parameters

(There are many definitions, I choose one).

with $\vec\sigma$ the Pauli matrices $\sigma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, and $\mathbb{1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

any 2x2 hermitian matrix

$$\rho = \tfrac12 a_0 \mathbb{1} + \tfrac12 \vec a\cdot\vec\sigma \qquad . \qquad \mathrm{Tr}\,\rho = 1 \;\Rightarrow\; a_0 = 1$$

a_1, a_2, a_3 are "Stoke parameters".

$$\det\rho = \det\tfrac12\begin{pmatrix}1+a_3 & a_1 - ia_2\\ a_1+ia_2 & 1-a_3\end{pmatrix} = \tfrac14(1-\vec a^2) \geq 0 \;\Rightarrow\; \vec a^2 \leq 1$$

The degree of Polarization $\underline{P} \equiv a_1^2 + a_2^2 + a_3^2$ has $P \leq 1$, with $P=1$ for complete polarization.

Additionally

(i) For $E_{0y} = 0$ ($\vec E_0 \propto \hat x$) $a_3 = 1$; more generally ρ is normalized to $\overline{|E|^2} \propto \bar u \propto \overline{|\vec S|}$, ie, beam intensity, so $\frac12(1+a_3) = \rho_{xx}$ is the fraction of intensity that passes through an $\hat x$-polarization filter, and $\frac12(1-a_3) = \rho_{yy}$ through a $\hat y$-polarizer

(ii) Similarly $\frac12(1+a_1)$ = fraction of intensity through $\hat x + \hat y$ filter

(iii) Setting $\vec E_0 = E_0\vec\epsilon_+$ gives $a_2 = 1$: $\frac12(1+a_2)$ fraction of intensity that passes through right circular polarizer.

Also useful

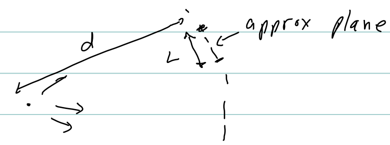
$a_L = \sqrt{a_1^2 + a_3^2}$ = degree of linear polarization

$a_C = a_2$ = 〃 〃 circular

L.L: Chap 7: 53
Scheck: 4.5.2
See also Chaichian et al ; 4.8.4

Eikonal Approximation

For non-plane waves (maybe spherical waves) we expect that at a distance d from sources, and over a region of size L, if $d \gg L$ (and $L \gg \lambda$) the wave is approximately a plane wave

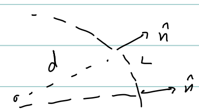


The Eikonal approximation formalizes this, and is the basis for geometrical optics. Take

$$\vec{E} = \vec{E}_0(\vec{x})\, e^{ik\psi(\vec{x}) - ikx^0}$$

$$\vec{B} = \vec{B}_0(\vec{x})\, e^{ik\psi(\vec{x}) - ikx^0}$$

We are assuming still monochromatic waves, with $k = 2\pi/\lambda$. But now what used to be $\exp(ikz) = \exp(ik\hat{n}\cdot\vec{x})$ with $\hat{n}$ giving the direction of propagation, became a function $\psi(\vec{x})$. Now we expect ψ to be like $\hat{n}\cdot\vec{x}$ with $\hat{n}$ varying as in



so $\Delta\psi \sim \frac{L}{d}\psi$ or $\frac{\Delta\psi}{\Delta L} \sim \frac{1}{d}\psi \sim \frac{d}{d} \sim 1$ (like $\hat{n}$).

That $\vec{E}_0$ and $\vec{B}_0$ have $\vec{x}$ dependence is also clear, since the polarization must change. But let's just follow the math. Let's use the ansatz in Maxwell's equations for A_μ:

$$A_\mu = a_\mu e^{ik\psi - ikx^0} \qquad a_\mu = a_\mu(\vec{x}) \qquad \psi = \psi(\vec{x})$$

$$\partial_\mu A^\mu = 0 \qquad \partial_\mu a^\mu + ik\left(\partial_\mu\psi - \delta^0_\mu\right)a^\mu = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{a} + ik\left(\vec{\nabla}\psi\cdot\vec{a} - a^0\right) = 0$$

and

$$\partial^2 A_\mu = 0 \quad \Rightarrow \quad -\vec{\nabla}^2 a_\mu - 2ik\,\vec{\nabla}a_\mu\cdot\vec{\nabla}\psi + a_\mu\left(k^2(\vec{\nabla}\psi)^2 - ik\nabla^2\psi\right) - k^2 a_\mu = 0$$

