Steepening of Alfven waves due to parallel compressibility

(Ref : "Relaxation Dynamics in Laboratory and Astrophysical Plasmas"
Ch 4.4 , P.H. Diamond , et. al. 2010)

shocks?

Alfven waves <-> $\nabla \cdot \underline{V} = 0$ $\not\longrightarrow$ Steepening

No

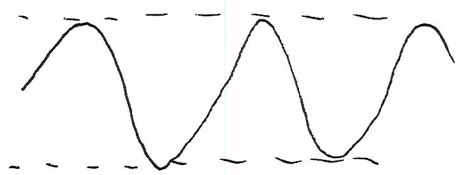
However, a little compressibility will lead to steepening of wave ~~[illegible]~~ packet.

Approach : Modulational instability

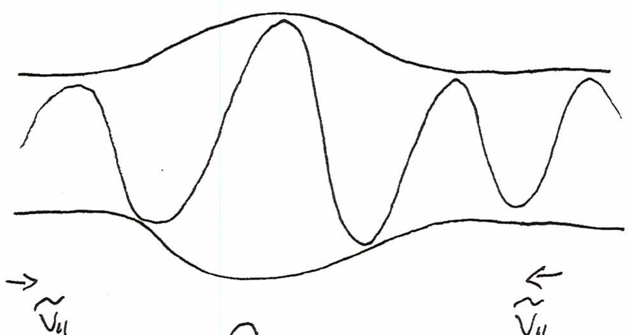
n.b. time scale disparity
(2) (3) (1)

$$\frac{V_A}{L_{\parallel}} , \quad \frac{C_s}{L_{\parallel}} , \quad \frac{V_{MS}}{L_{\perp}}$$

Alfven wave :



Small compression :
expansion



$\Rightarrow$ Parallel flow $\tilde{V}_{\parallel}$

$\tilde{V}_{\parallel} \sim \tilde{B}^2$

$\downarrow$

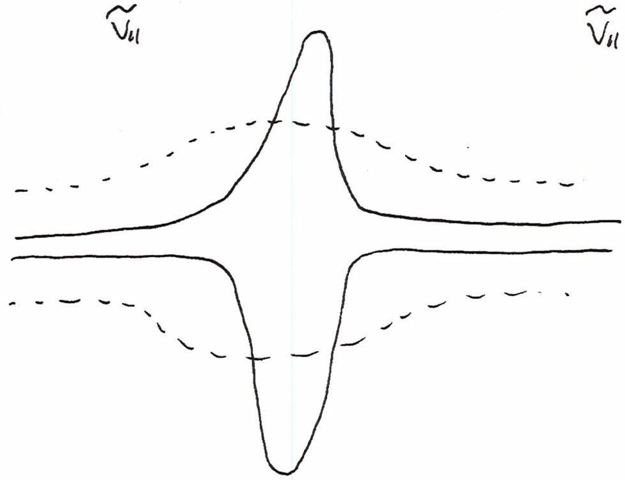
$\tilde{B}\uparrow$, $\frac{\partial}{\partial t}\tilde{B} = -\underline{V}\cdot\nabla\tilde{B}$

feedback loop

$\partial_z \tilde{B}^2 \rightarrow \{ \tilde{U}_{\parallel}, \tilde{\rho}$

$\rightarrow$ change in V_A

Steepening :



Result.

Until : Saturated by - dissipation η

- dispersion $d_i \sim \frac{c}{\omega_{pi}}$ $\Leftarrow$ ion inertial scale

Physical derivation :

Need equation for envelope . $\Longleftrightarrow$ Scale separation $\nearrow$ fast $\searrow$ slow

Dispersion relation for Alfven wave with perturbed density $\rho = \rho_0 + \tilde{\rho}$:

$$\omega = k_{\parallel} V_A = k_{\parallel} \frac{B_0}{\sqrt{4\pi(\rho_0+\tilde{\rho})}} \cong \underbrace{k_{\parallel} V_A}_{\text{fast}} - \underbrace{\frac{k_{\parallel} V_A}{2} \frac{\tilde{\rho}}{\rho_0}}_{\text{slow}} , \quad V_A = \frac{B_0}{\sqrt{4\pi \rho_0}}$$

$$\Rightarrow \boxed{\omega = \omega^{(0)} + i \left(\frac{\partial}{\partial t}\right)_{slow} , \quad k_{\parallel} = k_{\parallel}^{(0)} - i \left(\frac{\partial}{\partial x}\right)_{slow}}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \delta B = - \frac{V_A}{2} \boxed{\frac{\tilde{\rho}}{\rho_0}} \left(\frac{\partial}{\partial x} \delta B\right)} , \quad x, t \text{ are slow variables.}$$

