PHYS 203A - Winter 2020

Hw 1:

1. The first problem is calisthenics, to loosen your brain muscles.
   Let ψ be a scalar function and $\vec{F}$ and $\vec{G}$ be vector fileds. Prove the identities:
   (a) $\vec{\nabla}\cdot(\psi\vec{F}) = \psi\vec{\nabla}\cdot\vec{F} + \vec{\nabla}\psi\cdot\vec{F}$
   (b) $\vec{\nabla}\times(\psi\vec{F}) = \psi\vec{\nabla}\times\vec{F} + \vec{\nabla}\psi\times\vec{F}$
   (c) $\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$
   (d) $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}$

2. Let $\vec{p}$ and $\vec{m}$ be constant vectors, and $r^2 \equiv |\vec{x}|^2 = \sum_{i=1}^{3}(x^i)^2$. Let
   $$\psi(\vec{x}) = \frac{\vec{p}\cdot\vec{x}}{r^3} \qquad \text{and} \qquad \vec{A}(\vec{x}) = \frac{\vec{m}\times\vec{x}}{r^3}$$
   (a) Calculate $\vec{\nabla}\psi$
   (b) Calculate $\vec{\nabla}\times\vec{A}$
   (c) *Bonus:* The integral curves of a vector field are such that the field is everywhere tangent to the curves. Assuming both $\vec{p}$ and $\vec{m}$ point in the third (say "$\hat{z}$") direction, sketch the integral curves of $\vec{\nabla}\psi$ and $\vec{\nabla}\times\vec{A}$ (on a suitable defined 2-dimensional plane). What is the relation of $\vec{\nabla}\psi$ to surfaces of constant ψ?

3. The first part is practice using Stoke's theorem, the second Gauss's. In these, $\hat{z}$ stands for a unit vector in the 3rd direction (Cartesian coordinates):
   (a) Sketch the vector field $\vec{A} = \hat{z}\times\vec{x}$. In whatever coordinate system you choose, calculate directly $\oint \vec{A}\cdot d\vec{\ell}$ around a circle of radius R in the xy plane centered at the origin. Calculate this again, but now using Stoke's theorem.
   (b) Sketch the vector field $\vec{B} = \vec{x}$. Calculate directly the surface integral $\int_{\mathcal{S}} da\ \vec{B}\cdot\hat{n}$, where $\mathcal{S}$ is the boundary to the cube $0 \leq x, y, z, \leq L$ and $\hat{n}$ is an outward pointing unit vector normal to $\mathcal{S}$. Calculate this again, but now using Gauss's theorem.

4. Calculate $\vec{\nabla}\psi$ and $\nabla^2\psi$ for each of the following fields:
   $$(a)\ \psi = \sqrt{x^2+y^2+z^2} \qquad\qquad (b)\ \psi = \frac{1}{\sqrt{x^2+y^2+z^2}}$$
   Do each calculation three ways: in Cartesian, cylindrical and spherical coordinates.

5. Consider the vector field (here we use $\vec{x} = (x, y, z)$)
   $$\vec{A} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$
   Show that both the divergence and the curl of this field vanish. Show that $\oint \vec{A}\cdot d\vec{\ell}$ calculated around a circle in the xy plane centered at the origin is non-vanishing.
   This seems to contradict Stoke's theorem. Can you figure out what went wrong?