→ Reduced Models

- Reduced MHD I
- Reduced MHD II
- Further applications – DW's.

For Posting:

Reduced Models

Reduced MHD → Simplifying the representation:

→ strong magnetization – anisotropy.

$\delta B_{\parallel} \to 0$; $\delta \underline{B}_{\perp} = \nabla A_{\parallel} \times \hat{z}$

$\delta \underline{V}_{\perp} = \nabla \phi \times \hat{z}$.

→ $\tau > \tau_{MS}$

Aside

→ $\boxed{\text{Reduced MHD}}$ → Reduced Representation for strong ⊗ straight B_0 – eliminates fast mode

① Note: full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " $(\nabla \cdot \underline{B} = 0)$
ρ, p

⇒ 7 components

② if $\nabla \cdot \underline{V} = 0$ ⇒ 4 components
($\rho = \text{const}$, p from $\nabla \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics) → later

"strong" ⟷ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ order : $B_z \sim \nabla_{\perp} \sim 1$

$B_{\perp} \sim \partial_z \sim O(\epsilon)$

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Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z:

$V_\perp^2 \sim p \sim B_\perp^2$ (i.e. equipartition of energy)
(springiness)

$\Rightarrow$ $V_\perp \sim \epsilon$, $p \sim \epsilon^2$, $\partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ / incompressibility)

$\delta(B_z^2) \sim 2B_z \delta(B_z) \sim p$

(esbrm)
i.e. $\omega \ll k(c_s^2 + V_A^2)^{1/2}$
[idea is to order out the fast mode]

$\Rightarrow$ $\delta B_z \sim \epsilon^2$.

$\therefore$ to lowest order $\Rightarrow$ $B_z = \text{const}$.

Now then: $(\underline{\nabla} \cdot \underline{B} = 0)$

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla}\psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \to 0$.

Similarly,

$\partial_z p \sim O(\epsilon^3)$, $\Rightarrow$ $V_z \ll V_\perp$
$J_\perp B_\perp \sim \epsilon^3$ neglect V_z.

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Now, $$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \nabla\phi = -\frac{\underline{v}\times\underline{B}}{c}$$

$$\Rightarrow \quad +\frac{1}{c}\frac{\partial A}{\partial t} = \frac{\underline{v}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$$

$$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\underline{\partial_z \rho_z}$)

$\therefore$ $\nabla_\perp \phi \cong \left(\frac{\underline{v}\times\underline{B}}{c}\right)_\perp$, in $(*)$ $V_\perp \leftrightarrow \nabla\phi$.

{Inductive piece negligible.

$$\Rightarrow \boxed{\underline{V}_\perp = \frac{c\hat{z}\times\nabla\phi}{B_z}}$$

$\perp$ velocity

$\rightarrow$ {motion $\perp$ is $\underline{E}\times\underline{B}$.

Now, taking parallel component of $(*)$.

(units!)

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\psi + \underline{v}\cdot\underline{\nabla}\psi = B_z\partial_z\phi}$$

so have (vector potential / flux) equation:

$$\boxed{\frac{\partial \psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$\underline{v}\cdot\nabla\psi$ from

$\underline{B}\cdot\nabla\phi \rightarrow$

$B_0\,\partial_z\phi + \nabla\psi\times\hat{z}\cdot\nabla\phi$

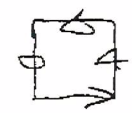
equation of evolution of magnetic flux.

$$= B_z\hat{z} + \hat{z}\times\nabla\psi$$

or, alternatively, $$\boxed{\frac{\partial\psi}{\partial t} - \mathbf{B}\cdot\nabla\phi = 0}\ . \qquad 94.$$

Finally, for ϕ, write:

$\perp$ motion



cells of $E\times B$ drift, ('spin an' rate?)

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{\nabla p}{\rho_0} + \frac{\mathbf{J}\times\mathbf{B}}{c}$$

$(\nabla\times)\cdot\hat{z} \Rightarrow$ vorticity component ($\parallel \hat{z}$) evolution.

Dynamics in $\perp$ plane, $\nabla\phi\times\hat{z} = \mathbf{V}_\perp$; so $\Rightarrow \hat{z}\cdot\omega$

$$\frac{\partial}{\partial t}\omega_z + \mathbf{V}_\perp\cdot\nabla\omega_z = -\cancelto{0}{\nabla\times\frac{\nabla p}{\rho_0}} + \hat{z}\cdot\nabla\times\left(\frac{\mathbf{J}\times\mathbf{B}}{c}\right)$$

$$= \mathbf{B}\cdot\nabla J_z - \cancel{\mathbf{J}\cdot\nabla B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \mathbf{B}\cdot\nabla J_z$$

$\therefore$

$$\boxed{\frac{\partial}{\partial t}\omega_z + \mathbf{V}\cdot\nabla\omega_z = \mathbf{B}\cdot\nabla J}$$

but:

$$\omega_z = \hat{z}\cdot\nabla\times\mathbf{V} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\nabla\times\mathbf{B})\frac{c}{\cancel{4\pi}} = \nabla^2\psi$$

75.