Envelope eqn

Set by $\tilde{V}_{\parallel}$ dynamics

$$\frac{\partial}{\partial t} \tilde{\rho} = - \rho_0 \frac{\partial}{\partial x} \tilde{V}_{\parallel}$$

$$\rho_0 \frac{\partial}{\partial t} \tilde{V}_{\parallel} = - \frac{\partial}{\partial x} \tilde{P} - \frac{\partial}{\partial x} \left(\rho_0 \frac{\tilde{V}^2}{2} + \frac{\delta B^2}{8\pi} \right)$$

For Alfven wave , $\rho_0 \frac{\tilde{V}^2}{2} = \frac{\delta B^2}{8\pi}$

and $\tilde{P} = c_s^2 \tilde{\rho}$, $c_s^2 = \gamma \frac{\tilde{P}}{\rho}$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \tilde{V}_{\parallel} = c_s^2 \frac{\partial^2}{\partial x^2} \tilde{V}_{\parallel} - \frac{\partial^2}{\partial t \partial x} \frac{\delta B^2}{4\pi \rho_0}$$

Transfer to the frame of wave moving , $\tilde{V}_{\parallel} = \tilde{V}_{\parallel}(x - V_A t)$

$$\boxed{\tilde{V}_{\parallel} = \frac{V_A}{V_A^2 - c_s^2} \frac{\delta B^2}{4\pi \rho_0}} , \quad \boxed{\frac{\tilde{\rho}}{\rho_0} = \frac{\tilde{V}_{\parallel}}{V_A} = \frac{1}{1-\beta} \frac{\delta B^2}{B_0^2}} , \quad \beta = \frac{c_s^2}{V_A^2}$$

Plug $\frac{\tilde{\rho}}{\rho_0}$ into the envelope eqn :

$$\boxed{\frac{\partial}{\partial t} \delta B = - \frac{V_A}{2} \frac{1}{1-\beta} \underbrace{\boxed{\frac{\delta B^2}{B_0^2}}}_{\text{Steepening}} \frac{\partial}{\partial x} \delta B} + \boxed{\ ???\ }$$

Steepening v.s. ?? (What stops steepening?)

DNLS

compare

NLS

Steepening is stopped by :

1. Dissipation : $\eta \frac{\partial^2}{\partial x^2} \delta B$ ← small → resistive ole' shock.

2. Dispersion : $i d_i^2 \Omega_i \frac{\partial^2}{\partial x^2} \delta B$ ← ion inertial scale physics.

↘ Quasi-parallel Alfvenic "collisionless shock"

Full envelope eqn :

$$\boxed{\frac{\partial}{\partial t}\delta B = -\frac{v_A}{2}\frac{1}{1-\beta}\frac{\delta B^2}{B_0^2}\frac{\partial}{\partial x}\delta B + \eta \frac{\partial^2}{\partial x^2}\delta B + i d_i^2 \Omega_i \frac{\partial^2}{\partial x^2}\delta B}$$

which is a derivative nonlinear Schrödinger (DNLS) equation.

This brings us to . . .

<u>Collisionless Shocks</u>!

Ion-Acoustic Shocks and Solitons – {simplest form c-shock}

In quasi-neutral system $(k^2 \lambda_{De}^2 \ll 1)$

$$n_e = n_0 \exp\left[|e|\phi / T_e\right]$$

$$\frac{\partial n_i}{\partial t} + v \frac{\partial n_i}{\partial x} = -n_i \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -|e| \frac{\partial \phi}{\partial x}$$

$$\phi = \frac{T_e}{|e|} \ln (n_e/n_0) = \frac{T_e}{|e|} \ln (n_i/n_0)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{T_e}{|e|} \frac{n_0}{n_e} \frac{1}{n_0} \frac{\partial n_i}{\partial x}$$

$$\Rightarrow \quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{T_e}{n_i}\frac{\partial n_i}{\partial x}$$