Now $k \sim \frac{2\pi}{\lambda} \gg \vec{\nabla}$ for any of these. So we must have

$$(\partial_\mu A^\mu = 0) \rightarrow \quad a^0 = \vec{\nabla}\psi\cdot\vec{a} \qquad \text{and} \qquad (\partial^2 A = 0) \quad (\vec{\nabla}\psi)^2 = 1$$

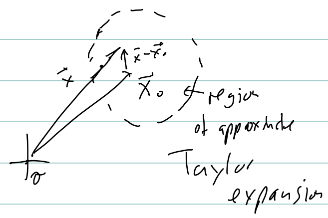
We still can redefine $A_\mu \to A_\mu + \partial_\mu\Phi$ with $\partial^2\Phi = 0$. Taking $\Phi = \varphi(\vec{x})\, e^{ik(\psi - x^0)}$

$A_\mu \to A_\mu + (\partial_\mu\varphi + ik(\partial_\mu\psi - \delta^0_\mu)\varphi)\, e^{ik(\psi - x^0)}$. In particular $A_0 \to A_0 - ik\varphi\, e^{ik(\psi - x^0)}$

or $a_0 \to a_0 - ik\varphi \Rightarrow$ set $a_0 = 0$

Summary: $(\vec{\nabla}\psi)^2 = 1 \qquad \bar{a}\cdot\vec{\nabla}\psi = 0 \qquad a^0 = 0$

Now $\vec{\nabla}\psi$ is a unit vector $\vec{\nabla}\psi = \hat{n}$ which depends on $\vec{x}$, $\hat{n} = \hat{n}(\vec{x})$;
Expanding $\psi(\vec{x}) = \psi(\vec{x}_0) + (\vec{x}-\vec{x}_0)\cdot\vec{\nabla}\psi(\vec{x}_0) + \cdots = \psi(\vec{x}_0) + \hat{n}\cdot(\vec{x}-\vec{x}_0) + \cdots = \alpha + \hat{n}\cdot(\vec{x}-\vec{x}_0) + \cdots$
we see that the wave in the region around $\vec{x}_0$ is



$$\vec{A} \simeq \vec{a}(\vec{x})\, e^{ik(\hat{n}\cdot\vec{x} - x^0) + i\alpha}$$

$$A^0 = 0 \qquad \vec{a}(\vec{x})\cdot\hat{n}(\vec{x}) = 0$$

$\vec{E}$ & $\vec{B}$:

We have $\vec{E} = -\dfrac{\partial\vec{A}}{\partial x^0} = ik\vec{A} = \vec{E}_0(\vec{x})\, e^{ik(\psi - x^0)}$

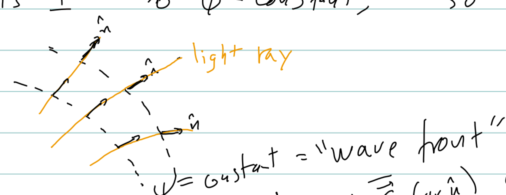
and $\vec{B} = \vec{\nabla}\times\vec{A} = ik\,\hat{n}\times\vec{A} = \hat{n}\times\vec{E}$

$$\bar{u} = \frac{1}{8\pi}\overline{(E^2 + B^2)} = \frac{1}{16\pi}(\vec{E}_0\cdot\vec{E}_0^* + \vec{B}_0\cdot\vec{B}_0^*) = \frac{1}{8\pi}|\vec{E}_0|^2$$

$$\overline{\vec{S}} = \frac{c}{8\pi}(\vec{E}_0\times\vec{B}_0^*) = \frac{c}{8\pi}\hat{n}|\vec{E}_0|^2 \qquad \Rightarrow \quad \overline{\vec{S}} = c\,\bar{u}\,\hat{n}$$

or $\bar{u} = \frac{1}{c}\hat{n}\cdot\overline{\vec{S}}$

The interpretation of $\hat{n} = \vec{\nabla}\psi$ is clear: it gives the direction of energy flow.
Note that $\vec{\nabla}\psi$ is $\perp$ to $\psi = \text{constant}$, so



Find those.

Let light ray be $\vec{x}(\lambda)$ with λ a parameter of the curve (much like $x^\mu(\lambda)$ for world-line).