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

Finally, have reduced MHD equation:

$B = B_0\hat{e}_\parallel$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi$$

$E_\parallel = \eta J_\parallel$

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi$$

vorticity in plane $\perp \hat{e}_\parallel$.

- note have reduced MHD to 2 scalar evolution equations

- does this look familiar?

- 2D dynamics + shear Alfven wave.

- nonlinearity → 2D dynamics.

75.

<u>even stronger!</u>

$$B_0 \partial_z \to 0, \quad \partial_z \psi = 0, \quad \partial_z \phi = 0$$

l. O. + [illegible]

- for $\underline{\underline{2D}}$ MHD:

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi &= \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi \\ \frac{\partial \psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi &= \eta\nabla^2\psi \end{aligned}}$$

- ① Conservation Laws, etc. (HW)

<u>3D</u>

$$\frac{d}{dt} E = \underset{(\text{to } \eta,\ \nu)}{0}, \quad E = \int d^3x \left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right]$$

② $\mathcal{H} = A\cdot B \cong B_z \psi$, $B_z \to$ const.

<u>3D</u>

$$\int d^2x\, A^2 = \text{MSMP} \ (2D)$$

$$\Rightarrow H = \int d^3x\, B_z \psi, \quad \frac{dH}{dt} = 0, \text{ to } O(\eta)$$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\begin{cases} H \text{ conserved} \\ E_M \text{ dissipated} \end{cases}$

③ $\underline{K = \int d^3x\, \underline{V}\cdot\underline{B} = \int d^3x\, (\nabla\phi\cdot\nabla\psi)}$

also conserved, to dissipation.

Alfven wave [illegible] imbalance.

Reduced MHD – Brief

See Strauss, '76 for full details (PoF)

→ the point: – strong $\langle \underline{B} \rangle$, ≈ straight

– low frequency $(\omega < \omega_{MS})$

– $\langle B \rangle$ ≈ unperturbed

– $\nabla \cdot \underline{V} = 0$

$\underline{\nabla} \cdot \underline{V} = 0 \Rightarrow$ 2 component $\underline{V}$

$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow$ 2 components $\underline{B}$

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} \simeq 0 \quad \Rightarrow \quad \underline{V}_\perp = +\frac{c}{B}\, \underline{E}_\perp \times \hat{z}$$

$$\underline{E}_\perp = -\underline{\nabla}_\perp \phi \cancel{- \frac{1}{c}\frac{\partial A_\perp}{\partial t}}$$

$E \cdot B_\parallel \to 0$

$$V_\perp = -\frac{c}{B}\, \nabla_\perp \phi \times \hat{z}$$

$\underline{\nabla} \times \underline{A} = 0$

$\underline{B}_\perp = \nabla A_\parallel \times \hat{z}$ $\qquad B_\perp = \nabla A_\parallel \times \hat{z}$

Then,

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = \eta \underline{J}$$

$$E_{\parallel} = -\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \nabla_{\parallel}\phi = \eta J_{\parallel}$$

$$\underline{B} = B_0 \hat{z} + \underline{B}_{\perp}$$

$\Rightarrow$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \frac{(B_0\hat{z} + \underline{\tilde{B}}_{\perp})\cdot\nabla\phi}{|B_0\hat{z} + \underline{\tilde{B}}_{\perp}|} = \eta J_{\parallel}$$

$\therefore$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \underline{\tilde{B}}_{\perp}\cdot\nabla\phi = \eta J_{\parallel}$$

and

$$\boxed{-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \underline{\nabla} A_{\parallel}\times\hat{z}\cdot\nabla\phi = \eta J_{\parallel}}$$

so $\Rightarrow$

$$\boxed{\frac{\partial \psi}{\partial t} + \underline{V}\cdot\nabla\psi = \partial_z\phi + \eta\nabla^2\psi}$$

reduced Ohm's Law

Now,  strong field

$+$

$v_\perp$ only → set by ϕ

so, $\frac{B_0 \cdot \nabla \times v}{|B_0|} = \hat{z} \cdot \nabla \times v$ → is the key to dynamics

Now, $\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p + \frac{J \times B}{c}$

$$v \cdot \nabla v = \nabla \frac{v^2}{2} - v \times \omega$$

$$\rho\left(\frac{\partial v}{\partial t}\right) = -\nabla\left(p + \rho \frac{v^2}{2}\right) + \rho\, v \times \omega + \frac{J \times B}{c}$$