→ isomorphic to 1D gas-dynamic equations (albeit isothermal)

→ steepening, shock formation will occur

→ but, as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. isomorphism to gas dynamics ⇒ $k^2\lambda_D^2 \ll 1$ regime (quasi-neutrality)

Shock structure limited when $L \sim \lambda_{De}$
→ Quasi-neutrality violated!

i.e. allowing for dispersion:

$$n_e = \exp[|e|\phi/T_e] \qquad \text{Boltzmann Electrons}$$

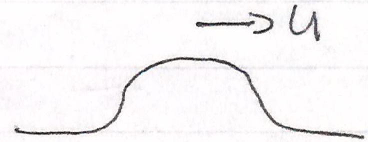
$$\left.\begin{aligned} &\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v) = 0 \\ &\partial v_i/\partial t + v_i \partial v_i/\partial x = -\frac{|e|}{m_i}\frac{\partial \phi}{\partial x} \end{aligned}\right\} \text{fluid ions}$$

$$\tilde{n}_e = \exp(q\phi/T_e)$$

$\tilde{n}_i$:
$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i V) = 0$$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} = -\frac{q}{m_i}\frac{\partial \phi}{\partial x}$$

$$\begin{Bmatrix} n_i \\ V_i \\ \phi \end{Bmatrix} = F(x-ut)$$



i.e. localized solution, moving at u.

<u>pulse</u>

$$-u n_i' + (n_i V)' = 0$$

$$(V-u)V' = -\frac{q}{m_i}\phi'$$

Now, integrating with
$$\left.\begin{matrix} \phi \to 0 \\ V \to 0 \\ n \to \underset{\equiv 1}{n_0} \end{matrix}\right\} \quad x \to \infty$$

⇒

$$-u n_i + n_i V = -u$$

$$\Rightarrow (u-V)n_i = u \quad \to \text{ to ensure } n \to n_0$$

$$n_i = u/(u-V)$$

Similarly,

$$\frac{q\phi}{m_i} = -\frac{1}{2}(u-v)^2 + \frac{u^2}{2} \qquad \text{(to ensure } \phi \to 0\text{)}$$

$$\Rightarrow \left(\frac{1}{2}u^2 - \frac{q}{m_i}\phi\right) = \frac{1}{2}(u-v)^2$$

$$\Rightarrow (u-v) = \left(u^2 - \frac{2q\phi}{m_i}\right)^{1/2}$$

so
$$\frac{\partial^2\phi}{\partial x^2} = -4\pi n_0 q\left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp(q\phi/T_e)\right)$$

$$\phi'\phi'' = -4\pi n_0 q\,\phi'\left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp\left(\frac{q\phi}{T_e}\right)\right)$$

integrating $\Rightarrow$

$$\frac{1}{2}\phi'^2 + V(\phi) = 0$$

$$\boxed{V(\phi) = -4\pi n_0\left\{ m u\left(u^2 - \frac{2q\phi}{m}\right)^{1/2} + T_e e^{q\phi/T_e}\right\} + C}$$

$\hookrightarrow$ Sagdeev Potential

$$\phi'' = dV/d\phi$$

Collisionless shock $\Leftrightarrow$ solitary wave

$\rightarrow$ Ion acoustic soliton reduced to particle orbit pblm!

Define Mach # $\mathcal{M} = u/c_s$

$$u = \mathcal{M} c_s$$

$\Rightarrow$

$$V(\phi) = -4\pi n_0 \left\{ m \mathcal{M} c_s \left(\mathcal{M}^2 c_s^2 - \frac{2q\phi}{m}\right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

$$= -4\pi n_0 \left\{ T_e \mathcal{M} \left(\mathcal{M}^2 - \frac{2q\phi}{T_e}\right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

Thus:

$\rightarrow$ need $\mathcal{M}^2 > 2q\phi_M/T_e$ for soliton to exist (reality)

$\therefore$ $\frac{u^2}{c_s^2} > 2q\phi_M/T_e$ (critical velocity)