Then $\dfrac{d\vec{x}}{d\lambda} \propto \hat{n}$ where $\hat{n} = \hat{n}(\vec{x}(\lambda))$. The proportionality is such that

$\hat{n}^2 = 1$: $\dfrac{d\vec{x}}{d\lambda} = \left|\dfrac{d\vec{x}}{d\lambda}\right|\hat{n}$. If $\lambda = s$, the lenght along the trajectory ($ds^2 = d\vec{x}\cdot d\vec{x}$)

then $\dfrac{d\vec{x}}{ds} = \hat{n} = \vec{\nabla}\psi$

But $\dfrac{d^2\vec{x}}{ds^2} = \dfrac{d}{ds}\vec{\nabla}\psi = \dfrac{dx^i}{ds}\partial_i(\vec{\nabla}\psi) = \hat{n}\cdot\vec{\nabla}\hat{n}$, but $\hat{n}\cdot\vec{\nabla}n^i = n^j\partial_j n^i = n^j\partial_j\partial_i\psi = n^j\partial_i n^j = \frac{1}{2}\vec{\nabla}_i(\hat{n}^2) = \frac{1}{2}\vec{\nabla}(1) = 0$

Acceleration $= 0 \Rightarrow$ constant velocity: $\dfrac{d\vec{r}}{ds} = \text{constant} \Rightarrow \vec{r}(s) = $ straight line

$\Longrightarrow$ geometrical optics.

In a medium replace $k^2(\vec\nabla\psi)^2 = k^2 = \frac{\omega^2}{c^2}$ by $k^2(\nabla\psi)^2 = \frac{\omega^2}{v^2} = n^2k^2$ or $(\nabla\psi)^2 = n^2$. If medium is not homogeneous $n = n(\vec x)$ and now rays satisfy

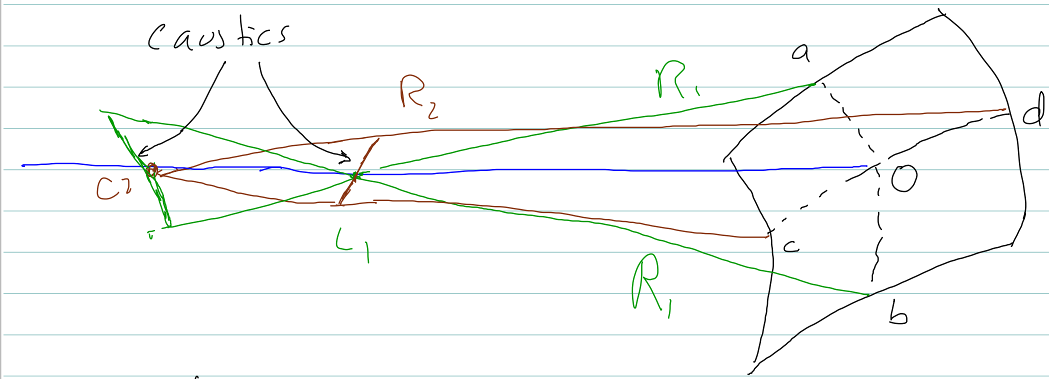
$$\frac{d\vec r}{ds} = \vec\nabla\psi \quad \text{with} \quad \left|\frac{d\vec r}{ds}\right|^2 = n^2 \qquad \text{(this "}n\text{" is index of refraction).}$$

Ref: L.L 54

Note that for $n(\vec x)$ = constant light rays are still straight. But they can curve for non-homogeneous ($n \neq$ constant) media.

Additional information is in $\vec a$ (or $\vec E_0$ & $\vec B_0$): light is also characterized by intensity. One can derive equations for $\vec a$ from next order in Eikonal expansion (so-called transport equations). We will settle for simple energy considerations (as done in most texts).

Consider wave-front ψ = const. Take a sufficiently small portion that we can characterize it by two principal sections of curvature:



Centers of curvature C_1 & C_2
Radii: R_1 & R_2

With fixed angles from C_1 & C_2 we get as we vary ψ a collection of surface segments with area $\propto R_1R_2$, all having the same light rays crossing which is equivalent to having constant $\int \vec S\cdot\hat n\, ds$. So the collection of surfaces has light intensity $I R_1 R_2$ = constant (I = energy flux) or

$$I = \frac{\text{Constant}}{R_1 R_2}$$

As the surfaces approach C_1 (C_2) the radius R_1 (R_2) shrinks to zero and the surface degenerates into a burning line or "caustic". If $R_1 = R_2$ it's a single point, the "focal point".

