PHYS 203A: Problems

Week 1

1. ~~(a) Prove~~ Gauss's theorem: $\int_R d^3x \, \mathrm{div}(\vec{a}) = \int_{\partial R} dA \, \hat{n} \cdot \vec{a}$, where R is a region in $\mathbb{R}_3$ and $\vec{a}$ is a vector field. Prove it is true for an infinitesimal cube and then express the integral over R as a sum of integrals over such cubes that partition R.

   (b) Prove two alternative forms of Gauss's theorem, $\int_R d^3x \, \nabla \psi = \int_{\partial R} d\vec{A} \, \psi$ and $\int_R d^3x \, \nabla \times \vec{F} = \int_{\partial R} d\vec{A} \times \vec{F}$, by applying the thm. for $\vec{a} = \vec{b}\psi$ and $\vec{a} = \vec{b} \times \vec{F}$ respectively, where $\vec{b}$ is a fixed vector.

2. Show that in 3d $\nabla^2\left(\frac{-1}{4\pi r}\right) = \delta^{(3)}(\vec{r})$, i.e. $\frac{-1}{4\pi r}$ is the Green's function for the Laplacian. This means that $\int d\vec{r} \, \nabla^2\left(\frac{-1}{4\pi r}\right) \Phi = \Phi(0)$ for any function Φ.

~~for~~ (hint: ~~integrate by~~ Note that $\nabla^2(\frac{1}{r})=0$ for $r \neq 0$, so you can take the integral over an infinitesimal ball centered at the origin. Then integrate by parts twice using Gauss's thm.)

3. Consider the integral $\int d^3y\, \epsilon^{ijk} a_j \partial_k \phi$, where a is a vector field and ϕ a function which vanishes at infinity. Show that the quantity under the integral transforms as a contra-variant vector. Integrate by parts and conclude that $\frac{1}{\sqrt{g}} \epsilon^{ijk} \partial_j a_k$ is a contra-variant vector, and therefore $\mathrm{Curl}(a)^i = \frac{1}{\sqrt{g}} \epsilon^{ijk} \partial_j a_k$ (where g is the metric determinant). Specify to the case of curvilinear coordinates, and convert to the components of a in an orthonormal basis, A^i, and show that in this case

$$\mathrm{Curl}(A)^i = \frac{1}{\sqrt{g_{jj}\, g_{kk}}} \epsilon_{ijk} \partial_j (\sqrt{g_{kk}} A^k).$$

4. Consider prolate spherical coordinates, defined ~~[illegible]~~ in terms of Cartesian by:

$$x = \sinh\mu \cos\nu \cos\varphi \qquad \mu \in [0,\infty)$$
$$y = \cosh\mu \sin\nu \sin\varphi \qquad \nu \in [0,\pi]$$
$$z = \cosh\mu \cos\nu \qquad \varphi \in [0,2\pi)$$

(a) find the metric in these coordinates. Verify that it is a curvilinear coordinate system.

(b) find expressions for divergence, gradient, curl, and Laplacian in these coordinates

$\boxed{\text{Week 2}}$

1. Show that $\lim_{N\to\infty}\left(1+\frac{1}{N}J\right)^N = \sum_{n=0}^{\infty}\frac{1}{n!}J^n = \exp(J)$, for any operator J. This shows how a finite transformation is constructed from the generating function of an infinitesimal one. (Use the binomial thm. to expand the LHS).

~~2~~.

2. (a) find the generating functions for rotations in 3d (group $SO(3)$). Show that they can be written as $(J_i)_{jk} = -\epsilon_{ijk}$

(b) Verify that they satisfy the $SO(3)$ algebra: $[J_i, J_j] = \epsilon_{ijk} J_k$

(c) find the rotation matrix, ~~in terms of~~

$$R_{\hat{n}}(\theta) = \exp(-\theta\, \hat{n}\cdot \vec{J}),$$

(d) Show that, if S is a rotation which takes $\hat{e}_3 \to \hat{n}$, then $S J_3 S^{-1} = \hat{n}\cdot\vec{J}$.

Therefore, $R_{\hat{n}}(\theta) = \exp(-\theta\, \hat{n}\cdot\vec{J}) = \exp(-\theta\, S J_3 S^{-1})$

$= S \exp(-\theta J_3) S^{-1} = S R_{\hat{e}_3}(\theta) S^{-1}$ is indeed a rotation about $\hat{n}$ through angle θ.

(e) Show that for a vector $\vec{r}$,

$$\vec{r}\,' = R_{\hat{n}}(\theta)\,\vec{r} = \vec{r}\cos\theta + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\theta) + (\vec{r}\times\hat{n})\sin\theta$$

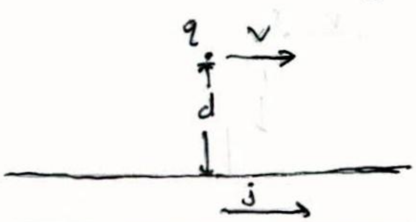
3. (a) Consider the integral $\int d^3x\, x^i x^j f(|\vec{x}|)$, where i and j are coordinate indices.

Show that it is a second rank tensor invariant under rotations. Conclude that $\int d^3x\, x^i x^j f(|\vec{x}|) \propto \delta_{ij}$, since δ_{ij} is the only invariant rank two tensor for $SO(3)$. Find the coefficient of proportionality by taking the trace of both sides to show that $\int d^3x\, x^i x^j f(|\vec{x}|) = \frac{1}{3}\delta_{ij} \int d^3x\, |\vec{x}|^2 f(|\vec{x}|)$

(b) Express $\int d^3x\, x^i x^j x^k x^l f(|\vec{x}|)$ in terms of invariant tensors by the procedure of part (a).

Week 3

I. An infinite straight wire has current j. A charge q moves with velocity v a distance d away from it.



(a) Calculate the magnetic field of the wire and the corresponding force on the charge.

(b) Noting that $(\rho, \vec{J})$ ~~is~~ is a four-vector, transform to the rest frame of the charge and find the the charge and current ~~of~~ in the wire in this frame. Calculate the resulting fields of the wire and force on the particle.

(c) Verify that your results in parts (a) and (b) are consistent with eachother.

2. Practice making covariant generalizations of various relation in electromagnetism (e.g. Ohm's law). See in particular Jackson, problems 11.16 and 11.17.