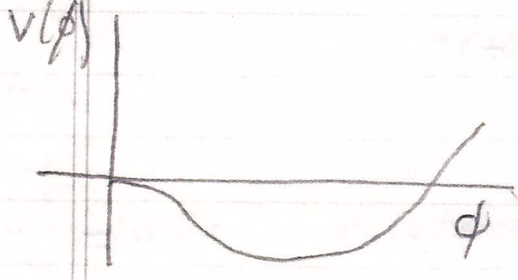
$\rightarrow$ speed – amplitude connection

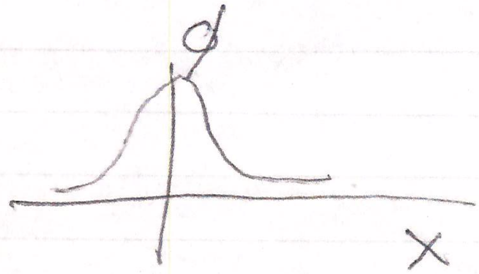
$\rightarrow$ Similarly, for small ϕ,

$$V(\phi) \simeq -4\pi n_0 \left\{ T_e \mathcal{M}^2 \left(1 - \frac{q\phi}{\mathcal{M}^2 T_e} - \frac{1}{2}\left(\frac{q\phi}{T_e \mathcal{M}^2}\right)^2\right) + T_e + q\phi + \frac{T_e}{2}\left(\frac{q\phi}{T_e}\right)^2 \right\}$$

$$\simeq -4\pi n_0 \left\{ T_e(1+\mathcal{M}^2) + \cancel{q\phi} - \cancel{q\phi} + \frac{T_e}{2}\left(\frac{q\phi}{T}\right)^2 \left(-\frac{1}{\mathcal{M}^2} + 1\right) \right\}$$

Now, for soliton $\Rightarrow$ need bound state





Then $V''(\phi)\big|_{\phi\to 0} < 0 \Rightarrow \mathcal{M}^2 > 1$

So need $\mathcal{M} > 1$ for soliton formation.

$\rightarrow$ Similarly, need $\mathcal{M} \lesssim 1.6$

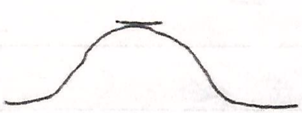
$\therefore$ for soliton, need $1 < \mathcal{M} < 1.6$
($e\phi/T$ small)

i.e. have

$$V(\phi) = -4\pi n_0 \left\{ mu\left(u^2 - \frac{2e\phi}{m}\right)^{1/2} + T_e\, e^{e\phi/T} \right\}$$

$$= -\phi'^2$$

take ϕ_{max} when $V(\phi) = 0 \rightarrow \phi' = 0$



$\Rightarrow$ defines ϕ_{max}

More Generally:

→ as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. quasi-neutrality ⟺ $k^2\lambda_{De}^2 \lll 1$

when $L_{shock} \sim \lambda_{De}$ ⇒ quasineutrality violated!

∴ ion-acoustic shock limited by dispersion

i.e. $\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_{De}^2$

→ Generally, sub-classify shocks into:

collisional → old standard hydrodynamics
L_{shock} limited by dissipation

collisionless → a/a' ion-acoustic in plasma
L_{shock} limited by dispersion
⇒ forms soliton

Aside: Some Generic Properties of Solitons

Contrast → Sound Wave $\quad \omega = k c_s$
$x = (c_s + v)t + f(v)$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + c_s \frac{\partial v}{\partial x} = 0$$

→ Dispersive Ion Acoustic Wave

$$\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_D^2$$

$$k\lambda_D < 1 \Rightarrow \quad \omega = k c_s \left(1 - k^2 \lambda_D^2 / 2\right)$$ (ω odd in k)

suggests model equation of form:

$$\frac{\partial \mathcal{E}}{\partial t} + (c_s + \mathcal{E}) \frac{\partial \mathcal{E}}{\partial x} + \frac{c_s \lambda_D^2}{2} \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

of generic form:

$$\frac{\partial \mathcal{E}}{\partial t} + u_0 \frac{\partial \mathcal{E}}{\partial x} + \alpha \mathcal{E} \frac{\partial \mathcal{E}}{\partial x} + \beta \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

$$a = \alpha \mathcal{E}$$
$$y = x - u_0 t$$

$$\Rightarrow \boxed{\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} + \beta \frac{\partial^3 a}{\partial y^3} = 0}$$ (Korteweg - deVries Eqn. (KdV))

dispersion vs dissipation

contrast

$$\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} - \bar{\nu} \frac{\partial^2 a}{\partial y^2} = 0$$ (Burgers Eqn.)