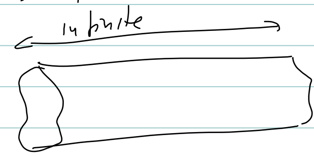
We'll come back to optics when we study diffraction.

Jackson: chap 8, secs 2-4
Schwartz: Chap 9

Wave guides and cavities

There are many practical uses for waves in regions of space bounded by conductors. The prime examples are

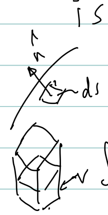
cylindrical wave guides:



symmetric under translations in $\longleftrightarrow$ direction
(ie: constant cross section shape)

cylindrical cavities: as above with end caps).
("resonant" cavities)

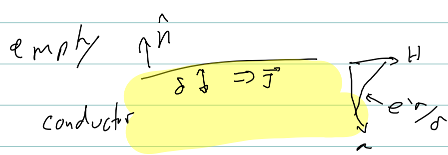
We will assume perfect conductors. For these, $\vec{E}=0$ in the interior. Any excess charge resides on the surface. If the surface charge density is σ then



$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \implies \hat{n}\cdot\vec{D} = \sigma$$

$$\int_V \vec{\nabla}\cdot\vec{D} = \int_{\partial V} \hat{n}\cdot D\, ds = 4\pi Q$$

In addition, one can show that harmonic ($\sim e^{i\omega t}$) magnetic fields $\vec{H}$ vanish inside the perfect conductor and for realistic conductors there is a "skin depth" over which the field exponentially dies



There is a current density $\vec{J}$ to the depth δ of the skin.

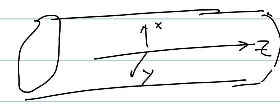
For perfect conductors $\delta\to 0$ and $\vec{J}$ becomes a surface current $\vec{K}$.

The b.c. is $\hat{n}\times\vec{H} = \frac{4\pi}{c}\vec{K}$

The homogeneous eqs give $n\cdot(\vec{B}_{in}-\vec{B}_{out})=0$ $\quad \hat{n}\times(\vec{E}_{in}-\vec{E}_{out})=0$ always. We take $\vec{E}_{in}=0=\vec{B}_{in}$, and drop the label: $\hat{n}\cdot\vec{B}=0$ and $\hat{n}\times\vec{E}=0$.

Put z-axis along cylindrical axis. Look for solutions to source free Maxwell equations of the form

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}(x,y)e^{i(kz-\omega t)}$$



$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}(x,y)\,e^{i(kz-\omega t)}$$

As before $$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

which now implies

$$\left[\nabla_\perp^2 + \left(\frac{\mu\epsilon}{c^2}\omega^2 - k^2\right)\right]\begin{pmatrix}\vec{\mathcal{E}}\\ \vec{\mathcal{B}}\end{pmatrix} = 0 \qquad \text{where } \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

One would guess that the solutions are plane waves with $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ perpendicular to $\hat{z}$ and $\vec{\mathcal{B}} \perp \vec{\mathcal{E}}$. But we will find solutions with non-vanishing z-component:

- Transverse electric modes (TE): $E_z = 0$ but $B_z \neq 0$
- Transverse magnetic modes (TM): $B_z = 0$ but $E_z \neq 0$
- Transverse electromagnetic mode (TEM): $E_z = 0$ and $B_z = 0$.

In preparation for this we separate $\vec{E}_z$ from $\vec{E}_\perp$

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_\parallel + \vec{\mathcal{E}}_\perp$$

where $\hat{z}\cdot\vec{\mathcal{E}}_\perp = 0$ $\quad \vec{\mathcal{E}}_\parallel = \hat{z}\,\hat{z}\cdot\vec{\mathcal{E}}_\parallel = \hat{z}\,\mathcal{E}_z$, $\quad \vec{\mathcal{E}}_\perp = \vec{\mathcal{E}} - \hat{z}\,\hat{z}\cdot\vec{\mathcal{E}} = (\hat{z}\times\vec{\mathcal{E}})\times\hat{z}$

and idem for $\vec{\mathcal{B}}$. Simplify discussion by assuming the cavity or wave guide are empty, so that $\mu\epsilon = 1$. The general $\mu\epsilon$ case is left as homework.