$\rho \equiv \rho_0$ (incompr/Boussinesq)

$$\frac{\partial v}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right) + v \times \omega + \frac{J \times B}{c\rho}$$

$$= -\nabla\left(\frac{p}{\rho} + \frac{B^2}{8\pi\rho} + \frac{v^2}{2}\right) + v \times \omega + \frac{B \cdot \nabla B}{4\pi\rho}$$

$$= -\nabla\left(p/\rho + \frac{B^2}{8\pi} + \frac{v^2}{2}\right) + v \times \omega + \frac{B_0 \partial_z \tilde{B}_\perp}{4\pi\rho} + \frac{\tilde{B}_\perp \cdot \nabla_\perp \tilde{B}_\perp}{4\pi\rho}$$

(no B_0 curv)

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so $\nabla \times$

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{\omega} + \frac{B_0}{4\pi\rho_0} \partial_z \underline{\nabla} \times \underline{\tilde{B}_\perp}$$

$$+ \frac{\underline{\tilde{B}_\perp}}{4\pi\rho_0} \cdot \nabla_\perp \left(\underline{\nabla} \times \underline{\tilde{B}_\perp} \right)$$

$$= -\underline{V} \cdot \underline{\nabla} \omega + \underline{\omega} \cdot \underline{\nabla} \underline{V} + \frac{B_0}{4\pi\rho} \partial_z \underline{\nabla} \times \underline{\tilde{B}_\perp}$$

$$+ \frac{\underline{\tilde{B}_\perp}}{4\pi\rho} \cdot \underline{\nabla_\perp} \left(\underline{\nabla} \times \underline{\tilde{B}_\perp} \right)$$

$\hat{z} \cdot ( ) \Rightarrow$

$$\Rightarrow \frac{d}{dt} \omega_z = \cancel{\underline{\omega} \cdot \underline{\nabla} V_z} + \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_{z\parallel} + \frac{\nabla \tilde{A}_\parallel \times \hat{z}}{4\pi\rho_0} \cdot \nabla \tilde{J}_{\parallel}$$

$$\omega_z \rightarrow \nabla^2 \phi$$

so

$$\boxed{\frac{d}{dt} \nabla^2 \phi = \partial_t \nabla^2 \phi + \underline{V} \cdot \underline{\nabla} \nabla^2 \phi = \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_z + \frac{\underline{\tilde{B}}}{4\pi\rho_0} \cdot \nabla \tilde{J}_z}$$

$\rightarrow$ Vorticity eqn!

Alternative Approach:

1.

$$\hat{\parallel} \cdot (\underline{E} = \eta \underline{J})$$

→ as before!

2.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0 \quad , \qquad \text{continuity!}$$

$$\rho = (n_i - n_e) e$$

and QN ⇒

$$\boxed{\underline{\nabla} \cdot \underline{J} = 0} \quad \rightarrow \text{generic!}$$

⇒

$$\underline{\nabla}_\perp \cdot \underline{J}_\perp = -\nabla_\parallel J_\parallel$$

advective

$$\left[n_0 M_i \frac{d\underline{V}_0}{dt} = n_0 \underline{E} - \underline{\nabla} P + n_0 e \frac{\underline{V} \times \underline{B}_0}{c} \right.$$

→ $O(\omega/\Omega_i)$ expansion, low $P(T_i)$. Ω

$$\underline{J}_\perp \simeq (n_i \underline{V}_E - n_0 \underline{V}_E) e + n_i e \underline{V}_{pol}$$

E×B current cancels. polarization current → cons ($M_i \gg m_e$)

$$\boxed{\underline{\nabla}_\perp \cdot (n_0 \underline{V}_{pol}) = -\frac{1}{e} \nabla_\parallel J_\parallel} \quad \rightarrow \text{vort eqn}$$

$$= -1/e \left(\partial_z \tilde{J}_\parallel + \tilde{\underline{B}}_\perp \cdot \underline{\nabla}_\perp \tilde{J}_\parallel \right)$$

and back to vorticity eqn.!

⇒ can extend to H-W, H-M, 3 field, ITG...

→ Now, can relate routes to RMHD:

So can come to RMHD by different orders of strong field and fluid approx.

Now, extensions:

(14)

Outline

A) A Look Back and A Look Around: Basic Ideas of the Drift Wave-Zonal Flow System

Drift Wave Models

Drift - Alfven.

B) A Look Ahead: Current Applications to Selected Problems of Interest

A) A Look Back and A Look Around

Basic Ideas of the Drift Wave – Zonal Flow System

i) Physics of Zonal Flow Formation

ii) Shearing Effects on Turbulence Transport

iii) Closing the Feedback Loops: Predator(s) Meet Prey

"The difference between an idea and a theory is that the first can generate a call to action and the second cannot."