Burgers → dissipative ($\bar{\gamma}$ limits steepening)

$$L_{shock} \sim \bar{\gamma}/a$$

KdV → dispersive (ω variation with k ⇒ u variation with k limits steepening - diff'nt scale comp.)

$$L_{soliton} \sim (\beta/a)^{1/2}$$

Solution of KdV Equation:

$$\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial y} + \beta\frac{\partial^3 a}{\partial y^3} = 0$$

$$a = a(y - V_0 t) \qquad \Rightarrow \quad V_{wave} = U_0 + V_0$$

⇒

$$\beta a''' + a a' - V_0 a' = 0$$

Invariant
$a \to a + V$
$V_0 \to V_0 + V$

∴

$$\beta a'' + \frac{1}{2} a^2 - V_0 a = \frac{1}{2} C_1$$

$$2\beta a' a'' + a' a^2 - 2V_0 a' a = C_1 a' \qquad (* \ 2a')$$

$$\Rightarrow \quad \beta a'^2 = -\frac{1}{3} a^3 + V_0 a^2 + C_1 a + C_2$$

→ can reduce to quadrature

Convenient to factorize:

$V_0,\ C_1,\ C_2 \rightarrow a_1,\ a_2,\ a_3$

$$\Rightarrow \quad \beta a'^2 = -\frac{1}{3}(a-a_1)(a-a_2)(a-a_3)$$

where $V_0 = \frac{1}{3}(a_1 + a_2 + a_3)$

For $\rightarrow$ bounded $|a(y)|$

$\rightarrow$ need a_1, a_2, a_3 real

if $a_1 > a_2 > a_3$

$$\Rightarrow a_1 \geq a \geq a_2 \qquad (\beta a'^2 > 0)$$

$\therefore$ $a_3 = 0$ is no loss generality

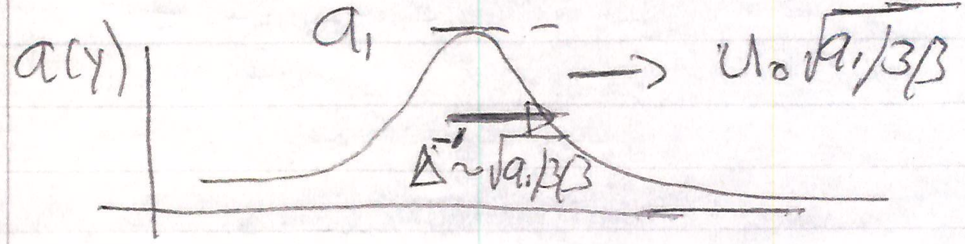
$$\Rightarrow \quad \beta a'^2 = \frac{1}{3}(a_1 - a)(a - a_2)\, a$$

if $a_2 = 0$ — Exact solution of NL KdV Eqn.

$$\therefore \quad \boxed{a(y) = a_1 \cosh^{-2}\left(\frac{1}{2} y \sqrt{a_1/3\beta}\right)}$$

$$= a_1 \cosh^{-2}\left(\frac{1}{2}(x - u_0 t)\sqrt{a_1/3\beta}\right)$$

$\Rightarrow$



$\Rightarrow$ soliton has finite width

$\Delta \sim \sqrt{3\beta / a_1}$

$\beta \sim \lambda_{De}^2$ for IA

$\Rightarrow \Delta \sim \lambda_D$

$\longleftrightarrow$ contrast zero-width shock

$\rightarrow$ soliton has finite amplitude a_1

with $V \sim u_0 \sqrt{a_1/3\beta}$

$\therefore$ bigger solitons move faster!

Note: $a_2 \neq 0 \Rightarrow$ non-localized, oscillatory solution

General comments:

$\rightarrow$ Collisional shock $\Delta \sim \nu/a$

Collisionless shock $\Delta \sim \lambda_D \sqrt{c_s/a}$

$\therefore$ Debye length sets "discontinuity" scale

$\rightarrow$ Can treat collisionless shock via

$$\nabla^2 \phi = -4\pi n_0 q (\tilde{n}_i - \tilde{n}_e)$$

etc $\Rightarrow$ Sagdeev Potential