To separate $\perp$ from $\parallel$ modes in Faraday $\left(\vec{\nabla}\times\vec{E} - i\frac{\omega}{c}\vec{B} = 0\right)$ take $\hat{z}\times(\text{eq})$ using

$$\hat{z}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}_\perp E_z - ik\vec{E}_\perp \qquad (\text{factor of } e^{i(kz-\omega t)} \text{ understood})$$

(Algebra, not for class: $\epsilon^{n3i}\left(\epsilon^{ijk}\partial_j E_k\right) = \left(\delta^{nj}\delta^{k3} - \delta^{nk}\delta^{j3}\right)\partial_j E_k = \vec{\nabla}(\hat{z}\cdot\vec{E}) - \frac{\partial}{\partial z}\vec{E}$

and $\vec{\nabla}E_z - \frac{\partial}{\partial z}\vec{E} = \left(\partial_x E_z - \partial_z E_x, \partial_y E_z - \partial_z E_y, 0\right) = \vec{\nabla}_\perp E_z - \partial_z\vec{E}_\perp$).

$$\Rightarrow \boxed{ik\vec{E}_\perp + i\frac{\omega}{c}\hat{z}\times\vec{B}_\perp = \vec{\nabla}_\perp E_z}$$

Similarly from Ampere's law $\left(\vec{\nabla}\times\vec{B} + \underbrace{\mu\epsilon}_{=1} i\frac{\omega}{c}\vec{E} = 0\right)$

using $(\vec{\nabla}\times\vec{B})_\perp = ik\hat{z}\times\vec{B}_\perp - \hat{z}\times\vec{\nabla}_\perp B_z$

[Algebra, not for class:

$$\epsilon^{ijn}\partial_j(B_n e^{ikz}) = e^{ikz}\left[\epsilon^{ijn}\partial_j B_n + ik\epsilon^{i3n}B_n\right]$$

with $i=1,2$ and noting that $B = B(x,y)$ only, $\epsilon^{ijn}\partial_j B_n = \epsilon^{ij3}\partial_j B_z = \left[(\vec{\nabla}B_z)\times\hat{z}\right]^i$

so $(\vec{\nabla}\times\vec{B})_\perp = -\hat{z}\times\vec{\nabla}B_z + ik\,\hat{z}\times\vec{B}_\perp$].

$$\Rightarrow \boxed{i\frac{\omega}{c}\vec{E}_\perp + ik\hat{z}\times\vec{B}_\perp = \hat{z}\times\vec{\nabla}_\perp B_z}$$

These 2 equations give $\vec{E}_\perp$ & $\hat{z}\times\vec{B}_\perp$ in terms of $\vec{\nabla}_\perp E_z$ & $\vec{\nabla}_\perp B_z$.
Solve the simultaneous equations

$$M\begin{pmatrix}\vec{E}_\perp\\ \hat{z}\times\vec{B}_\perp\end{pmatrix} = \begin{pmatrix}\vec{\nabla}_\perp E_z\\ \hat{z}\times\vec{\nabla}_\perp B_z\end{pmatrix} \qquad \text{with } M = \begin{pmatrix} ik & i\frac{\omega}{c}\\ i\frac{\omega}{c} & ik\end{pmatrix}$$

$$\text{so} \quad \begin{pmatrix}\vec{E}_\perp\\ \hat{z}\times\vec{B}_\perp\end{pmatrix} = M^{-1}\begin{pmatrix}\vec{\nabla}_\perp E_z\\ \hat{z}\times\vec{\nabla}_\perp B_z\end{pmatrix} \qquad \text{where } M^{-1} = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2}\begin{pmatrix} ik & -i\omega/c\\ -i\omega/c & ik\end{pmatrix}$$

(Of course $\vec{B}_\perp$ is determined, $\vec{B}_\perp = (\hat{z}\times\vec{B}_\perp)\times\hat{z}$).

So we solve the wave equation for the z-component:

$$\left(\vec{\nabla}_\perp^2 + \gamma^2\right)\psi = 0 \qquad \text{where } \psi = E_z \text{ or } B_z \text{ and } \gamma^2 = \left(\frac{\omega}{c}\right)^2 - k^2$$

Boundary conditions: we want them in terms of $\mathcal{E}_z$ & $\mathcal{B}_z$ only.

