Week 6

1. The wavefunction in Q.M. can be viewed as a classical field. The action for the Schrödinger field is given by:

$$S = \int d^3x\, dt \left[i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V(\vec{x})\psi^*\psi \right]$$

(a) ψ and ψ^* may be thought of as independent fields. Show that varying S w.r.t. ψ^* yields the Schrödinger eq. for ψ.

(b) Show that S is invariant under the $U(1)$ symmetry $\psi \to e^{i\alpha}\psi$. Compute the associated Noether current and check that it is conserved. This is just the probability current for the particle.

(c) Compute the energy and momentum density using the Noether procedure. Integrate over space to obtain the total energy and momentum.

(d) Show that S is invariant under Galilean transformations: $t \to t$, $\vec{x} \to \vec{x} - \vec{v}t$, provided ψ also transforms: $\psi \to e^{i\alpha(t,\vec{x};\vec{v})}\psi$ and find $\alpha(t,\vec{x};\vec{v})$.

2. <u>Bonus problem</u>: Given that the Green's function for the wave equation is a Lorentz scalar, it should be possible to derive it in a manifestly covariant way. (i.e. explicitly preserving Lorentz invariance at each step of the calculation). In this problem, we outline a method to do this.

(a) Show that the function $\frac{1}{k^2 - i\epsilon\,\mathrm{sgn}(k_0)}$ (ϵ infinitesimal) has its poles in k_0 positioned slightly above the real axis. Conclude that $G_r(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 - i\epsilon\,\mathrm{sgn}(k_0)}$ is an expression for the retarded Green's function of the 3+1 d wave eq.

(b) To re-express the denominator, we use the technique of Schwinger parametrization. Show that $\frac{1}{A} = \int_0^\infty ds\, e^{-sA}$ provided $\mathrm{Re}\,A > 0$ and use this to re-write $\frac{1}{k^2 - i\epsilon\,\mathrm{sgn}(k_0)}$ as an integral over ~~and~~ exponential.

(c) Interchange the order of integration and do the Gaussian integrals over $\mathbf{k}$. To do this, it is useful to shift the value of x_0 by a Lorentz transformation to $x_0 \to \mathrm{sgn}(x_0)\infty$. (this can always be done since the Green's function is a Lorentz invariant).

(d) Do the remaining integral over s ~~to~~ find $G_r(x)$ in covariant form: $G_r(x) = \frac{1}{2\pi}\Theta(x_0)\delta(x^2)$. Show that this is equivalent to the form derived in the lecture.

(e) Super bonus: repeat the steps above to find the retarded covariant Green's function for the Klein Gordon eq.: $(\partial^2 + \mu^2) G_r(x) = \delta^{(4)}(x)$. The answer may be expressed in terms of a Bessel function using the integral representation

$$J_\nu(x) = \frac{2}{\pi}\int_0^\infty dt\, \sin(x\cosh t - \tfrac{1}{2}\nu\pi)\cosh(\nu t).$$

Show that your answer reduces to that of part (d) as $\mu \to 0$.