— Stanley Fish

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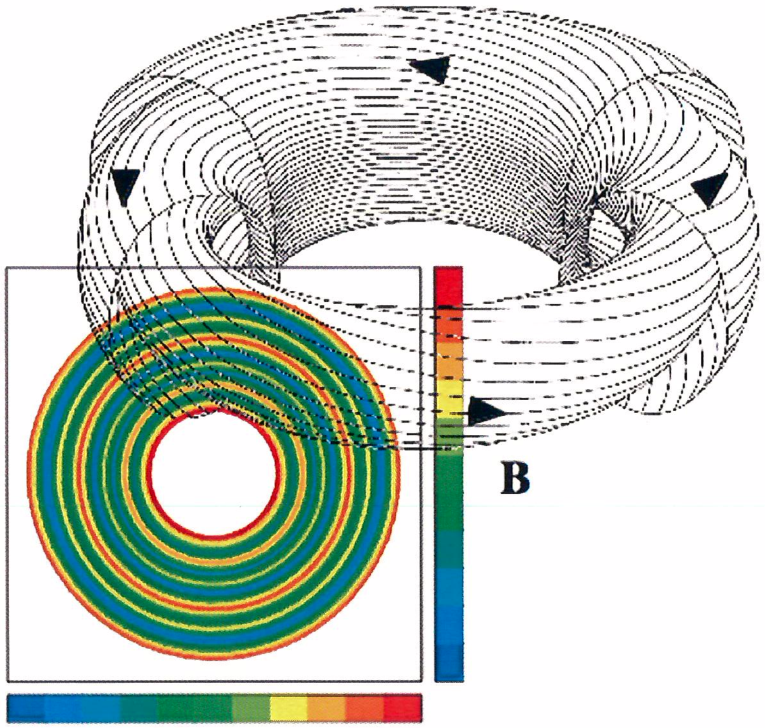
Preamble I

Tokamaks



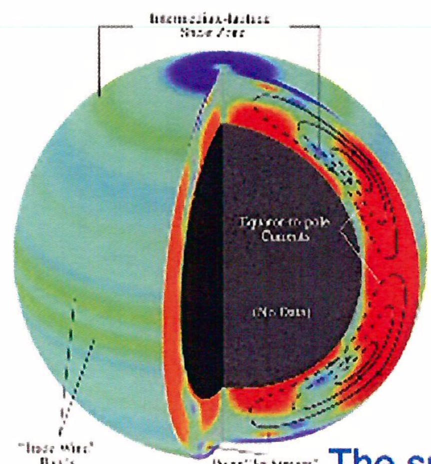
Zonal Flows:

$$m = n = 0$$

finite k_r

potential fluctuations

planets



The sun

Preamble II

→ Re:Plasma?

→ 2 Simple Models

a.) Hasegawa-Wakatani (collisional drift inst.)

b.) Hasegawa-Mima (DW)

a.) $\mathbf{V} = \frac{c}{B}\hat{z} \times \nabla\phi + \mathbf{V}_{pol}$

$\qquad\qquad\qquad\qquad \to m_s$

$L > \lambda_D \rightarrow \nabla \cdot \mathbf{J} = 0 \rightarrow \nabla_\perp \cdot \mathbf{J}_\perp = -\nabla_\| J_\|$

$J_\perp = n|e|V_{pol}^{(i)}$

$J_\| : \eta J_\| = -(1/c)\cancel{\partial_t A_\|} - \nabla_\| \phi + \nabla_\| p_e$

e.s.

linear, if conduction dropped.

n.b.

– MHD: $\partial_t A_\|$ v.s. $\nabla_\| \phi$

– DW: $\nabla_\| p_e$ v.s. $\nabla_\| \phi$

can correlated Drift Alfven

b.) $dn_e/dt = 0$

$$\rightarrow \frac{dn_e}{dt} + \frac{\nabla_\| J_\|}{-n_0|e|} = 0$$

$J_\| \to$ electrons.

$$\rightarrow \begin{cases} \frac{\partial f}{\partial t} + v_\| \nabla_\| f - \frac{c}{B}\nabla\phi \times \hat{z} \cdot \nabla f \\ - \frac{q}{m}\nabla_\| \phi \frac{\partial f}{\partial v_\|} = C(f) \end{cases}$$

Ohm's Law – From DKE

Ohm's Law is key to model zoology.