$\hat{n}\times\vec{E}=0 \Rightarrow \mathcal{E}_z=0$

$\hat{n}\cdot\vec{B}=0$: Take (Ampere's law) $\cdot(\hat{n}\times\hat{z}) \Rightarrow \hat{n}\cdot\nabla_\perp\mathcal{B}_z=\frac{\partial\mathcal{B}_z}{\partial n}=0$

(Algebra: not in class

$$(\hat{n}\times\hat{z})\cdot\left(i\frac{\omega}{c}\vec{\mathcal{E}}_\perp+ik\hat{z}\times\vec{\mathcal{B}}_\perp-\hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z\right)=0$$

$\vec{\mathcal{E}}_\perp\cdot\hat{n}\times\hat{z}=(\vec{\mathcal{E}}_\perp\times\hat{n})\cdot\hat{z}=0$ on boundary

$(\hat{n}\times\hat{z})\cdot(\hat{z}\times\vec{\mathcal{B}}_\perp)=-\epsilon^{ij3}\epsilon^{ik3}\hat{n}^j\mathcal{B}_\perp^k=\hat{n}\cdot\vec{\mathcal{B}}_\perp=0$ on boundary).

So

<u>TE</u>: $\mathcal{E}_z=0$. Then solve $(\nabla_\perp^2+\gamma^2)\mathcal{B}_z=0$

and from this compute

$$\vec{\mathcal{E}}_\perp=-i\frac{\omega}{c\gamma^2}\hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z \quad\text{and}\quad \vec{\mathcal{B}}_\perp=\frac{ik}{\gamma^2}\vec{\nabla}_\perp\mathcal{B}_z$$

<u>TM</u>: $\mathcal{B}_z=0$. Solve $(\vec{\nabla}_\perp^2+\gamma^2)\mathcal{E}_z=0$ and then

$$\vec{\mathcal{E}}_\perp=i\frac{k}{\gamma^2}\vec{\nabla}_\perp\mathcal{E}_z \qquad \vec{\mathcal{B}}_\perp=i\frac{\omega}{c\gamma^2}\hat{z}\times\vec{\nabla}_\perp\mathcal{E}_z$$

<u>TEM</u>: $\mathcal{E}_z=\mathcal{B}_z=0$. Then

$$M\begin{pmatrix}\vec{\mathcal{E}}_\perp\\ \hat{z}\times\vec{\mathcal{B}}_\perp\end{pmatrix}=0$$ has solutions only if $\det M=0 \Leftrightarrow \frac{\omega^2}{c^2}=k^2$

Then, from Ampere's law (above) $\vec{\mathcal{B}}_\perp=(\operatorname{sgn}k)\,\hat{z}\times\vec{\mathcal{E}}_\perp$

Faraday $\Rightarrow \vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp=0$

(Algebra: $\vec{\nabla}\times(\vec{\mathcal{E}}_\perp e^{ikz})=e^{ikz}\left[\vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp+ik\hat{z}\times\vec{\mathcal{E}}_\perp\right]$ and $ik\hat{z}\times\vec{\mathcal{E}}_\perp-i\frac{\omega}{c}\vec{\mathcal{B}}_\perp=0$).

Gauss: $\vec{\nabla}_\perp\cdot\vec{\mathcal{E}}_\perp=0$

So $\vec{\nabla}_\perp\cdot\vec{\mathcal{E}}_\perp=0$, $\vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp=0$ and $\vec{\mathcal{E}}_\perp\times\hat{n}\big|_{\text{boundary}}=0$ is a problem in 2-dimensional electrostatics.

Notes:

(1) For TM & TE modes one must have $\gamma^2>0$ to satisfy b.c.'s

Exercise: show this!

Now $(\nabla_\perp^2+\gamma^2)\psi=0$ with $\psi\big|_{\partial V}=0$ or $\frac{\partial\psi}{\partial n}\big|_{\partial V}=0$ is an

eigenvalue problem: expect discrete solutions $\psi_n(x,y)$ for $\gamma^2=\gamma_n^2$, $n=1,2,\ldots$

This means ω and k are related by

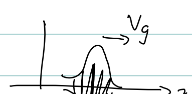
$$\left(\frac{\omega}{c}\right)^2-k^2=\gamma_n^2 \qquad \text{or} \qquad \omega=c\sqrt{k^2+\gamma_n^2}$$

Cut-off frequency: If $\gamma_1<\gamma_2<\gamma_3<\cdots$ (ie, we organize γ_n's by their magnitude) then there is a minimum possible frequency for transmission, $\omega_{min}=c\gamma_1$

For $\omega\in[c\gamma_1,c\gamma_2]$ the only propagating mode is ψ_1. Similarly, for $c\gamma_2<\omega<c\gamma_3$ one can have $\psi(x,y)=c_1\psi_1(x,y)+c_2\psi_2(x,y)$ propagating in the wave guide.

Since $E,B\sim e^{ik(x-\frac{\omega}{k}t)}$, the phase velocity is $v_p=\frac{\omega}{k}=c\frac{\sqrt{k^2+\gamma_n^2}}{k}$ for mode ψ_n. So $v_p>c$ and diverges as $k\to0$ $(\lambda\to\infty)$.

The group velocity is defined [for $\omega=\omega(k)$ a slowly varying function] to give the velocity of the peak of a wave-packet, which is where the energy density is localized:



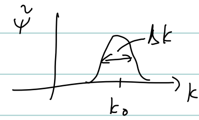
(ignoring x,y dependence) $\psi(z,t)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde\psi(k)\,e^{i(kz-\omega(k)t)}$

$\tilde\psi$ can be determined from snapshot at fixed time, say $t=0$:

$$\psi(z,0)=\int_{-\infty}^{\infty}\frac{dk}{(2\pi)}e^{ikz}\tilde\psi(k) \Rightarrow \tilde\psi(k)=\int_{-\infty}^{\infty}dz\,e^{-ikz}\psi(z,0)$$

For $\omega=$ constant $\psi(z,t)=\psi(z-\frac{\omega}{k}t,0)$ so $\frac{\omega}{k}=$ constant has $v_p=v_g$.

If $\tilde\psi(k)$ is localized around k_0



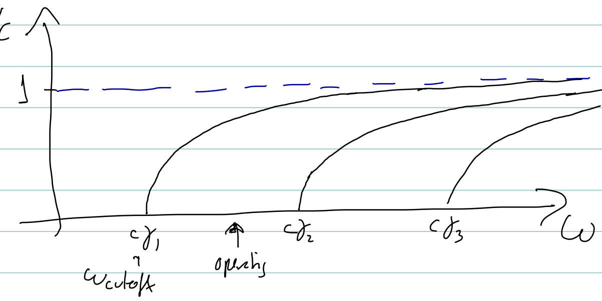
so $\Delta z\sim\frac{1}{\Delta k}$ is the width of the wave-packet, then

$$\omega(k)\simeq\omega(k_0)+(k-k_0)\frac{d\omega}{dk}\Big|_{k_0}+\cdots \quad\text{and}\quad \psi(z,t)=\int\frac{dk}{2\pi}\tilde\psi(k)\,e^{i\left[kz-\left(\omega(k_0)+(k-k_0)\frac{d\omega}{dk}\big|_{k_0}\right)t\right]}$$

$$\text{so } \psi(z,t)\simeq e^{-i(\omega_0-k_0\frac{d\omega}{dk}|_0)t}\,\psi\!\left(z-\frac{d\omega}{dk}\Big|_{k_0}t,0\right) \Rightarrow v_g=\frac{d\omega}{dk}\Big|_{k_0}$$

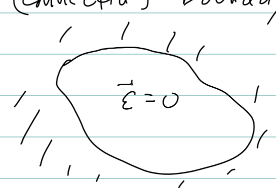
For wave-guides $v_g=\frac{d\omega}{dk}=c\frac{k}{\sqrt{k^2+\gamma_n^2}}<c$ $\left(\text{or } \frac{v_g}{c}=\frac{\sqrt{\omega^2-c^2\gamma_n^2}}{\omega}\right)$

Possible group velocities in wave-guide

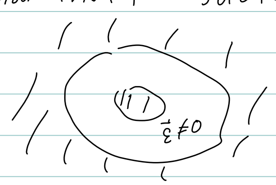


Often design is such that operating frequency ω is between ω_{cutoff} and $\omega_{2,cutoff}$.

(ii) For TEM there are no solutions other than trivial ($\vec{E}=0$) for a single (connected) boundary

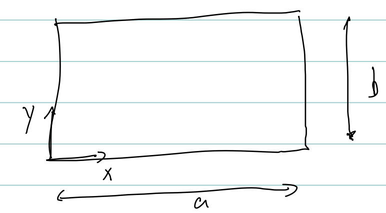


One can have non-trivial solutions with ≥ 2 disconnected boundaries



This is why co-axial cables or two-wire transmission lines are used in practice.