$\underline{\nabla} \cdot \underline{J} = 0$ is universal. Ohm's Law varies

__DKE:__ (for electrons)

$$\frac{\partial f}{\partial t} + v_{\parallel} \nabla_{\parallel} f - \frac{c}{B_0} \nabla \phi \times \hat{z} \cdot \nabla f$$

$$- \frac{|e|}{m_e} E_{\parallel} \frac{\partial f}{\partial v_{\parallel}} = C(f_e)$$

$$\nabla_{\parallel} = \partial_z + \frac{\delta \underline{B}_{\perp}}{B_0} \cdot \underline{\nabla} \rightarrow \underline{\text{nonlinear}}\ \underline{\text{operator}}$$

$$E_{\parallel} = -\frac{1}{c} \frac{\partial A_{\parallel}}{\partial t} - \nabla_{\parallel} \phi \rightarrow \text{nonlinear} + \text{induction}$$

Ohm's Law: $\int v_{\parallel} f \rightarrow$ electron momentum equation and $\int (v_\parallel^2 + v_\perp^2) \to P_e$.

so, $v_\parallel$ moment:

current advection → electron inertia

$$m_e \left[\frac{\partial J_\parallel}{\partial t} - \frac{c}{B} \nabla \phi \times \hat{z} \cdot \nabla J_\parallel \right]$$ usually neglected

collisions

$$+ \frac{1}{c} \frac{\partial A_\parallel}{\partial t} - \nabla_\parallel \left[\frac{|e|\hat{\phi}}{T} - T \frac{\hat{n}}{n_0} \right] = \eta J_\parallel$$

$E_\parallel$ $E_\parallel$

from ∥ pressure gradient ($\nabla_\parallel P_e$)
(can include $\tilde{T}$)

Point:

- electrostatics → drop $\partial A_\parallel / \partial t$
- MHD - like → drop $\nabla_\parallel P$
- Drift waves → drop induction
  retain ∥ P.

MHD

$\nabla_\parallel \phi$ vs $\eta J_\parallel$, $\frac{1}{c} \frac{\partial A_\parallel}{\partial t}$

all → drift Alfven.

DW

$\nabla_\parallel \phi$ vs $T \nabla_\parallel n$

So H-W

$D_\parallel = \frac{v_{The}^2}{\nu_{e,e}}$

$\hat{\phi} \to e\tilde{\phi}/T$

$\hat{n} \to \tilde{n}/n_0$

$-\nabla_\parallel \phi = -\nabla_\parallel \hat{n} T + \eta J_\parallel$

$$\rho_s^2 \frac{d}{dt}\nabla^2\hat{\phi} = -D_\parallel \nabla_\parallel^2(\hat{\phi} - \hat{n}/n_0) + \nu\nabla^2\nabla^2\hat{\phi}$$

from $\nabla_\parallel J_\parallel$

$\parallel$ diffn vs freq.

$\left[D_\parallel k_\parallel^2/\omega \right.$ is key parameter

$$\frac{d}{dt}n - D_0\nabla^2\hat{n} = -D_\parallel \nabla_\parallel^2(\hat{\phi} - \hat{n}/n_0)$$

from $\nabla_\parallel J_\parallel$

— decouple for Z.F.

n.b. $\quad \underline{\mathrm{PV}} = n - \rho_s^2\nabla^2\phi \qquad \frac{d}{dt}(\mathrm{PV}) = 0$ <u>inviscid</u>.

→ total ~~density~~/ charge

akin conserved phase space density.

b.) $D_\parallel k_\parallel^2/\omega \gg 1 \to \hat{n}/n_0 \sim e\hat{\phi}/T_e \qquad (m, n \neq 0)$

$$\left[\frac{d}{dt}(\phi - \rho_s^2\nabla^2\phi) + v_*\partial_y\phi = 0 \right. \qquad \to \text{H-M}$$

[derive via continuity eqn.

$\underline{\nabla}\cdot\underline{V}_i = \cancel{\underline{\nabla}\cdot\underline{V}_E} + \underline{\nabla}\cdot\underline{V}_{pol}$

$\frac{\hat{n}_i}{n} = \frac{|e|\hat{\phi}}{T}$

n.b. $\quad \mathrm{PV} = \phi - \rho_s^2\nabla^2\phi + \ln n_0(x)$

n.b. **Zonal Flows:** $\rho_s^2 \frac{d}{dt}\nabla^2\phi = -\mu\nabla^2\phi + \nu\nabla^2\nabla^2\phi$

$\nabla_\parallel \to 0 \qquad$ decouple.

An infinity of models follow:

- MHD: ideal ballooning — + curvature
  resistive → RBM — curvature, e.s

- HW + $A_{\|}$: drift - Alfven —

- HW + curv. : drift - RBM

- HM + curv. + Ti: Fluid ITG

- gyro-fluids

- GK

N.B.: Most Key advances appeared in consideration of simplest possible models

Lecture Notes of Non-linear Plasma Theory
Chapter 6
Hasegawa-Wakatani, Hasegawa-Mima and Quasi-Geostrophic Models

Xiang Fan — Notes
Prof. Patrick H Diamond — Lecture

February 28, 2014

1 Introduction

Hasegawa-Wakatani model and Hasegawa-Mima model, which will be called H-W model and H-M model for short in the rest of the lecture notes, are useful to describe drift-wave turbulence, and also important for understanding zonal flow. These two models have very similar form to quasi-geostrophic models.

In section 2 and section 3, we will talk in detail about Hasegawa-Wakatani model and Hasegawa-Mima model respectively. Section 4 is some discussion about H-W and H-M models. In section 5, comparison with quasi-geostrophic is shown. In section 6, the relationship between H-W H-M model and MHD will be discussed.

2 Hasegawa-Wakatani Model

H-W model is a consistent model of drift wave turbulence, no forcing added by hand. It is a model describing two variables, density n and electric potential ϕ. It is a simple limit of 2-fluid system. Now we are going to derive H-W equations[1], some assumptions are made during this process.

[1] Thanks to D. Strintzi's lecture note "Simple models of plasma turbulence", see http://www2.ipp.mpg.de/~fsj/PAPERS_1/tutorial_3.pdf

Start from ion force balance equation:

$$\frac{d\mathbf{u}}{dt} = -\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u}\times\mathbf{B} - \nabla p_i - \nabla\Pi + \mathbf{F} \tag{1}$$

Now we assume cold ions, $T_i \ll T_e$, thus $\nabla p_i = 0$. Assume $\nabla\Pi$ has the form $-\nabla\Pi = \mu\nabla^2\mathbf{u}$, where μ is the ion viscosity coefficient. Also we have $\mathbf{F} = 0$.

Then let's deal with $\mathbf{u}$. The 0th order of it is the $\mathbf{E}\times\mathbf{B}$ drift:

$$\mathbf{u}_0 = \mathbf{u}_E = -\nabla\phi \times \frac{\mathbf{B}}{B_0^2} \tag{2}$$

where B_0 is the homogeneous background magnetic field. Now let $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, and substitute it into the ion force balance equation (1), then we can get the 1st order of $\mathbf{u}$:

$$\mathbf{u}_1 = \mathbf{u}_p + \mathbf{u}_{visc} = -\frac{1}{\omega_{ci}B_0}\frac{d\nabla\phi}{dt} - \frac{\mu}{\omega_{ci}B_0}\nabla^2(\nabla\phi) \tag{3}$$

where ω_{ci} is the usual $\omega_{ci} = eB_0/m$. The first term $\mathbf{u}_p$ is the polarization drift, and the second term $\mathbf{u}_{visc}$ is an additional term caused by viscosity. Here we have introduced a small parameter $\epsilon = \frac{1}{\omega_{ci}}\frac{\partial}{\partial t}$, $\epsilon \ll 1$ means the strong magnetic field approximation. It is easy to verify $\frac{\mathbf{u}_1}{\mathbf{u}_0} \sim \epsilon$.

Note that, although $\mathbf{u}_0 \gg \mathbf{u}_1$, we have

$$\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{u}_E = 0 \tag{4}$$

$$\nabla\cdot\mathbf{u}_1 \neq 0 \tag{5}$$

So the divergence of polarization drift and the viscosity term is not neglectable.

Now we can substitute $\mathbf{u}$ into the charge conservation equation $\nabla\cdot\mathbf{J} = 0$:

$$\nabla_\perp\cdot\mathbf{J}_\perp = -\nabla_\parallel J_\parallel \tag{6}$$

where $\mathbf{J}_\perp$ comes from $\mathbf{u}_1$

$$\mathbf{J}_\perp = n|e|\mathbf{u}_1 \tag{7}$$

and $J_\parallel$ comes from the parallel force balance equation (i.e. Ohm's Law):

$$\nabla_\parallel\phi + \eta\mathbf{J}_\parallel - \frac{1}{en_0}\nabla_\parallel p_e = 0 \tag{8}$$

After some substitution and normalization of the variables, we get the first part of the H-W equations:

$$\rho_s^2\frac{d}{dt}\nabla^2\phi = -D_\parallel\nabla_\parallel^2(\phi - \frac{n}{n_0}) + \nu\nabla^2\nabla^2\phi \tag{9}$$

In order to give the corresponding equation for electron, we start from the force balance equation, but for electron we put: $\nabla\Pi = 0$, $m_e n_e \frac{d\mathbf{u}}{dt} = 0$, but we keep the electron-ion friction:

$$\mathbf{F}_{e\parallel} = -m_e n_e \nu_{ei} \mathbf{u}_{e\parallel} = \frac{m_e \nu_{ei}}{e} \mathbf{J}_\parallel \tag{10}$$

And the parallel force balance equation (i.e. Ohm's Law) can give us

$$\nabla_\parallel \phi + \eta \mathbf{J}_\parallel - \frac{1}{e n_0} \nabla_\parallel p_e = 0 \tag{11}$$