Example: rectangular cavity



TM: $B_z = 0$, $E_z\big|_{\partial V} = 0$ $\quad (\nabla_\perp^2 + \gamma^2) E_z(x,y) = 0$

Separation of variables: $E = E_0 X(x) Y(y)$

$$-\gamma^2 = \frac{1}{E}\nabla_\perp^2 E = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad \text{with } X(0) = X(a) = 0 \quad Y(0) = Y(b) = 0$$

$$\Rightarrow X(x) = \sin\left(\frac{m\pi}{a}x\right) \quad Y = \sin\left(\frac{n\pi}{b}y\right) \quad \Rightarrow E_z(x,y) = E_{0z}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

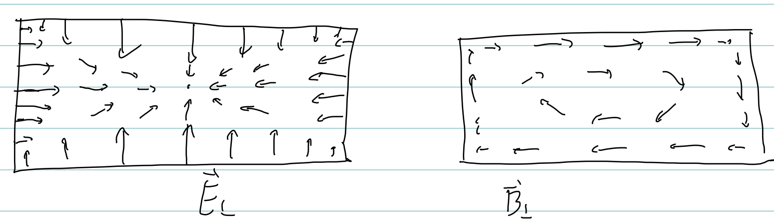
$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad m, n \in \mathbb{Z} \quad mn \neq 0.$$

The transverse fields are

$$\vec{E}_\perp = i\frac{k}{\gamma^2}\vec{\nabla}_\perp E_z = i\frac{k}{\gamma_{mn}^2}E_{0z}\left(\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right), 0\right)$$

$$\vec{B}_\perp = \frac{\omega}{ck}\hat{z}\times\vec{E}_\perp = i\frac{\omega}{c\gamma_{mn}^2}E_{0z}\left(-\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), \frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), 0\right)$$

TM_{11}

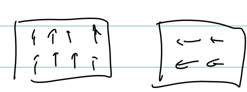


TE: $E_z = 0$ $\quad \frac{\partial B_z}{\partial n}\Big|_{\partial V} = 0$ $\quad (\nabla_\perp^2 + \gamma^2) B_z(x,y) = 0$

Same but with $\frac{dX}{dx}\Big|_{0,a} = 0$ $\quad \frac{dY}{dy}\Big|_{0,b} = 0$ $\quad \Rightarrow$ cosines

$$B_z(x,y) = B_{z0}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad \gamma_{mn}^2 \text{ as above} \quad \text{and } m, n \in \mathbb{Z} \text{ except } m = n = 0.$$

Exercise: compute $\vec{E}_\perp, \vec{B}_\perp$, and plot TE_{10}



Circular cross section: need Bessel functions → problem session.

And in notes in "Appendix: Bessel functions".

Resonant cavities.

The discussion is as above, but in addition we have boundary conditions at $z = 0, L$ (assuming both ends are capped, and the capping surface is plane).

We now have standing waves so $e^{ikz} \to \cos(kz)$ or $\sin(kz)$

TE: This had $B_z \neq 0$, $E_z = 0$, but now $B_z = 0$ at $z = 0, L$ (recall $\vec{B}\cdot\hat{n} = 0$).

$$B_z(\vec{x}) = \psi(x,y)\sin\left(\frac{\pi p}{L}z\right) \qquad p \in \mathbb{Z}_+$$

TM: We need $\vec{E}_\perp = 0$ at $z = 0, L$, but $\vec{E}_\perp = \frac{1}{\gamma^2}\vec{\nabla}_\perp \frac{\partial E_z}{\partial z} \Rightarrow \left.\frac{\partial E_z}{\partial z}\right|_{0,L} = 0$

$$E_z(\vec{x}) = \psi(x,y)\cos\left(\frac{\pi p}{L}z\right) \qquad p = 0, 1, \dots$$

Note that now $\frac{\omega^2}{c^2} = k^2 + \gamma^2 = \left(\frac{\pi p}{L}\right)^2 + \gamma^2$ completely discretized

For example, box $a \times b \times L$ has

$$\omega_{mnp} = c\sqrt{\left(\frac{\pi p}{L}\right)^2 + \gamma_{mn}^2} = c\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}$$