The electron polarization drift is neglectable because of the mass ratio, so the continuity equation is simple:

$$\frac{dn}{dt} + n \nabla_\parallel \cdot \mathbf{u}_{e\parallel} = 0 \tag{12}$$

Combine them together, we can get the other part of H-W equations. After changing some notations, we can write the *Hasegawa-Wakatani Equations* in a beautiful way:

$$\boxed{\begin{aligned} \rho_s^2 \frac{d}{dt} \nabla^2 \phi &= -D_\parallel \nabla_\parallel^2 (\phi - \frac{n}{n_0}) + \nu \nabla^2 \nabla^2 \phi \\ \frac{d}{dt} n - D_0 \nabla^2 n &= -D_\parallel \nabla_\parallel^2 (\phi - \frac{n}{n_0}) \end{aligned}} \tag{13}$$

3 Hasegawa-Mima Model

From H-W equations (13), we can see that the key parameter is $D_\parallel k_\parallel^2/\omega$. If $D_\parallel k_\parallel^2/\omega \gg 1$, we can get $\frac{\tilde{n}}{n_0} \sim \frac{e\phi}{T_e}$, which means the electrons are adiabatic. In both electron adiabatic limit and collisionless limit (i.e. $\nu = 0$), the H-W equations will become the *Hasegawa-Mima Equation*:

$$\boxed{\frac{d}{dt}(\phi - \rho_s^2 \nabla^2 \phi) + v_* \partial_y \phi = 0} \tag{14}$$

It is important to note that, the H-M equation can be written as the form of potential vorticity conservation:

$$\boxed{\frac{d}{dt}(PV) = 0} \tag{15}$$

where

$$PV = \phi - \rho_s^2\nabla^2\phi + \ln n_0 \tag{16}$$

PV is short for potential vorticity. What exactly is a potential vorticity? We can say it is a kind of generalization of vorticity. Actually potential vorticity has different form in different systems, for example, the above expression is the potential vorticity in plasmas; another example, is in quasi-geographic model, which will be discussed in section 5. Actually what's important is that, we find a conserved quantity!

4 Discussion about H-W and H-M Models

4.1 Cartoon image of drift wave

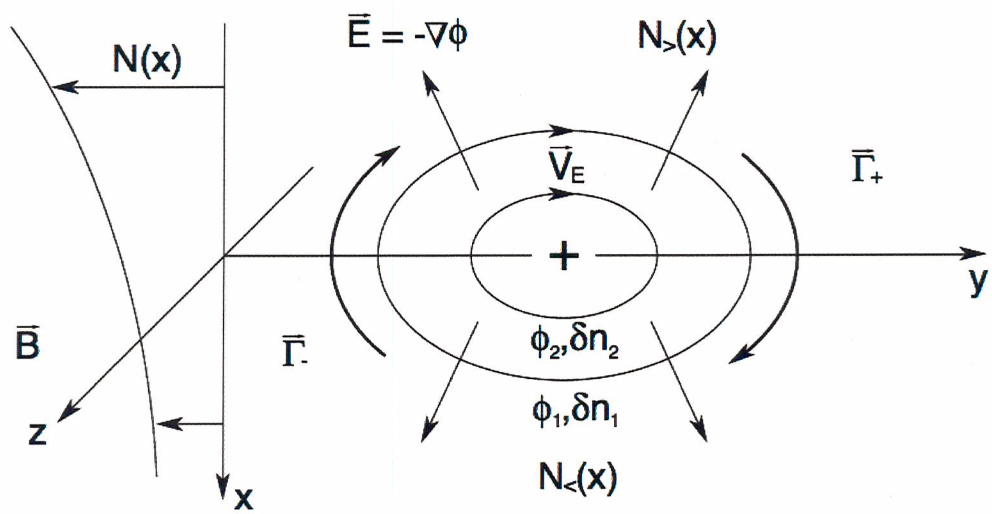


Figure 1: Cartoon image of drift wave.
From `http://peaches.ph.utexas.edu/ifs/ifsreports/Review.pdf`

H-W and H-M equations can support drift waves. Drift wave means density or potential wave caused by drift flows such as $\mathbf{E}\times\mathbf{B}$ flow. Figure 1 demonstrates a simple example of drift wave. Here is the set up of this simple system (for simplicity, let's assume it's the positive charges that move):

(1) The particle density $N(x)$ is homogeneous along y and z direction, but has a gradient along x. As shown in the figure, let ∇N face -x direction.

(2) $\mathbf{B} = \mathbf{B}_0$ is a constant everywhere and its direction is +z.

(3) There is a positive density perturbation at the "+" point in the figure.

Let's see what happens next. The electric potential ϕ will rise following the density, then it will produce radiation-shaped electric field $\mathbf{E} = -\nabla\phi$. Electrical field $\mathbf{E}$ along with $\mathbf{B}$ will produce $\mathbf{E} \times \mathbf{B}$ flow, and the direction of this flow is a clockwise circle around the "+" position. Note that there are more particles on the upper half compared to the lower half because of the density gradient, so more particles will comes to the right, which means the density perturbation is propagating along +y direction. This is a simple cartoon of the drift wave.

4.2 Linear dispersion relations of H-W H-M equations

For H-M equation, let's assume a plane wave solution $\phi = \phi_{\mathbf{k}} exp(i\mathbf{k}\cdot\mathbf{x} - \omega t)$, and plug it in (14), we can get

$$\omega = \frac{v_* k_y}{1 + \rho_s^2 k^2} \tag{17}$$

For H-W equation, let $\phi = \phi_{\mathbf{k}} exp(i\mathbf{k}\cdot\mathbf{x} - \omega t)$ and $n = n_{\mathbf{k}} exp(i\mathbf{k}\cdot\mathbf{x} - \omega t)$, and plug it in (13). This gives us two equations, and the determinant=0 will give us the dispersion relation:

$$\omega(1 + \frac{k^2}{1 - i\omega/c_1}) = -\mathbf{k} \times \hat{z} \cdot \nabla \ln n_0 - ic_2 k^4 \tag{18}$$

4.3 Conservation quantities

In H-M model, generalized energy W is conserved:

$$W = \int dV \frac{\phi^2 + (\nabla\phi)^2}{2} \tag{19}$$

And also the generalized enstrophy U is conserved:

$$U = \int dV \frac{(\nabla\phi)^2 + (\nabla^2\phi)^2}{2} \tag{20}$$

5 Comparison to quasi-geostrophic model

Hasegawa-Mima model is quite similar to the quasi-geostrophic model (we will call it Q-G for short in this section), and the latter is less abstract, so let's discuss a little bit to have a more clear picture about these models.

Table 1: Comparison between H-M model and quasi-geostrophic model

Hasegawa-Mima Model	quasi-geostrophic model
Lorentz force	Coriolis force
$\mathbf{E} \times \mathbf{B}$ flow	geostrophic current
potential vorticity conservation	potential vorticity conservation
drift wave	Rossby wave
zonal flow	zonal flow

The basic comparison is in Table 1. In Q-G model, the basic variable to solve ϕ is the geopotential height, while in H-M model ϕ is the electric potential. The basic study object in Q-G model is the motion of wind (or ocean flow). The motion of wind will be affected by Coriolis force, just as the Lorentz force in H-M model. So geostrophic current will be formed, like the $\mathbf{E} \times \mathbf{B}$ flow. Thus Rossby wave, similar to drift wave, can propagate in this model. In the end, zonal flow will emerge in both models.

Kelvin's Theorem is the governing equation of Q-G model. Let ω be the relative angular velocity of the motion of wind, and Ω is the planetary angular velocity. Then Kelvin's Theorem says,

$$\oint_C \mathbf{v} \cdot d\mathbf{l} = \int d\mathbf{a} \cdot (\omega + 2\Omega) = Const \tag{21}$$

This theorem is easy to proof, so we omit it here. If we change some variables, it can be written in a better way. Let y be the velocity component towards the north pole, and define a constant $\beta = 2\Omega \sin\theta_0 R$, where $\sin\theta_0$ is the latitude. Then Kelvin's Theorem can also be written as:

$$\frac{d}{dt}(\omega + \beta y) = 0 \tag{22}$$

Here $PV = \omega + \beta y$, so this equation has the form of potential vorticity conservation, just the same as H-M equation (15).

Now let's talk a little about the Rossby wave. We can write the Kelvin's Theorem as following:

$$\frac{d}{dt}(\nabla^2\phi) + \beta\partial_x\phi = 0 \tag{23}$$

The linear dispersion relation of it is

$$\omega = -\frac{\beta k_x}{k^2} \tag{24}$$

which looks like (17). From this dispersion relation, we can get the group velocity:

$$v_{gy} = 2\beta \frac{k_x k_y}{(k^2)^2} \tag{25}$$

It has the structure of Reynolds stress! So Rossby wave is intimately connected to momentum transport.

6 Relationship to MHD

The generalized Ohm's Law is

$$\nabla_{\parallel}\phi - (\mathbf{u} \times \mathbf{B})_{\parallel} + \eta \mathbf{J}_{\parallel} - \frac{1}{en_0}\nabla_{\parallel} p_e = 0 \tag{26}$$

Compare to what we used to derive H-W and H-M equations (8), actually we neglected the second term.

On the other hand, in MHD, what we usually do is to neglect the last term:

$$\eta \mathbf{J}_{\parallel} = (\mathbf{E} + \mathbf{u} \times \mathbf{B})_{\parallel} \tag{27}$$

So we know that H-W/H-M and MHD are in different limit of the generalized Ohm's Law.